A four-input three-stage queuing network approach to model an industrial system

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A B S T R A C T

An industrial system is represented as a four-input, three-stage queuing network in this paper. The four-input queuing network receives orders from clients, and the orders are waiting to be served. Each order comprises (i) time of occurrence of the orders, and (ii) quantity of items to be delivered in each order. The objective of this paper is to compute the optimal path which produces the least response time for the delivery of items to the final destination along the three stages of the network. The average number of items that can be delivered with this minimum response time constitute the optimum capacity of the queuing network. After getting serviced by the last node (a queue and its server) in each stage of the queuing network, a decision is made to route the items to the appropriate node in the next stage which can produce the least response time. Performance measures such as average queue lengths, average response times, average waiting times of the jobs in the four-input network are derived and plotted. Closed-form expressions for the equivalent service rate, equivalent average queue lengths, equivalent response and waiting times of a single equivalent queue with a server representing the entire four-input queuing network are also derived and plotted.

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1. Introduction

Current generation enterprises such as global supply chains, virtual enterprises and e-businesses are driving research in the area of enterprise modeling framework suitable for a distributed environment. Supply chain (SC) is a concept which can be considered analogous to a pipeline of physical and informational flows between suppliers and customers. From an operational point of view, this pipeline works like a process of activities, and these activities are distributed. Each company is at the center of a network of suppliers and customers [1].

The supply chain could be defined as “a network of connected and interdependent organizations mutually and co-operatively working together to control, manage, and improve the flow of materials and information from suppliers to end users” [2]. Since the supply chain management (SCM) is a market-driven concept, it is necessary to adopt the requirements of the customer. When considering the performance of a SCM system, the inputs are the “orders”, and the outputs are the “goods”. The main challenge of the SCM system is to improve the performance while reducing the costs (generally in terms of trade-offs). One of the performance features is the responsiveness of the SCM system, and the corresponding key indicator is the gap between the order’s cycle (i.e., the delay for the order to be fulfilled), and the logistics (procurement, manufacturing, delivery, etc) lead-time. This gap is due to supply management inefficiency, bottleneck activities, setup times, and the inventory and transport activities of the interfaces.

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A widespread trend observable over many years is the globalization of the supply chain with few offshore manufacturing/sourcing and distribution centers which means that the SC extends from one side of the globe to the other. Components may be sourced from one or two countries, and assembled in other countries; and the final products are centralized in a global warehouse for distribution. These strategic decisions are generally justified by lower manufacturing costs, but the performance results are at a higher level of risk for extended lead times due to transport requirements (inventory activities at least). This is often the case of textile industries. This paper focuses on one such example.

1.1. Industrial system description

The industrial system constitutes some basic activities. They are (i) knitting, (ii) making, and (iii) distribution (the central warehousing). For performance evaluation, the supply chain is modeled by a process with these three activities (three stages). In fact, these activities may be supported by operational resources physically distributed in many sites and interlinked by transportation.

In the first two stages, textile designers work to knit and weave to create two-dimensional designs that can be produced in a design repeat for the production of textile products. Textile designers may also work in associated industry functions, for e.g., designing wrapping paper, packaging, greeting cards and ceramics.

Typical work activities in the textile industry include (i) producing sketches and design ranges for presentation to customers, (ii) using specialist Computer Aided Design (CAD) software to produce a range of designs, (iii) experimenting with color, fabric and texture, maintaining up-to-date knowledge of developing design and production techniques, (iv) developing...
new design concepts, (v) visiting sites and other sources of ideas for designs, assessing and approving completed items, (vi) ensuring that projects are completed on time, (vii) sourcing fabrics and other materials at trade fairs, markets, and (viii) developing a network of business contacts. Consider Fig. 1.

- **Knitting** locations are in L1 (France), L2 (Morocco), and C (Contractor).
- **Making** locations are in L1 and L2, L3 (Morocco), and L4 (Tunisia). The warehouse location is in L1.

There are routing choices for the physical flows at two steps of the processes. They are:

- **Knitting**: from S2 to (L1 or L2), from S3 to (L2 or C).
- **Making**: from L1 to (L1 or L3), (or) from L2 to (L2 or L4).

Each resource is modeled as a queue where batches are waiting to be processed (see Fig. 1). The routing decision may be performed considering the estimated throughput delay (From S1 to S4). This delay includes the manufacturing delay (depending on the batch quantities to be processed) and also the total waiting times in all the downstream queues. Comparing with the routing problem in telecommunication networks (IP networks), the problem is not a hop by hop problem [3], but we consider the whole route to make the decision.

1.2. Literature review

There have been quite a number of research papers published in the area of modeling e-businesses, enterprize systems, assembly and manufacturing systems using queuing networks. In [4], the fundamental matrix of the Discrete-Time Markov Chain (DTMC) is used to obtain the average number of visits an email makes to a particular node before getting resolved. Performance measures such as throughput, utilization, and response time using efficient algorithms such as convolution and mean-value analysis are derived in [5].

Two approaches developed in [6] generate a queuing network model from a (i) business process markup language and (ii) formal Petri-net based business process representation. In [7], a closed queuing network model with state-dependent routing probabilities is developed for the study of interactive computing systems, and an algorithm is proposed to obtain an approximate solution of the mathematical model. Chandy et al. [8] discussed a queuing network of M service stations and N customers, and deduce an “equivalent” network of a subsystem, where all the queues in a subsystem are replaced by a single composite queue.

In [9], a technique is presented whereby queuing network models and generalized stochastic Petri nets are combined in such a way as to exploit the best features of both modeling techniques. Muppala et al. [10] discussed the construction and solution of finite-state continuous-time Markov chain using a variation of stochastic Petri nets called Stochastic Reward Nets (SRN). The general approach of the queuing network analyzer in [11], is to approximately characterize the arrival process by two moments and then analyze individual queues individually. Arbitrary configurations of open or closed network of single server or multiple-server queues with finite or infinite capacity are also analyzed in [11]. Melamed and Yadin [12] presented a numerical method based on the tagged customer approach for evaluating the response time distribution in a discrete-state Markovian queuing network. In [13], queuing network models of supply chains are developed to model supply, transportation, and distribution operations. Leung and Kamath [14] studied the performance evaluation of an assembly system with components or sub-assemblies feeding into a kitting and assembly stage framework.

1.3. Organization of the paper

The objective of this paper is to compute the minimum response time, and the average number of items that can be delivered with this response time, for the delivery of an item to the final destination along the three stages of the queuing network. Section 2 describes the queuing network modeling the industrial system, and derives expressions for utilizations of each node (queue and a server) in the network. Section 3 discusses performance measures like average response times, average queue lengths, and average waiting times of individual nodes and different paths in the network. Section 4 describes the numerical results. Finally, Section 5 presents the conclusions.

2. Queuing network description

The analysis of the four-input, three-stage queuing network is made as follows:

2.1. Stage 1

There are four-inputs in the queuing network considered in Fig. 1. The arrival rates at the four-inputs are λ1, λ2, λ3 and λ4, respectively. The arrival rate at source (S1) is (λ + δ). The probability of arrivals at S2 and S3 are s1 and s2, respectively. The
For satisfactory management and control requirements, it is required that the overall system in terms of (i) production, and (ii) delivery.

Let $\lambda_1(= \lambda q_1)$ be the arrival rate of jobs at $Q_4$, and let $\lambda_2(= \lambda q_2)$ be the arrival rate at $Q_5$. Let the service rates of servers $A4$ and $A5$ be $\mu_1$ and $\mu_2$, respectively. After getting serviced at server $A4$, the jobs arrive at the queues $Q_8$ and $Q_{15}$ with probabilities $p_1$ and $p_2$, respectively. So, the arrival rate at $Q_8$ is $\lambda_1 p_1$, and the arrival rate at $Q_{15}$ is $\lambda_1 p_2$. Jobs which get serviced by server $A5$ arrive at the queues $Q_{17}$ and $Q_{18}$ with probabilities $p_3$ and $p_4$, respectively. So, the arrival rate at $Q_{17}$ is $\lambda_2 p_3$, and the arrival rate at $Q_{18}$ is $\lambda_2 p_4$.

Let $\lambda_3(= \delta q_1)$ be the arrival rate of jobs at $Q_{21}$, and let $\lambda_4(= \delta q_2)$ be the arrival rate at $Q_{22}$. Let the service rate of servers $A21$ and $A22$ be $\mu_3$ and $\mu_4$, respectively. Let the service rates of servers $A3$, $A6$, $A22$ be $\mu_5$, $\mu_6$, and $\mu_4$, respectively. Jobs which get serviced by server $A3$ enter the queues $Q_{17}$ and $Q_{19}$ with probabilities $p_5$ and $p_6$, respectively. Similarly, jobs which get serviced by server $A22$ enter the queues $Q_8$ and $Q_{16}$ with probabilities $p_7$ and $p_8$, respectively.

2.2. Stage II

- The total arrival rate of jobs at queue, $Q_8$, is $(\lambda_1 p_1 + \lambda_4 p_7)$. Jobs with this arrival rate get serviced by server $A8$ whose service rate is $\mu_8$.
- The total arrival rate at $Q_{17}$ is $(\lambda_2 p_3 + \lambda_3 p_5)$. The jobs get serviced by server $A17$, whose service rate is $\mu_{17}$. Jobs with arrival rate, $(\lambda_2 p_3 + \lambda_3 p_5)$, get serviced by server $A9$ whose service rate is $\mu_9$.
- The arrival rate at $Q_{15}$ is $(\lambda_1 p_2 + \lambda_2 p_4)$. Jobs at queue, $Q_{15}$, get serviced by server $A15$ whose service rate is $\mu_{15}$. Since $Q_{15}$ and $Q_3$ are in serial connection, the arrival rate at $Q_3$ is also $(\lambda_1 p_2 + \lambda_2 p_4)$. The service rate of server $A7$ is $\mu_7$.
- The arrival rate at $Q_{18}$ is $(\lambda_2 p_4 + \lambda_3 p_6)$. Jobs at queue, $Q_{18}$, get serviced by server $A18$ whose service rate is $\mu_{18}$. Since $Q_{18}$ and $Q_{10}$ are in serial connection, the arrival rate at $Q_{10}$ is also $(\lambda_2 p_4 + \lambda_3 p_6)$. The service rate of server $A10$ is $\mu_{10}$.

2.3. Stage III

- Jobs getting serviced by server $A8$, arrive at queue, $Q_{11}$, with arrival rate, $(\lambda_1 p_1 + \lambda_4 p_7)$, and get serviced by server $A11$, whose service rate is $\mu_{11}$.
- Jobs getting serviced by server $A9$, arrive at $Q_{12}$ with arrival rate, $(\lambda_2 p_3 + \lambda_3 p_5)$, and get serviced by server $A12$, whose service rate is $\mu_{12}$.
- Jobs getting serviced by server $A7$, arrive at $Q_{13}$ with arrival rate, $(\lambda_2 p_4 + \lambda_3 p_6)$, and get serviced by server $A13$, whose service rate is $\mu_{13}$.
- Jobs getting serviced by server $A10$, arrive at $Q_{14}$ with arrival rate, $(\lambda_2 p_4 + \lambda_3 p_6)$, and get serviced by server $A14$, whose service rate is $\mu_{14}$.

Finally, jobs after service completion at servers $A11$, $A12$, $A13$, and $A14$ arrive at the sink, $S_4$, with departure rate, $(\lambda + \delta)$. For satisfactory management and control requirements, it is required that $\mu_4 = \mu_1$ and $\mu_6 = \mu_3 = \mu_7 = \mu_{10}$. Fig. 2 represents the overall system in terms of (i) production, and (ii) delivery.

The utilizations of the servers in Stage I are:

$$
\rho_1^{(A4)} = \frac{\lambda_1}{\mu_1},
$$
$$
\rho_1^{(A5)} = \frac{\lambda_2}{\mu_1} = \frac{\lambda_2}{\mu_2},
$$
$$
\rho_{21}^{(A21)} = \frac{\lambda_3}{\mu_2},
$$
$$
\rho_3^{(A3)} = \frac{\lambda_2}{\mu_5},
$$
$$
\rho_2^{(A20)} = \frac{\lambda_4}{\mu_2},
$$
$$
\rho_6^{(A6)} = \frac{\lambda_4}{\mu_6},
$$
$$
\rho_{22}^{(A22)} = \frac{\lambda_4}{\mu_{22}},
$$

(1)
The utilizations of the servers in Stage II are:

\[ \rho_8^{(A8)} = \frac{\lambda_1 p_1 + \lambda_4 p_7}{\mu_8}, \]
\[ \rho_7^{(A17)} = \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_7}, \]
\[ \rho_9^{(A9)} = \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_9}, \]
\[ \rho_9^{(A15)} = \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_9}, \]
\[ \rho_7^{(A7)} = \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_7}, \]
\[ \rho_7^{(A18)} = \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_7}, \]
\[ \rho_{10}^{(A10)} = \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{10}}. \] (2)

The utilizations of the servers in Stage III are:

\[ \rho_{11}^{(A11)} = \frac{\lambda_1 p_1 + \lambda_4 p_7}{\mu_{11}}, \]
\[ \rho_{12}^{(A12)} = \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_{12}}, \]
\[ \rho_{13}^{(A13)} = \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{13}}, \]
\[ \rho_{14}^{(A14)} = \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{14}}. \] (3)

A node is defined by a queue and its corresponding server.
The nodes in Stage I are: (Q_4, A4), (Q_5, A5), (Q_{21}, A21), (Q_3, A3), (Q_{20}, A20), (Q_6, A6), (Q_{22}, A22).
The nodes in Stage II are: (Q_8, A8), (Q_{17}, A17), (Q_9, A9), (Q_{15}, A15), (Q_7, A7), (Q_{18}, A18), (Q_{10}, A10).
The nodes in Stage III are: (Q_{11}, A11), (Q_{12}, A12), (Q_{13}, A13), (Q_{14}, A14).

Each activity belongs to a specific process. Each activity is an object which describes a specific task that the resource has to do. Here, A_i’s are the differential activities carried out in this industrial system. The activities, A_4, A_5, A_3, and A_6 are called “Knitting”, activities A_7, A_8, A_9 and A_10 are called “Making”, and activities A_11, A_12, A_13, A_14, A_15, A_17, A_18, A_20, A_21, and A_22 are called “Transporting”.

3. Performance measures

The performance measures of a single server is measured by the average queue lengths, average waiting times, average response times, and the average number of jobs in the system. All the queues in the model of Fig. 1 are assumed to be M/M/1.
The order arrivals occurring in a given interval of time is assumed to be Poisson distributed, and the service times are assumed to be exponentially distributed. At first glance, the exponential distribution seems to be unrealistic. But it turns out that this is an extremely robust distribution and approximates closely a large number of order arrivals and breakdown patterns in practice in the textile industrial system.

A Poisson input implies that arrivals are independent of one another or the state of the system. The probability of an arrival in any interval of time, t does not depend on the starting point of the arrival or on the specific history of arrivals preceding it, but depends only on the length of it. Thus the queuing systems with Poisson input can be considered as a Markovian process (The reason for using M in the notation).
The average queue length in the ith node (both number of jobs waiting in the queue and those in service) of the M/M/1 queue is \( E[N_i] = \frac{\rho_i}{1 - \rho_i} \), where \( \rho_i \) is the utilization of the concerned server whose average queue length is \( E[N_i] \) [15].
Performance measures such as (a) average response times, (b) average queue lengths, and (c) average waiting times are derived in this section.

3.1. Average queue lengths

Now, the average queue lengths of servers in Stage I are:

\[
\begin{align*}
E[N_1^{(A4)}] & = \frac{\rho_1^{(A4)}}{1 - \rho_1^{(A4)}} = \frac{\lambda_1}{\mu_1 - \lambda_1}, \\
E[N_1^{(A5)}] & = \frac{\rho_1^{(A5)}}{1 - \rho_1^{(A5)}} = \frac{\lambda_2}{\mu_1 - \lambda_2}, \\
E[N_{21}^{(A21)}] & = \frac{\rho_{21}^{(A21)}}{1 - \rho_{21}^{(A21)}} = \frac{\lambda_3}{\mu_2 - \lambda_3}, \\
E[N_2^{(A3)}] & = \frac{\rho_2^{(A3)}}{1 - \rho_2^{(A3)}} = \frac{\lambda_3}{\mu_2 - \lambda_3}, \\
E[N_2^{(A20)}] & = \frac{\rho_2^{(A20)}}{1 - \rho_2^{(A20)}} = \frac{\lambda_4}{\mu_2 - \lambda_4}, \\
E[N_6^{(A6)}] & = \frac{\rho_6^{(A6)}}{1 - \rho_6^{(A6)}} = \frac{\lambda_4}{\mu_6 - \lambda_4}, \\
E[N_{22}^{(A22)}] & = \frac{\rho_{22}^{(A22)}}{1 - \rho_{22}^{(A22)}} = \frac{\lambda_4}{\mu_{22} - \lambda_4}.
\end{align*}
\]

(4)

The average queue lengths of servers in Stage II are:

\[
\begin{align*}
E[N_4^{(A8)}] & = \frac{\rho_4^{(A8)}}{1 - \rho_4^{(A8)}} = \frac{\lambda_1 p_1 + \lambda_4 p_7}{\mu_8 - (\lambda_1 p_1 + \lambda_4 p_7)}, \\
E[N_{17}^{(A17)}] & = \frac{\rho_{17}^{(A17)}}{1 - \rho_{17}^{(A17)}} = \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_{17} - (\lambda_2 p_3 + \lambda_3 p_5)}, \\
E[N_9^{(A9)}] & = \frac{\rho_9^{(A9)}}{1 - \rho_9^{(A9)}} = \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_9 - (\lambda_2 p_3 + \lambda_3 p_5)}, \\
E[N_{15}^{(A15)}] & = \frac{\rho_{15}^{(A15)}}{1 - \rho_{15}^{(A15)}} = \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{15} - (\lambda_1 p_2 + \lambda_4 p_8)}, \\
E[N_7^{(A7)}] & = \frac{\rho_7^{(A7)}}{1 - \rho_7^{(A7)}} = \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_8 - (\lambda_1 p_2 + \lambda_4 p_8)}, \\
E[N_{18}^{(A18)}] & = \frac{\rho_{18}^{(A18)}}{1 - \rho_{18}^{(A18)}} = \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{18} - (\lambda_2 p_4 + \lambda_3 p_6)}, \\
E[N_{10}^{(A10)}] & = \frac{\rho_{10}^{(A10)}}{1 - \rho_{10}^{(A10)}} = \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_8 - (\lambda_2 p_4 + \lambda_3 p_6)}.
\end{align*}
\]

(5)

The average number of queue lengths of servers in Stage III are:

\[
\begin{align*}
E[N_{11}^{(A11)}] & = \frac{\rho_{11}^{(A11)}}{1 - \rho_{11}^{(A11)}} = \frac{\lambda_1 p_1 + \lambda_4 p_7}{\mu_{11} - (\lambda_1 p_1 + \lambda_4 p_7)}, \\
E[N_{12}^{(A12)}] & = \frac{\rho_{12}^{(A12)}}{1 - \rho_{12}^{(A12)}} = \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_{12} - (\lambda_2 p_3 + \lambda_3 p_5)}, \\
E[N_{13}^{(A13)}] & = \frac{\rho_{13}^{(A13)}}{1 - \rho_{13}^{(A13)}} = \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{13} - (\lambda_1 p_2 + \lambda_4 p_8)}, \\
E[N_{14}^{(A14)}] & = \frac{\rho_{14}^{(A14)}}{1 - \rho_{14}^{(A14)}} = \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{14} - (\lambda_2 p_4 + \lambda_3 p_6)}.
\end{align*}
\]

(6)
3.2. Average response times

The average response times in Stage I are:

\[
E[R^{(A4)}_4] = \frac{1}{\lambda_4} E[N^{(A4)}_4] = \frac{1}{\mu_1 - \lambda_1},
\]

\[
E[R^{(A5)}_1] = \frac{1}{\lambda_2} E[N^{(A5)}_5] = \frac{1}{\mu_1 - \lambda_2},
\]

\[
E[R^{(A21)}_{21}] = \frac{1}{\lambda_3} E[N^{(A21)}_{21}] = \frac{1}{\mu_2 - \lambda_3},
\]

\[
E[R^{(A3)}_3] = \frac{1}{\lambda_3} E[N^{(A3)}_3] = \frac{1}{\mu_5 - \lambda_3},
\]

\[
E[R^{(A20)}_2] = \frac{1}{\lambda_4} E[N^{(A20)}_2] = \frac{1}{\mu_2 - \lambda_4},
\]

\[
E[R^{(A6)}_6] = \frac{1}{\lambda_4} E[N^{(A6)}_6] = \frac{1}{\mu_6 - \lambda_4},
\]

\[
E[R^{(A22)}_{22}] = \frac{1}{\lambda_4} E[N^{(A22)}_{22}] = \frac{1}{\mu_6 - \lambda_4}.
\]

(7)

The average response times in Stage II are:

\[
E[R^{(A8)}_8] = \frac{E[N^{(A8)}_8]}{\lambda_1 p_1 + \lambda_2 p_7} + \frac{1}{\mu_8 - (\lambda_1 p_1 + \lambda_4 p_7)}
\]

\[
E[R^{(A17)}_{17}] = \frac{E[N^{(A17)}_{17}]}{\lambda_2 p_3 + \lambda_3 p_5} = \frac{1}{\mu_{17} - (\lambda_2 p_3 + \lambda_3 p_5)}
\]

\[
E[R^{(A9)}_9] = \frac{E[N^{(A9)}_9]}{\lambda_2 p_3 + \lambda_3 p_5} = \frac{1}{\mu_8 - (\lambda_2 p_3 + \lambda_3 p_5)}
\]

\[
E[R^{(A15)}_{15}] = \frac{E[N^{(A15)}_{15}]}{\lambda_1 p_2 + \lambda_4 p_8} = \frac{1}{\mu_{15} - (\lambda_1 p_2 + \lambda_4 p_8)}
\]

\[
E[R^{(A7)}_7] = \frac{E[N^{(A7)}_7]}{\lambda_1 p_2 + \lambda_4 p_8} = \frac{1}{\mu_8 - (\lambda_1 p_2 + \lambda_4 p_8)}
\]

\[
E[R^{(A18)}_{18}] = \frac{E[N^{(A18)}_{18}]}{\lambda_2 p_4 + \lambda_3 p_6} = \frac{1}{\mu_{18} - (\lambda_2 p_4 + \lambda_3 p_6)}
\]

\[
E[R^{(A10)}_{10}] = \frac{E[N^{(A10)}_{10}]}{\lambda_2 p_4 + \lambda_3 p_6} = \frac{1}{\mu_8 - (\lambda_2 p_4 + \lambda_3 p_6)}
\]

(8)

The average response times in Stage III are:

\[
E[R^{(A11)}_{11}] = \frac{E[N^{(A11)}_{11}]}{\lambda_1 p_1 + \lambda_2 p_7} = \frac{1}{\mu_{11} - (\lambda_1 p_1 + \lambda_4 p_7)}
\]

\[
E[R^{(A12)}_{12}] = \frac{E[N^{(A12)}_{12}]}{\lambda_2 p_3 + \lambda_3 p_5} = \frac{1}{\mu_{12} - (\lambda_2 p_3 + \lambda_3 p_5)}
\]

\[
E[R^{(A13)}_{13}] = \frac{E[N^{(A13)}_{13}]}{\lambda_1 p_2 + \lambda_4 p_8} = \frac{1}{\mu_{13} - (\lambda_2 p_3 + \lambda_3 p_5)}
\]

\[
E[R^{(A14)}_{14}] = \frac{E[N^{(A14)}_{14}]}{\lambda_2 p_4 + \lambda_3 p_6} = \frac{1}{\mu_{14} - (\lambda_2 p_3 + \lambda_3 p_5)}
\]

(9)
3.3. Average waiting times

The average waiting times in Stage I are:

\[
E[W_{1}^{(A4)}] = E[R_{1}^{(A4)}] - \frac{1}{\mu_1} = \frac{\lambda_1}{\mu_1 (\mu_1 - \lambda_1)},
\]

\[
E[W_{2}^{(A5)}] = E[R_{2}^{(A5)}] - \frac{1}{\mu_2} = \frac{\lambda_2}{\mu_2 (\mu_2 - \lambda_2)},
\]

\[
E[W_{3}^{(A21)}] = E[R_{21}^{(A21)}] - \frac{1}{\mu_3} = \frac{\lambda_3}{\mu_3 (\mu_3 - \lambda_3)},
\]

\[
E[W_{4}^{(A3)}] = E[R_{3}^{(A3)}] - \frac{1}{\mu_4} = \frac{\lambda_4}{\mu_4 (\mu_4 - \lambda_4)},
\]

(10)

\[
E[W_{5}^{(A6)}] = E[R_{6}^{(A6)}] - \frac{1}{\mu_5} = \frac{\lambda_4}{\mu_5 (\mu_5 - \lambda_4)},
\]

\[
E[W_{6}^{(A20)}] = E[R_{20}^{(A20)}] - \frac{1}{\mu_6} = \frac{\lambda_4}{\mu_6 (\mu_6 - \lambda_4)},
\]

(11)

The average waiting times in Stage II are:

\[
E[W_{8}^{(A8)}] = E[R_{8}^{(A8)}] + \frac{1}{\mu_8} + \Delta_{11} = \frac{\lambda_1 p_1 + \lambda_3 p_7}{\mu_8 [\mu_8 - (\lambda_1 p_1 + \lambda_3 p_7)]} + \Delta_{11},
\]

\[
E[W_{9}^{(A9)}] = E[R_{9}^{(A9)}] - \frac{1}{\mu_9} = \frac{\lambda_2 p_2 + \lambda_3 p_5}{\mu_9 [\mu_9 - (\lambda_2 p_2 + \lambda_3 p_5)]},
\]

\[
E[W_{15}^{(A15)}] = E[R_{15}^{(A15)}] - \frac{1}{\mu_{15}} + \Delta_{13} = \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{15} [\mu_{15} - (\lambda_1 p_2 + \lambda_4 p_8)]} + \Delta_{13},
\]

\[
E[W_{7}^{(A7)}] = E[R_{7}^{(A7)}] - \frac{1}{\mu_7} = \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_7 [\mu_7 - (\lambda_1 p_2 + \lambda_4 p_8)]},
\]

(12)

where \(\Delta_{11}, \Delta_{13}, \Delta_{22}, \) and \(\Delta_{24}\) are the transport delays between \(L_1 \& L_1, L_1 \& L_3, L_2 \& L_2,\) and \(L_2 \& L_4,\) respectively. Transport delays have to be incurred when the items have to be moved from Stage I to Stage II.

The average waiting times in Stage III are:

\[
E[W_{11}^{(A11)}] = E[R_{11}^{(A11)}] - \frac{1}{\mu_{11}} = \frac{\lambda_1 p_1 + \lambda_4 p_7}{\mu_{11} [\mu_{11} - (\lambda_1 p_1 + \lambda_4 p_7)]},
\]

\[
E[W_{12}^{(A12)}] = E[R_{12}^{(A12)}] - \frac{1}{\mu_{12}} = \frac{\lambda_2 p_2 + \lambda_3 p_5}{\mu_{12} [\mu_{12} - (\lambda_2 p_2 + \lambda_3 p_5)]},
\]

\[
E[W_{13}^{(A13)}] = E[R_{13}^{(A13)}] - \frac{1}{\mu_{13}} = \frac{\lambda_2 p_2 + \lambda_4 p_6}{\mu_{13} [\mu_{13} - (\lambda_2 p_2 + \lambda_4 p_6)]},
\]

\[
E[W_{14}^{(A14)}] = E[R_{14}^{(A14)}] - \frac{1}{\mu_{14}} = \frac{\lambda_2 p_2 + \lambda_4 p_6}{\mu_{14} [\mu_{14} - (\lambda_2 p_2 + \lambda_4 p_6)]}.
\]

(13)

3.4. Average queue lengths in different paths

The average number of jobs in path \(X_1\) (\(A4, A8, A11\)) is

\[
E[N_{x_1}] = E[N_{x_1}^{(A4)}] + E[N_{x_1}^{(A8)}] + E[N_{x_1}^{(A11)}] = \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_1 p_1 + \lambda_4 p_7}{\mu_8 - (\lambda_1 p_1 + \lambda_4 p_7)} + \frac{\lambda_1 p_1 + \lambda_4 p_7}{\mu_{11} - (\lambda_1 p_1 + \lambda_4 p_7)},
\]

(14)
The average number of jobs in path $X_2$ (A4, A15, A7, A13) is

$$E[N_{X_2}] = E[N_{1}^{(A4)}] + E[N_{15}^{(A15)}] + E[N_{7}^{(A7)}] + E[N_{13}^{(A13)}]$$

$$= \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_4 p_2 + \lambda_4 p_3}{\mu_1 - (\lambda_1 p_2 + \lambda_4 p_3)} + \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_1 - (\lambda_1 p_2 + \lambda_4 p_8)} + \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_1 - (\lambda_1 p_2 + \lambda_4 p_8)}.$$

(15)

The average number of jobs in path $X_3$ (A5, A17, A9, A12) is

$$E[N_{X_3}] = E[N_{1}^{(A5)}] + E[N_{17}^{(A17)}] + E[N_{9}^{(A9)}] + E[N_{12}^{(A12)}]$$

$$= \frac{\lambda_2}{\mu_1 - \lambda_2} + \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_1 - (\lambda_2 p_3 + \lambda_3 p_5)} + \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_1 - (\lambda_2 p_3 + \lambda_3 p_5)} + \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_1 - (\lambda_2 p_3 + \lambda_3 p_5)}.$$

(16)

The average number of jobs in path $X_4$ (A5, A18, A10, A14) is

$$E[N_{X_4}] = E[N_{1}^{(A5)}] + E[N_{18}^{(A18)}] + E[N_{10}^{(A10)}] + E[N_{14}^{(A14)}]$$

$$= \frac{\lambda_2}{\mu_1 - \lambda_2} + \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_1 - (\lambda_2 p_4 + \lambda_3 p_6)} + \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_1 - (\lambda_2 p_4 + \lambda_3 p_6)} + \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_1 - (\lambda_2 p_4 + \lambda_3 p_6)}.$$

(17)

The average number of jobs in path $X_5$ (A21, A3, A17, A9, A12) is

$$E[N_{X_5}] = E[N_{21}^{(A21)}] + E[N_{3}^{(A3)}] + E[N_{17}^{(A17)}] + E[N_{9}^{(A9)}] + E[N_{12}^{(A12)}]$$

$$= \frac{\lambda_3}{\mu_5 - \lambda_3} + \frac{\lambda_3 p_3 + \lambda_5 p_5}{\mu_5 - (\lambda_3 p_3 + \lambda_5 p_5)} + \frac{\lambda_3 p_3 + \lambda_5 p_5}{\mu_5 - (\lambda_3 p_3 + \lambda_5 p_5)} + \frac{\lambda_3 p_3 + \lambda_5 p_5}{\mu_5 - (\lambda_3 p_3 + \lambda_5 p_5)} + \frac{\lambda_3 p_3 + \lambda_5 p_5}{\mu_5 - (\lambda_3 p_3 + \lambda_5 p_5)}.$$

(18)

The average number of jobs in path $X_6$ (A21, A3, A18, A10, A14) is

$$E[N_{X_6}] = E[N_{21}^{(A21)}] + E[N_{3}^{(A3)}] + E[N_{18}^{(A18)}] + E[N_{10}^{(A10)}] + E[N_{14}^{(A14)}]$$

$$= \frac{\lambda_3}{\mu_5 - \lambda_3} + \frac{\lambda_3 p_3 + \lambda_5 p_6}{\mu_5 - (\lambda_3 p_3 + \lambda_5 p_6)} + \frac{\lambda_3 p_3 + \lambda_5 p_6}{\mu_5 - (\lambda_3 p_3 + \lambda_5 p_6)} + \frac{\lambda_3 p_3 + \lambda_5 p_6}{\mu_5 - (\lambda_3 p_3 + \lambda_5 p_6)} + \frac{\lambda_3 p_3 + \lambda_5 p_6}{\mu_5 - (\lambda_3 p_3 + \lambda_5 p_6)}.$$

(19)

The average number of jobs in path $X_7$ (A20, A6, A22, A8, A11) is

$$E[N_{X_7}] = E[N_{2}^{(A20)}] + E[N_{6}^{(A6)}] + E[N_{22}^{(A22)}] + E[N_{4}^{(A4)}] + E[N_{11}^{(A11)}]$$

$$= \frac{\lambda_4}{\mu_6 - \lambda_4} + \frac{\lambda_4 p_1 + \lambda_4 p_7}{\mu_6 - (\lambda_4 p_1 + \lambda_4 p_7)} + \frac{\lambda_4 p_1 + \lambda_4 p_7}{\mu_6 - (\lambda_4 p_1 + \lambda_4 p_7)} + \frac{\lambda_4 p_1 + \lambda_4 p_7}{\mu_6 - (\lambda_4 p_1 + \lambda_4 p_7)} + \frac{\lambda_4 p_1 + \lambda_4 p_7}{\mu_6 - (\lambda_4 p_1 + \lambda_4 p_7)}.$$

(20)

The average number of jobs in path $X_8$ (A20, A6, A22, A15, A7, A13) is

$$E[N_{X_8}] = E[N_{2}^{(A20)}] + E[N_{6}^{(A6)}] + E[N_{22}^{(A22)}] + E[N_{15}^{(A15)}] + E[N_{7}^{(A7)}] + E[N_{13}^{(A13)}]$$

$$= \frac{\lambda_4}{\mu_6 - \lambda_4} + \frac{\lambda_4 p_1 + \lambda_4 p_8}{\mu_6 - (\lambda_4 p_1 + \lambda_4 p_8)} + \frac{\lambda_4 p_1 + \lambda_4 p_8}{\mu_6 - (\lambda_4 p_1 + \lambda_4 p_8)} + \frac{\lambda_4 p_1 + \lambda_4 p_8}{\mu_6 - (\lambda_4 p_1 + \lambda_4 p_8)}.$$

(21)

### 3.5. Average response times in different paths

The global throughput delay from $S_1$ to $S_4$ in Fig. 1 can be chosen to be the minimum of the response times of the eight paths shown below. The global throughput delay represents the order’s cycle. This can be done by

- considering that orders are independently and equally routed from $S_1$ to $S_4$,
- optimizing the route by taking into account the present state of the network.

The average response time in path $X_1$ (A4, A8, A11) is

$$E[R_{X_1}] = E[R_{1}^{(A4)}] + E[R_{8}^{(A8)}] + E[R_{11}^{(A11)}]$$

$$= \frac{1}{\mu_1 - \lambda_1} + \frac{1}{\mu_6 - (\lambda_1 p_1 + \lambda_4 p_7)} + \frac{1}{\mu_11 - (\lambda_1 p_1 + \lambda_4 p_7)}.$$

(22)

The average response time in path $X_2$ (A4, A15, A7, A13) is

$$E[R_{X_2}] = E[R_{1}^{(A4)}] + E[R_{15}^{(A15)}] + E[R_{7}^{(A7)}] + E[R_{13}^{(A13)}]$$

$$= \frac{1}{\mu_1 - \lambda_1} + \frac{1}{\mu_15 - (\lambda_1 p_2 + \lambda_4 p_8)} + \frac{1}{\mu_8 - (\lambda_1 p_2 + \lambda_4 p_8)} + \frac{1}{\mu_{13} - (\lambda_1 p_2 + \lambda_4 p_8)}.$$

(23)
The average response time in path $X_3$ (A5, A17, A9, A12) is

$$E[R_{X_3}] = E[R_{17}^{(A5)}] + E[R_{17}^{(A17)}] + E[R_{12}^{(A9)}] + E[R_{12}^{(A12)}]$$

$$= \frac{1}{\mu_1} - \frac{1}{\mu_2} + \frac{1}{\mu_{17} - (\lambda_2 p_3 + \lambda_3 p_5)} + \frac{1}{\mu_8 - (\lambda_2 p_3 + \lambda_3 p_5)} + \frac{1}{\mu_{12} - (\lambda_2 p_3 + \lambda_3 p_5)}$$  \hspace{1cm} (24)

The average response time in path $X_4$ (A5, A18, A10, A14) is

$$E[R_{X_4}] = E[R_{17}^{(A5)}] + E[R_{18}^{(A18)}] + E[R_{10}^{(A10)}] + E[R_{14}^{(A14)}]$$

$$= \frac{1}{\mu_1} - \frac{1}{\mu_2} + \frac{1}{\mu_{18} - (\lambda_2 p_4 + \lambda_3 p_5)} + \frac{1}{\mu_8 - (\lambda_2 p_4 + \lambda_3 p_5)} + \frac{1}{\mu_{14} - (\lambda_2 p_4 + \lambda_3 p_5)}$$  \hspace{1cm} (25)

The average response time in path $X_5$ (A21, A3, A17, A9, A12) is

$$E[R_{X_5}] = E[R_{21}^{(A21)}] + E[R_{3}^{(A3)}] + E[R_{17}^{(A17)}] + E[R_{12}^{(A9)}] + E[R_{12}^{(A12)}]$$

$$= \frac{1}{\mu_2} - \frac{1}{\mu_3} + \frac{1}{\mu_5 - \lambda_3} + \frac{1}{\mu_{17} - (\lambda_2 p_3 + \lambda_3 p_5)} + \frac{1}{\mu_8 - (\lambda_2 p_3 + \lambda_3 p_5)} + \frac{1}{\mu_{14} - (\lambda_2 p_3 + \lambda_3 p_5)}$$  \hspace{1cm} (26)

The average response time in path $X_6$ (A21, A3, A18, A10, A14) is

$$E[R_{X_6}] = E[R_{21}^{(A21)}] + E[R_{3}^{(A3)}] + E[R_{18}^{(A18)}] + E[R_{10}^{(A10)}] + E[R_{14}^{(A14)}]$$

$$= \frac{1}{\mu_2} - \frac{1}{\mu_3} + \frac{1}{\mu_5 - \lambda_3} + \frac{1}{\mu_{18} - (\lambda_2 p_4 + \lambda_3 p_5)} + \frac{1}{\mu_8 - (\lambda_2 p_4 + \lambda_3 p_5)} + \frac{1}{\mu_{14} - (\lambda_2 p_4 + \lambda_3 p_5)}$$  \hspace{1cm} (27)

The average response time in path $X_7$ (A20, A6, A22, A8, A11) is

$$E[R_{X_7}] = E[R_{22}^{(A20)}] + E[R_{6}^{(A6)}] + E[R_{22}^{(A22)}] + E[R_{4}^{(A48)}] + E[R_{4}^{(A11)}]$$

$$= \frac{1}{\mu_2} - \frac{1}{\mu_4} + \frac{1}{\mu_6 - \lambda_4} + \frac{1}{\mu_{22} - \lambda_4} + \frac{1}{\mu_8 - (\lambda_4 p_1 + \lambda_4 p_7)} + \frac{1}{\mu_{11} - (\lambda_4 p_1 + \lambda_4 p_7)}$$  \hspace{1cm} (28)

The average response time in path $X_8$ (A20, A6, A22, A15, A7, A13) is

$$E[R_{X_8}] = E[R_{22}^{(A20)}] + E[R_{6}^{(A6)}] + E[R_{22}^{(A22)}] + E[R_{4}^{(A15)}] + E[R_{4}^{(A7)}] + E[R_{4}^{(A13)}]$$

$$= \frac{1}{\mu_2} - \frac{1}{\mu_4} + \frac{1}{\mu_6 - \lambda_4} + \frac{1}{\mu_{22} - \lambda_4} + \frac{1}{\mu_{15} - (\lambda_4 p_1 + \lambda_4 p_8)} + \frac{1}{\mu_8 - (\lambda_4 p_1 + \lambda_4 p_8)}$$

$$+ \frac{1}{\mu_{13} - (\lambda_4 p_1 + \lambda_4 p_8)}$$  \hspace{1cm} (29)

3.6. Development of an equivalent network

We are interested in representing the queueing network in Fig. 1 as a single equivalent queue with a single server as shown in Fig. 3. So, it is important to consider nodes which are in serial or parallel connection in Fig. 1, and provide expressions for queue lengths, response and waiting times of the equivalent single queue-single server queueing system (Fig. 3) in terms of the specifications (arrival rates, service rates and probability of arrivals) in the original network. For simplicity, we ignore transport delays in this section. We proceed our analysis of development of an equivalent network as follows:

(1) Nodes (Q, A8) and (Q, A11): The average queue length of the equivalent node is

$$E[N_{eq}^{(1)}] = \frac{\rho_{eq}^{(1)}}{1 - \rho_{eq}^{(1)}} = E[N_8^{(A8)}] + E[N_{11}^{(A11)}]$$

$$= \frac{\lambda_4 p_1 + \lambda_4 p_7}{\mu_8 - (\lambda_4 p_1 + \lambda_4 p_7)} + \frac{\lambda_1 p_1 + \lambda_4 p_7}{\mu_{11} - (\lambda_4 p_1 + \lambda_4 p_7)}$$  \hspace{1cm} (30)

But, $\rho_{eq}^{(1)} = \frac{\lambda_4 p_1 + \lambda_4 p_7}{\mu_{eq}^{(1)}}$. So, the equivalent service rate of the nodes (Q, A8) and (Q, A11) is

$$\mu_{eq}^{(1)} = \lambda_4 p_1 + \lambda_4 p_7 + \frac{\mu_8 - (\lambda_4 p_1 + \lambda_4 p_7)}{\mu_8 + \mu_{11} - 2(\lambda_4 p_1 + \lambda_4 p_7)}$$  \hspace{1cm} (31)
Equivalent queue of the industrial system

The average response time of the equivalent node is

\[
E\left[R_{eq}^{(1)}\right] = \frac{E\left[N_{eq}^{(1)}\right]}{\mu_{eq}^{(1)}} = \frac{\mu_{eq}^{(1)} + \mu_{11} - 2(\lambda_{1}p_{1} + \lambda_{d}p_{7})}{\mu_{8} + \mu_{11} - (\lambda_{1}p_{1} + \lambda_{d}p_{7})}.
\]

The average waiting time of the equivalent node is

\[
E\left[W_{eq}^{(1)}\right] = \frac{1}{\mu_{eq}^{(1)}} = \frac{\mu_{eq}^{(1)} + \mu_{11} - 2(\lambda_{1}p_{1} + \lambda_{d}p_{7})}{\mu_{8} + \mu_{11} - (\lambda_{1}p_{1} + \lambda_{d}p_{7})} - \frac{1}{\mu_{eq}^{(1)}}.
\]

where \(\mu_{eq}^{(1)}\) is as shown in (31).

(2) Nodes (Q_{17}, A17), (Q_{9}, A9) and (Q_{12}, A12): The average queue length of the equivalent node is

\[
E\left[N_{eq}^{(2)}\right] = \frac{\rho_{eq}^{(2)}}{1 - \rho_{eq}^{(2)}} = E\left[N_{17}^{(A17)}\right] + E\left[N_{9}^{(A9)}\right] + E\left[N_{12}^{(A12)}\right] = \frac{\lambda_{2}p_{3} + \lambda_{3}p_{5}}{\mu_{17} - (\lambda_{2}p_{3} + \lambda_{3}p_{5})} + \frac{\lambda_{2}p_{3} + \lambda_{3}p_{5}}{\mu_{8} - (\lambda_{2}p_{3} + \lambda_{3}p_{5})} + \frac{\lambda_{2}p_{3} + \lambda_{3}p_{5}}{\mu_{12} - (\lambda_{2}p_{3} + \lambda_{3}p_{5})}.
\]

But, \(\rho_{eq}^{(2)} = \frac{\lambda_{2}p_{3} + \lambda_{3}p_{5}}{\mu_{eq}^{(2)}}\). So, the equivalent service rate of the nodes (Q_{17}, A17), (Q_{9}, A9) and (Q_{12}, A12) is

\[
\mu_{eq}^{(2)} = \lambda_{2}p_{3} + \lambda_{3}p_{5} + \frac{\lambda_{12} - (\lambda_{2}p_{3} + \lambda_{3}p_{5})}{D2a + D2b + D2c}.
\]

where
- \(D2a = (\lambda_{9} - (\lambda_{2}p_{3} + \lambda_{3}p_{5}))(\mu_{12} - (\lambda_{2}p_{3} + \lambda_{3}p_{5}))\),
- \(D2b = (\mu_{17} - (\lambda_{2}p_{3} + \lambda_{3}p_{5}))(\mu_{8} - (\lambda_{2}p_{3} + \lambda_{3}p_{5}))\) and
- \(D2c = (\mu_{17} - (\lambda_{2}p_{3} + \lambda_{3}p_{5}))(\mu_{12} - (\lambda_{2}p_{3} + \lambda_{3}p_{5}))\).

The average response time of the equivalent node is

\[
E\left[R_{eq}^{(2)}\right] = \frac{1}{\mu_{eq}^{(2)}} = \frac{\lambda_{2}p_{3} + \lambda_{3}p_{5}}{\mu_{17} - (\lambda_{2}p_{3} + \lambda_{3}p_{5})}.
\]
The average waiting time of the equivalent node is

\[
E[\mathcal{W}_{eq}^{(2)}] = E[\mathcal{R}_{eq}^{(2)}] - \frac{1}{\mu_{eq}^{(2)}} = \frac{1}{\mu_{eq}^{(2)}} \left[ \mu_{17} - (\lambda_2 p_3 + \lambda_3 p_5) \right] \left[ \mu_8 - (\lambda_2 p_3 + \lambda_3 p_5) \right] \\
\times \left[ (\mu_8 - (\lambda_2 p_3 + \lambda_3 p_5))(\mu_{12} - (\lambda_2 p_3 + \lambda_3 p_5)) + (\mu_{17} - (\lambda_2 p_3 + \lambda_3 p_5))(\mu_8 - (\lambda_2 p_3 + \lambda_3 p_5)) \right] + \frac{1}{\mu_{eq}^{(2)}}
\]

where \( \mu_{eq}^{(2)} \) is as shown in (35).

(3) Nodes (Q15, A15), (Q7, A7), and (Q13, A13): The average queue length of the equivalent node is

\[
E[N_{eq}^{(3)}] = \frac{\mu_{eq}^{(3)}}{1 - \rho_{eq}^{(3)}} = E[N_{k15}^{(4)}] + E[N_{k7}^{(4)}] + E[N_{k13}^{(4)}] = \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{15} - (\lambda_1 p_2 + \lambda_4 p_8)} + \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_8 - (\lambda_1 p_2 + \lambda_4 p_8)} + \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{13} - (\lambda_1 p_2 + \lambda_4 p_8)}
\]

But, \( \rho_{eq}^{(3)} = \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{eq}^{(3)}} \). So, the equivalent service rate of the nodes (Q15, A15), (Q7, A7), and (Q13, A13) is

\[
\mu_{eq}^{(3)} = \lambda_1 p_2 + \lambda_4 p_8 + \frac{1}{1 - \rho_{eq}^{(3)}} \left( \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{15} - (\lambda_1 p_2 + \lambda_4 p_8)} + \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_8 - (\lambda_1 p_2 + \lambda_4 p_8)} + \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{13} - (\lambda_1 p_2 + \lambda_4 p_8)} \right)
\]

The average response time of the equivalent node is

\[
E[R_{eq}^{(3)}] = E[N_{eq}^{(3)}] - \frac{1}{\mu_{eq}^{(3)}} = \frac{\mu_8 + \mu_{15} - 2(\lambda_1 p_2 + \lambda_4 p_8)}{\mu_{15} - (\lambda_1 p_2 + \lambda_4 p_8)} + \frac{1}{\mu_{13} - (\lambda_1 p_2 + \lambda_4 p_8)}
\]

The average waiting time of the equivalent node is

\[
E[W_{eq}^{(3)}] = E[R_{eq}^{(3)}] - \frac{1}{\mu_{eq}^{(3)}} = \frac{\mu_8 + \mu_{15} - 2(\lambda_1 p_2 + \lambda_4 p_8)}{\mu_{15} - (\lambda_1 p_2 + \lambda_4 p_8)} + \frac{1}{\mu_{13} - (\lambda_1 p_2 + \lambda_4 p_8)} - \frac{1}{\mu_{eq}^{(3)}}
\]

where \( \mu_{eq}^{(3)} \) is as shown in (39).

(4) Nodes (Q18, A18), (Q10, A10), and (Q14, A14): The average queue length of the equivalent node is

\[
E[N_{eq}^{(4)}] = \frac{\mu_{eq}^{(4)}}{1 - \rho_{eq}^{(4)}} = E[N_{k18}^{(4)}] + E[N_{k10}^{(4)}] + E[N_{k14}^{(4)}] = \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{18} - (\lambda_2 p_4 + \lambda_3 p_6)} + \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_8 - (\lambda_2 p_4 + \lambda_3 p_6)} + \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{14} - (\lambda_2 p_4 + \lambda_3 p_6)}
\]

But, \( \rho_{eq}^{(4)} = \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{eq}^{(4)}} \). So, the equivalent service rate of the nodes (Q18, A18), (Q10, A10), and (Q14, A14) is

\[
\mu_{eq}^{(4)} = \lambda_2 p_4 + \lambda_3 p_6 + \frac{1}{1 - \rho_{eq}^{(4)}} \left( \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{18} - (\lambda_2 p_4 + \lambda_3 p_6)} + \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_8 - (\lambda_2 p_4 + \lambda_3 p_6)} + \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{14} - (\lambda_2 p_4 + \lambda_3 p_6)} \right)
\]

The average response time of the equivalent node is

\[
E[R_{eq}^{(4)}] = E[N_{eq}^{(4)}] - \frac{1}{\mu_{eq}^{(4)}} = \frac{\mu_8 + \mu_{18} - 2(\lambda_2 p_4 + \lambda_3 p_6)}{\mu_{18} - (\lambda_2 p_4 + \lambda_3 p_6)} + \frac{1}{\mu_{14} - (\lambda_2 p_4 + \lambda_3 p_6)}
\]

The average waiting time of the equivalent node is

\[
E[W_{eq}^{(4)}] = E[R_{eq}^{(4)}] - \frac{1}{\mu_{eq}^{(4)}} = \frac{\mu_8 + \mu_{18} - 2(\lambda_2 p_4 + \lambda_3 p_6)}{\mu_{18} - (\lambda_2 p_4 + \lambda_3 p_6)} + \frac{1}{\mu_{14} - (\lambda_2 p_4 + \lambda_3 p_6)} - \frac{1}{\mu_{eq}^{(4)}}
\]

where \( \mu_{eq}^{(4)} \) is as shown in (43).

(5) Nodes (Q21, A21), (Q2, A3): The average queue length of the equivalent node is
The average response time of the equivalent node is
\[
E \left[ T^{(5)} \right] = \frac{\rho^{(5)}_{eq}}{1 - \rho^{(5)}_{eq}} = E \left[ N^{(21)}_{41} \right] + E \left[ N^{(33)}_{43} \right] = \frac{\lambda_3}{\mu_2 - \lambda_3} + \frac{\lambda_3}{\mu_5 - \lambda_3} = \frac{\lambda_3 (\mu_2 + \mu_5 - 2 \lambda_3)}{(\mu_2 - \lambda_3)(\mu_5 - \lambda_3)}. \tag{46}
\]

But, \( \rho^{(5)}_{eq} = \frac{\lambda_3}{\mu_2 - \lambda_3} \). So, the equivalent service rate of the nodes (Q_{21}, A_{21}), (Q_{33}, A_{33}) is
\[
\mu^{(5)}_{eq} = \lambda_3 + \frac{(\mu_2 - \lambda_3)(\mu_5 - \lambda_3)}{\mu_2 + \mu_5 - 2 \lambda_3}. \tag{47}
\]

The average response time of the equivalent node is
\[
E \left[ T^{(5)}_{eq} \right] = \frac{E \left[ N^{(5)}_{eq} \right]}{\lambda_3} = \frac{\mu_2 + \mu_5 - 2 \lambda_3}{(\mu_2 - \lambda_3)(\mu_5 - \lambda_3)} \tag{48}
\]

The average waiting time of the equivalent node is
\[
E \left[ W^{(5)}_{eq} \right] = E \left[ T^{(5)}_{eq} \right] - \frac{1}{\mu^{(5)}_{eq}} = \frac{\mu_2 + \mu_5 - 2 \lambda_3}{(\mu_2 - \lambda_3)(\mu_5 - \lambda_3)} - \frac{1}{\mu^{(5)}_{eq}}, \tag{49}
\]
where \( \mu^{(5)}_{eq} \) is as shown in (47).

(6) Nodes (Q_{20}, A_{20}), (Q_{6}, A_{6}), and (Q_{22}, A_{22}): The average queue length of the equivalent node is
\[
E \left[ N^{(6)}_{eq} \right] = \frac{\rho^{(6)}_{eq}}{1 - \rho^{(6)}_{eq}} = E \left[ N^{(20)}_{6} \right] + E \left[ N^{(46)}_{6} \right] + E \left[ N^{(422)}_{22} \right] = \frac{\lambda_4}{\mu_2 - \lambda_4} + \frac{\lambda_4}{\mu_6 - \lambda_4} + \frac{\lambda_4}{\mu_{22} - \lambda_4} \tag{50}
\]

But, \( \rho^{(6)}_{eq} = \frac{\lambda_4}{\mu_2 - \lambda_4} \). So, the equivalent service rate of the nodes (Q_{20}, A_{20}), (Q_{6}, A_{6}) and (Q_{22}, A_{22}) is
\[
\mu^{(6)}_{eq} = \lambda_4 + \frac{(\mu_2 - \lambda_4)(\mu_6 - \lambda_4)(\mu_{22} - \lambda_4)}{(\mu_2 - \lambda_4)(\mu_6 - \lambda_4)(\mu_{22} - \lambda_4)}. \tag{51}
\]

The average response time of the equivalent node is
\[
E \left[ T^{(6)}_{eq} \right] = \frac{E \left[ N^{(6)}_{eq} \right]}{\lambda_4} = \frac{(\mu_6 - \lambda_4)(\mu_{22} - \lambda_4) + (\mu_2 - \lambda_4)(\mu_{6} - \lambda_4) + (\mu_2 - \lambda_4)(\mu_6 - \lambda_4)}{(\mu_2 - \lambda_4)(\mu_6 - \lambda_4)(\mu_{22} - \lambda_4)}. \tag{52}
\]

The average waiting time of the equivalent node is
\[
E \left[ W^{(6)}_{eq} \right] = E \left[ T^{(6)}_{eq} \right] - \frac{1}{\mu^{(6)}_{eq}} = \frac{(\mu_6 - \lambda_4)(\mu_{22} - \lambda_4) + (\mu_2 - \lambda_4)(\mu_{6} - \lambda_4) + (\mu_2 - \lambda_4)(\mu_6 - \lambda_4)}{(\mu_2 - \lambda_4)(\mu_6 - \lambda_4)(\mu_{22} - \lambda_4)} - \frac{1}{\mu^{(6)}_{eq}}, \tag{53}
\]
where \( \mu^{(6)}_{eq} \) is as shown in (51).

(7) Nodes (Q_{m1}, A_{m1}), (Q_{eq}, A_{eq}): The average length of the equivalent node is
\[
E \left[ N^{(7)}_{eq} \right] = \frac{\rho^{(7)}_{eq}}{1 - \rho^{(7)}_{eq}} = E \left[ N^{(1)}_{eq} \right] + E \left[ N^{(3)}_{eq} \right] = \frac{(\lambda_1 p_1 + \lambda_4 p_4)(\mu_6 + \mu_1 - 2(\lambda_1 p_1 + \lambda_4 p_4))}{(\mu_6 - (\lambda_1 p_1 + \lambda_4 p_4))} + \frac{(\lambda_2 p_2 + \lambda_4 p_4)(\mu_6 + \mu_1 - 2(\lambda_2 p_2 + \lambda_4 p_4))}{(\mu_6 - (\lambda_2 p_2 + \lambda_4 p_4))} + \frac{\lambda_1 p_1 + \lambda_4 p_4}{\mu_3 + (\lambda_1 p_1 + \lambda_4 p_4)} D7a + \frac{\lambda_2 p_2 + \lambda_4 p_4}{\mu_3 + (\lambda_2 p_2 + \lambda_4 p_4)} D7b + \frac{\lambda_4 p_3 + \lambda_4 p_4}{\mu_3 + \lambda_4 p_3 + \lambda_4 p_4} D7c. \tag{54}
\]

But, \( \rho^{(7)}_{eq} = \frac{\lambda_1 + \lambda_4}{\mu_2 - \lambda_4} \). So, the equivalent service rate of the nodes (Q_{eq}, A_{eq}) and (Q_{eq}, A_{eq}) is
\[
\mu^{(7)}_{eq} = \lambda_1 + \lambda_4 + \frac{1}{\mu_2 + \mu_5 - 2 \lambda_3} D7a + \frac{1}{\mu_2 + \mu_5 - 2 \lambda_3} D7b + \frac{1}{\mu_2 + \mu_5 - 2 \lambda_3} D7c, \tag{55}
\]
where
- \( D7a = \frac{\mu_6 + \mu_1 - 2(\lambda_1 p_1 + \lambda_4 p_4)}{(\mu_6 - (\lambda_1 p_1 + \lambda_4 p_4))} \)
- \( D7b = \frac{\mu_6 + \mu_1 - 2(\lambda_2 p_2 + \lambda_4 p_4)}{(\mu_6 - (\lambda_2 p_2 + \lambda_4 p_4))} \)
- \( D7c = \frac{\mu_6 + \mu_1 - 2(\lambda_4 p_3 + \lambda_4 p_4)}{(\mu_6 - (\lambda_4 p_3 + \lambda_4 p_4))} \)

The average response time of the equivalent node is
\[
E \left[ T^{(7)}_{eq} \right] = \frac{E \left[ N^{(7)}_{eq} \right]}{\lambda_1 + \lambda_4}, \tag{56}
\]
where \( E \left[ N^{(7)}_{eq} \right] \) is as shown in (54), \( \lambda_1 = \lambda q_1 \), and \( \lambda_4 = \delta r_2 \).
The average waiting time of the equivalent node is

$$E\left[ W_{eq}^{(7)} \right] = E\left[ R_{eq}^{(7)} \right] - \frac{1}{\mu_{eq}^{(7)}}$$

where $E\left[ R_{eq}^{(7)} \right]$ and $\mu_{eq}^{(7)}$ are shown in (56) and (55) respectively.

(8) Nodes ($Q_{eq_2}$, $A_{eq_2}$), ($Q_{eq_4}$, $A_{eq_4}$): The average queue length of the equivalent node is

$$E\left[ N_{eq}^{(8)} \right] = \frac{\rho_{eq}^{(8)}}{1 - \rho_{eq}^{(8)}} = E\left[ N_{eq}^{(3)} \right] + E\left[ N_{eq}^{(4)} \right] = \frac{(\lambda_2p_3 + \lambda_3p_5)}{[\lambda_1 - (\lambda_2p_3 + \lambda_3p_5)][\mu_8 - (\lambda_2p_3 + \lambda_3p_5)] [\mu_{12} - (\lambda_2p_3 + \lambda_3p_5)]}

\times \left[ (\mu_{17} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{12} - (\lambda_2p_3 + \lambda_3p_5)) + (\mu_{17} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{14} - (\lambda_2p_3 + \lambda_3p_5)) + (\mu_{18} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{14} - (\lambda_2p_3 + \lambda_3p_5)) + (\mu_{18} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{12} - (\lambda_2p_3 + \lambda_3p_5)) \right]

\times \left[ (\mu_{17} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{14} - (\lambda_2p_3 + \lambda_3p_5)) + (\mu_{17} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{12} - (\lambda_2p_3 + \lambda_3p_5)) + (\mu_{18} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{14} - (\lambda_2p_3 + \lambda_3p_5)) + (\mu_{18} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{12} - (\lambda_2p_3 + \lambda_3p_5)) \right].$$

But, $\rho_{eq}^{(8)} = \frac{\lambda_2 + \lambda_3}{\mu_{eq}^{(8)}}$. So, the equivalent service rate of the nodes ($Q_{eq_2}$, $A_{eq_2}$) and ($Q_{eq_4}$, $A_{eq_4}$) is

$$\mu_{eq}^{(8)} = \lambda_2 + \lambda_3 + \frac{1}{\lambda_2 + \lambda_3} D8a + \frac{1}{\lambda_2 + \lambda_3} D8b,$$

where

- $D8a = \frac{1}{[\mu_{12} - (\lambda_2p_3 + \lambda_3p_5)][\mu_8 - (\lambda_2p_3 + \lambda_3p_5)] [\mu_{15} - (\lambda_2p_3 + \lambda_3p_5)]}

\times \left[ (\mu_{17} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{12} - (\lambda_2p_3 + \lambda_3p_5)) + (\mu_{17} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{14} - (\lambda_2p_3 + \lambda_3p_5)) + (\mu_{18} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{14} - (\lambda_2p_3 + \lambda_3p_5)) + (\mu_{18} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{12} - (\lambda_2p_3 + \lambda_3p_5)) \right]

\times \left[ (\mu_{17} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{14} - (\lambda_2p_3 + \lambda_3p_5)) + (\mu_{17} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{12} - (\lambda_2p_3 + \lambda_3p_5)) + (\mu_{18} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{14} - (\lambda_2p_3 + \lambda_3p_5)) + (\mu_{18} - (\lambda_2p_3 + \lambda_3p_5))(\mu_{12} - (\lambda_2p_3 + \lambda_3p_5)) \right].$$

The average response time of the equivalent node is

$$E\left[ R_{eq}^{(8)} \right] = E\left[ N_{eq}^{(8)} \right] = \frac{\lambda_2 + \lambda_3}{\mu_{eq}^{(8)}}$$

where $E\left[ N_{eq}^{(8)} \right]$ is as shown in (58), $\lambda_3 = \delta r_1$, and $\lambda_2 = \lambda q_2$.

The average waiting time of the equivalent node is

$$E\left[ W_{eq}^{(8)} \right] = E\left[ R_{eq}^{(8)} \right] - \frac{1}{\mu_{eq}^{(8)}}$$

(9) Nodes ($Q_4$, $A_4$), ($Q_{eq_7}$, $A_{eq_7}$): The average queue length of the equivalent node is

$$E\left[ N_{eq}^{(9)} \right] = \frac{\rho_{eq}^{(9)}}{1 - \rho_{eq}^{(9)}} = E\left[ N_{eq}^{(4)} \right] + E\left[ N_{eq}^{(5)} \right] = \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{(\lambda_1p_1 + \lambda_4p_4)}{[\mu_6 - (\lambda_1p_1 + \lambda_4p_4)][\mu_{11} - (\lambda_1p_1 + \lambda_4p_4)] [\mu_{15} - (\lambda_1p_1 + \lambda_4p_4)]}

\times \left[ (\mu_6 - (\lambda_1p_1 + \lambda_4p_4))(\mu_{11} - (\lambda_1p_1 + \lambda_4p_4)) + (\mu_6 - (\lambda_1p_1 + \lambda_4p_4))(\mu_{15} - (\lambda_1p_1 + \lambda_4p_4)) + (\mu_{11} - (\lambda_1p_1 + \lambda_4p_4))(\mu_{15} - (\lambda_1p_1 + \lambda_4p_4)) + (\mu_{11} - (\lambda_1p_1 + \lambda_4p_4))(\mu_{15} - (\lambda_1p_1 + \lambda_4p_4)) \right].$$

But, $\rho_{eq}^{(9)} = \frac{\lambda_1 + \lambda_4}{\mu_{eq}^{(9)}}$. So, the equivalent service rate of the nodes ($Q_4$, $A_4$) and ($Q_{eq_7}$, $A_{eq_7}$) is

$$\mu_{eq}^{(9)} = \lambda_1 + \lambda_4 + \frac{1}{\lambda_1 + \lambda_4} D9a + \frac{1}{\lambda_1 + \lambda_4} D9b + \frac{1}{\lambda_1 + \lambda_4} D9c,$$

where

- $D9a = \frac{\mu_{12} - (\lambda_1p_1 + \lambda_4p_4)}{[\mu_6 - (\lambda_1p_1 + \lambda_4p_4)][\mu_{11} - (\lambda_1p_1 + \lambda_4p_4)] [\mu_{15} - (\lambda_1p_1 + \lambda_4p_4)]}

\times \left[ (\mu_6 - (\lambda_1p_1 + \lambda_4p_4))(\mu_{11} - (\lambda_1p_1 + \lambda_4p_4)) + (\mu_6 - (\lambda_1p_1 + \lambda_4p_4))(\mu_{15} - (\lambda_1p_1 + \lambda_4p_4)) + (\mu_{11} - (\lambda_1p_1 + \lambda_4p_4))(\mu_{15} - (\lambda_1p_1 + \lambda_4p_4)) + (\mu_{11} - (\lambda_1p_1 + \lambda_4p_4))(\mu_{15} - (\lambda_1p_1 + \lambda_4p_4)) \right].$
The average response time of the equivalent node is
\[
E^\left[N^\left(R_{eq}^{(0)}\right)\right] = \frac{E\left[N^\left(R_{eq}^{(0)}\right)\right]}{\lambda_2 + \lambda_4},
\]
where \(E\left[N^\left(R_{eq}^{(0)}\right)\right]\) is as shown in (62), \(\lambda_4 = \delta_2\), and \(\lambda_1 = \delta q_1\). The average waiting time of the equivalent node is
\[
E^\left[W^\left(R_{eq}^{(0)}\right)\right] = E^\left[N^\left(R_{eq}^{(0)}\right)\right] - \frac{1}{\mu_{eq}^{(0)}},
\]
where \(E^\left[R_{eq}^{(0)}\right]\) and \(\mu_{eq}^{(0)}\) are shown in (64) and (63) respectively.

(10) Nodes \((Q_5, A_5), (Q_{eq}, A_{eq})\): The average queue length of the equivalent node is
\[
E^\left[N^\left(10\right)_{eq}\right] = \frac{\rho_{eq}^{(10)}}{1 - \rho_{eq}^{(10)}} = E^\left[N^\left(10\right)\right] = \frac{\lambda_2 + \lambda_3 + \frac{1}{D10a + \frac{2\rho_{eq}^{(10)} + \frac{1}{D10b} + \frac{1}{D10c}}}{\lambda_2 + \lambda_3}}{\lambda_2 + \lambda_3},
\]
where
\[
\begin{align*}
D10a &= \frac{1}{\frac{1}{D10a} - \frac{1}{D10b} - \frac{1}{D10c}}, \\
D10b &= \frac{1}{\frac{1}{D10a} - \frac{1}{D10b} - \frac{1}{D10c}} \times \left[(\mu_8 - (\lambda_2p + \lambda_3p))(\mu_1' - (\lambda_2p + \lambda_3p))((\mu_1' - (\lambda_2p + \lambda_3p))(\mu_8 - (\lambda_2p + \lambda_3p))\right] \\
D10c &= \frac{1}{\frac{1}{D10a} - \frac{1}{D10b} - \frac{1}{D10c}} \times \left[(\mu_8 - (\lambda_2p + \lambda_3p))(\mu_1' - (\lambda_2p + \lambda_3p))((\mu_1' - (\lambda_2p + \lambda_3p))(\mu_8 - (\lambda_2p + \lambda_3p))\right]
\end{align*}
\]

The average response time of the equivalent node is
\[
E^\left[R^\left(10\right)_{eq}\right] = \frac{E^\left[N^\left(10\right)_{eq}\right]}{\lambda_2 + \lambda_3},
\]
where \(E^\left[N^\left(10\right)_{eq}\right]\) is as shown in (66), \(\lambda_3 = \delta r_1\), and \(\lambda_2 = \delta q_2\). The average waiting time of the equivalent node is
\[
E^\left[W^\left(R_{eq}^{(10)}\right)\right] = E^\left[R^\left(R_{eq}^{(10)}\right)\right] - \frac{1}{\mu_{eq}^{(10)}},
\]
where \(E^\left[R_{eq}^{(10)}\right]\) and \(\mu_{eq}^{(10)}\) are shown in (66) and (67) respectively.

(11) Nodes \((Q_{eq}, A_{eq}), (Q_{eq}, A_{eq})\): The average queue length of the equivalent node is
\[
E^\left[N^\left(11\right)_{eq}\right] = \frac{\rho_{eq}^{(11)}}{1 - \rho_{eq}^{(11)}} = E^\left[N^\left(11\right)\right] = \frac{\lambda_2(\lambda_2 + \lambda_3 - 2\lambda_3)}{(\lambda_2 + \lambda_3)(\mu_1 - \lambda_3)} + \frac{\lambda_2(\lambda_2 + \lambda_3)}{\lambda_2 + \lambda_3} \times \left[(\mu_8 - (\lambda_2p + \lambda_3p))(\mu_1' - (\lambda_2p + \lambda_3p))((\mu_1' - (\lambda_2p + \lambda_3p))(\mu_8 - (\lambda_2p + \lambda_3p))\right] \\
+ \frac{\lambda_2(\lambda_2 + \lambda_3)}{\lambda_2 + \lambda_3} \times \left[(\mu_8 - (\lambda_2p + \lambda_3p))(\mu_1' - (\lambda_2p + \lambda_3p))((\mu_1' - (\lambda_2p + \lambda_3p))(\mu_8 - (\lambda_2p + \lambda_3p))\right]
\]
But, \( \mu_{eq}^{(11)} = \frac{\lambda_1 + \lambda_4}{\mu_{eq}^{(11)}} \). So, the equivalent service rate of the nodes \((Q_{eq}, A_{eq})\) and \((Q_{eq}, A_{eq})\) is
\[
\mu_{eq}^{(11)} = \lambda_2 + \lambda_3 + \frac{1}{q_2 + q_3} D11a + \frac{\lambda_2 p_1 + \lambda_3 p_2}{q_2 + q_3} D11b + \frac{\lambda_2 p_1 + \lambda_3 p_2}{q_2 + q_3} D11c,
\]
(71)
where
- \( D11a = \frac{\mu_1 + \mu_2 - \lambda_2}{\mu_{eq}^{(11)}} \)
- \( D11b = \frac{\mu_1 - (\lambda_2 p_1 + \lambda_3 p_2)}{\mu_{eq}^{(11)}} \times \left[ (\mu_8 - (\lambda_2 p_3 + \lambda_3 p_5)) (\mu_{12} - (\lambda_2 p_3 + \lambda_3 p_5)) (\mu_{17} - (\lambda_2 p_3 + \lambda_3 p_5)) (\mu_9 - (\lambda_2 p_3 + \lambda_3 p_5)) \right] \)
- \( D11c = \frac{1}{\mu_{eq}^{(11)}} \times \left[ (\mu_8 - (\lambda_2 p_4 + \lambda_3 p_6)) (\mu_{14} - (\lambda_2 p_4 + \lambda_3 p_6)) (\mu_{18} - (\lambda_2 p_4 + \lambda_3 p_6)) \right] \)

The average response time of the equivalent node is
\[
E[R_{eq}^{(11)}] = \frac{E[N_{eq}^{(11)}]}{\lambda_2 + \lambda_3},
\]
(72)
where \( E[N_{eq}^{(11)}] \) is as shown in (70), \( \lambda_2 = \lambda q_2 \), and \( \lambda_3 = \delta r_1 \).

The average waiting time of the equivalent node is
\[
E[W_{eq}^{(11)}] = E[R_{eq}^{(11)}] - \frac{1}{\mu_{eq}^{(11)}},
\]
(73)
where \( E[R_{eq}^{(11)}] \) and \( \mu_{eq}^{(11)} \) are shown in (72) and (71), respectively.

(12) Nodes \((Q_{eq}, A_{eq}), (Q_{eq}, A_{eq})\): The average queue length of the equivalent node is
\[
E[N_{eq}^{(12)}] = E[N_{eq}^{(6)}] + E[N_{eq}^{(7)}] = \lambda_4 \left( \frac{(\mu_6 - \lambda_4) (\mu_{22} - \lambda_4) (\mu_2 - \lambda_4) (\mu_6 - \lambda_4)}{(\mu_2 - \lambda_4) (\mu_6 - \lambda_4) (\mu_{22} - \lambda_4)} \right)
+ \frac{(\lambda_1 p_1 + \lambda_4 p_4) [\mu_8 + \mu_11 - 2(\lambda_1 p_1 + \lambda_4 p_4)]}{\mu_8 - (\lambda_1 p_1 + \lambda_4 p_4)}
+ \frac{(\lambda_1 p_2 + \lambda_4 p_4) [\mu_9 + \mu_15 - 2(\lambda_1 p_2 + \lambda_4 p_4)]}{\mu_9 - (\lambda_1 p_2 + \lambda_4 p_4)}
= \frac{\lambda_1 p_2 + \lambda_4 p_4}{\mu_13 - (\lambda_1 p_2 + \lambda_4 p_4)},
\]
(74)

But, \( \rho_{eq}^{(12)} = \frac{\lambda_1 + \lambda_4}{\mu_{eq}^{(12)}} \). So, the equivalent service rate of the nodes \((Q_{eq}, A_{eq})\) and \((Q_{eq}, A_{eq})\) is
\[
\mu_{eq}^{(12)} = \lambda_1 + \lambda_4 + \frac{1}{\lambda_1 + \lambda_4} D12a + \frac{\lambda_1 p_1 + \lambda_4 p_4}{\lambda_1 + \lambda_4} D12b + \frac{\lambda_1 p_2 + \lambda_4 p_4}{\lambda_1 + \lambda_4} D12c,
\]
(75)
where
- \( D12a = \frac{(\mu_6 - \lambda_4) (\mu_2 - \lambda_4) (\mu_6 - \lambda_4) (\mu_{22} - \lambda_4)}{(\mu_2 - \lambda_4) (\mu_6 - \lambda_4) (\mu_{22} - \lambda_4)} \)
- \( D12b = \frac{\mu_9 + \mu_11 - 2(\lambda_1 p_1 + \lambda_4 p_4)}{\mu_9 - (\lambda_1 p_1 + \lambda_4 p_4)} \)
- \( D12c = \frac{\mu_9 + \mu_15 - 2(\lambda_1 p_2 + \lambda_4 p_4)}{\mu_9 - (\lambda_1 p_2 + \lambda_4 p_4)} \)

The average response time of the equivalent node is
\[
E[R_{eq}^{(12)}] = \frac{E[N_{eq}^{(12)}]}{\lambda_1 + \lambda_4},
\]
(76)
where \( E[N_{eq}^{(12)}] \) is as shown in (74), \( \lambda_1 = \lambda q_1 \), and \( \lambda_4 = \delta r_2 \).

The average waiting time of the equivalent node is
\[
E[W_{eq}^{(12)}] = E[R_{eq}^{(12)}] - \frac{1}{\mu_{eq}^{(12)}},
\]
(77)
where \( E[R_{eq}^{(12)}] \) and \( \mu_{eq}^{(12)} \) are shown in (76) and (75), respectively.

(13) Nodes \((Q_{eq}, A_{eq}), (Q_{eq}, A_{eq})\): The average queue length of the equivalent node is
\[ E \left[ N_{eq}^{(13)} \right] = \frac{\mu_{eq}^{(13)}}{1 - \rho_{eq}^{(13)}} = E \left[ N_{eq}^{(9)} \right] + E \left[ N_{eq}^{(10)} \right] = \frac{\hat{\lambda}_1}{\mu_1 - \hat{\lambda}_1} + \frac{\hat{\lambda}_2}{\mu_1 - \hat{\lambda}_2} \left[ H_8 + \frac{\mu_{11} - 2(\hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_4 \hat{\lambda}_7)}{\mu_{11} - (\hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_4 \hat{\lambda}_7)} \right] \]

\[ + \left( \frac{\hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_4 \hat{\lambda}_7}{\mu_1 - \hat{\lambda}_1} \right) \left[ H_9 + \frac{\mu_{15} - 2(\hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_4 \hat{\lambda}_7)}{\mu_{15} - (\hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_4 \hat{\lambda}_7)} \right] \]

But, \( \rho_{eq}^{(13)} = \frac{\lambda + \delta}{D13} \). So, the equivalent service rate of the nodes \( (Q_{eq1}, A_{eq1}) \) and \( (Q_{eq2}, A_{eq2}) \) is

\[ \mu_{eq}^{(13)} = \delta + \frac{\lambda + \delta}{D13} \]  

(79)

where

- D13 = \( \lambda_1 D13a + (\lambda_1 p_1 + \lambda_4 p_7) D13b + (\lambda_1 p_2 + \lambda_4 p_8) D13c + \lambda_2 D13d + (\lambda_2 p_3 + \lambda_3 p_5) D13e + (\lambda_2 p_4 + \lambda_3 p_6) D13f \)
- D13a = \( \frac{1}{\mu_1 - \hat{\lambda}_1} \left[ \frac{1}{\mu_1 - (\hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_4 \hat{\lambda}_7)} + \frac{1}{\mu_1 - (\hat{\lambda}_1 p_1 + \hat{\lambda}_4 p_7)} \right] \)
- D13b = \( \frac{1}{\mu_1 - \hat{\lambda}_1} \left[ \frac{1}{\mu_1 - (\hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_4 \hat{\lambda}_7)} + \frac{1}{\mu_1 - (\hat{\lambda}_1 p_2 + \hat{\lambda}_4 p_8)} \right] \)
- D13e = \( \frac{1}{\mu_1 - \hat{\lambda}_1} \left[ \frac{1}{\mu_1 - (\hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_4 \hat{\lambda}_7)} + \frac{1}{\mu_1 - (\hat{\lambda}_1 p_3 + \hat{\lambda}_3 p_5)} \right] \)
- D13f = \( \frac{1}{\mu_1 - \hat{\lambda}_1} \left[ \frac{1}{\mu_1 - (\hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_4 \hat{\lambda}_7)} + \frac{1}{\mu_1 - (\hat{\lambda}_1 p_4 + \hat{\lambda}_3 p_6)} \right] \)

The average response time of the equivalent node is

\[ E \left[ R_{eq}^{(13)} \right] = E \left[ N_{eq}^{(13)} \right] \times \hat{\lambda}_3, \]  

(80)

where \( \hat{\lambda}_3 = \lambda q_1, \lambda_2 = \lambda q_2, \lambda_3 = \delta r_1, \) and \( \hat{\lambda}_4 = \delta r_2. \)

The average waiting time of the equivalent node is

\[ E \left[ W_{eq}^{(13)} \right] = E \left[ R_{eq}^{(13)} \right] - \frac{1}{\mu_{eq}^{(13)}}, \]  

(81)

where \( E [N_{eq}^{(13)}] \) and \( \mu_{eq}^{(13)} \) are shown in (80) and (79), respectively.

(14) Nodes \( (Q_{eq11}, A_{eq11}), (Q_{eq12}, A_{eq12}) \)

The average queue length of the equivalent node is

\[ E \left[ N_{eq}^{(14)} \right] = \frac{\rho_{eq}^{(14)}}{1 - \rho_{eq}^{(14)}} = E \left[ N_{eq}^{(11)} \right] + E \left[ N_{eq}^{(12)} \right] = \frac{\hat{\lambda}_3 \left( \mu_2 + \mu_5 - 2 \hat{\lambda}_3 \right)}{\mu_5 - \hat{\lambda}_3} \]

\[ + \left( \frac{\mu_{17} - (\hat{\lambda}_2 p_3 + \hat{\lambda}_3 p_5)}{\mu_8 - (\hat{\lambda}_2 p_3 + \hat{\lambda}_3 p_5)} \right) \left[ \frac{\mu_{12} - (\hat{\lambda}_2 p_3 + \hat{\lambda}_3 p_5)}{\mu_8 - (\hat{\lambda}_2 p_3 + \hat{\lambda}_3 p_5)} \right] \times \left[ (\hat{\lambda}_2 p_3 + \hat{\lambda}_3 p_5) \right] \]

\[ + \left( \frac{\mu_{17} - (\hat{\lambda}_2 p_3 + \hat{\lambda}_3 p_5)}{\mu_8 - (\hat{\lambda}_2 p_3 + \hat{\lambda}_3 p_5)} \right) \left[ \frac{\mu_{12} - (\hat{\lambda}_2 p_3 + \hat{\lambda}_3 p_5)}{\mu_8 - (\hat{\lambda}_2 p_3 + \hat{\lambda}_3 p_5)} \right] \times \left[ (\hat{\lambda}_2 p_3 + \hat{\lambda}_3 p_5) \right] \]

But, \( \rho_{eq}^{(14)} = \frac{\lambda + \delta}{D14} \). So, the equivalent service rate of the nodes \( (Q_{eq11}, A_{eq11}) \) and \( (Q_{eq12}, A_{eq12}) \) is

\[ \mu_{eq}^{(14)} = \delta + \frac{\lambda + \delta}{D14} \]  

(82)
\[
\mu_{eq}^{(14)} = \lambda + \delta + \frac{\lambda + \delta}{D14}, \quad (83)
\]

where

- **D14** = \(\lambda_3 D14a + (\lambda_2 p_3 + \lambda_3 p_5) D14b + (\lambda_2 p_4 + \lambda_3 p_6) D14c + \lambda_4 D14d + (\lambda_1 p_1 + \lambda_4 p_7) D14e + (\lambda_1 p_2 + \lambda_4 p_8) D14f,\)
- **D14a** = \(\frac{\mu_{eq}^{(14)}}{(\mu_{eq}^{(14)} - \lambda_1)}\),
- **D14b** = \(\frac{1}{(\mu_{eq}^{(14)} - \lambda_1)^2 (\mu_{eq}^{(14)} - \lambda_2 p_3) (\mu_{eq}^{(14)} - \lambda_2 p_4)} \times \{(\mu_{eq}^{(14)} - \lambda_2 p_3) (\mu_{eq}^{(14)} - \lambda_3 p_5) (\mu_{eq}^{(14)} - \lambda_3 p_6) (\mu_{eq}^{(14)} - \lambda_4 p_7) (\mu_{eq}^{(14)} - \lambda_4 p_8)\} \times \{(\mu_{eq}^{(14)} - \lambda_2 p_4) (\mu_{eq}^{(14)} - \lambda_3 p_6) (\mu_{eq}^{(14)} - \lambda_4 p_8)\},\)
- **D14c** = \(\frac{1}{(\mu_{eq}^{(14)} - \lambda_2 p_3) (\mu_{eq}^{(14)} - \lambda_3 p_5) (\mu_{eq}^{(14)} - \lambda_3 p_6) (\mu_{eq}^{(14)} - \lambda_4 p_8)\} \times \{(\mu_{eq}^{(14)} - \lambda_2 p_4) (\mu_{eq}^{(14)} - \lambda_3 p_6) (\mu_{eq}^{(14)} - \lambda_4 p_8)\},\)
- **D14d** = \(\frac{1}{(\mu_{eq}^{(14)} - \lambda_2 p_3) (\mu_{eq}^{(14)} - \lambda_3 p_5) (\mu_{eq}^{(14)} - \lambda_3 p_6) (\mu_{eq}^{(14)} - \lambda_4 p_8)\} \times \{(\mu_{eq}^{(14)} - \lambda_2 p_4) (\mu_{eq}^{(14)} - \lambda_3 p_6) (\mu_{eq}^{(14)} - \lambda_4 p_8)\},\)
- **D14e** = \(\frac{1}{(\mu_{eq}^{(14)} - \lambda_2 p_3) (\mu_{eq}^{(14)} - \lambda_3 p_5) (\mu_{eq}^{(14)} - \lambda_3 p_6) (\mu_{eq}^{(14)} - \lambda_4 p_8)\} \times \{(\mu_{eq}^{(14)} - \lambda_2 p_4) (\mu_{eq}^{(14)} - \lambda_3 p_6) (\mu_{eq}^{(14)} - \lambda_4 p_8)\},\)
- **D14f** = \(\frac{1}{(\mu_{eq}^{(14)} - \lambda_2 p_3) (\mu_{eq}^{(14)} - \lambda_3 p_5) (\mu_{eq}^{(14)} - \lambda_3 p_6) (\mu_{eq}^{(14)} - \lambda_4 p_8)\} \times \{(\mu_{eq}^{(14)} - \lambda_2 p_4) (\mu_{eq}^{(14)} - \lambda_3 p_6) (\mu_{eq}^{(14)} - \lambda_4 p_8)\}.

The average response time of the equivalent node is

\[
E[R_{eq}^{(14)}] = E[N_{eq}^{(14)}], \quad (84)
\]

where \(\lambda_1 = \lambda q_1, \lambda_2 = \lambda q_2, \lambda_3 = \delta r_1, \text{ and } \lambda_4 = \delta r_2.\)

The average waiting time of the equivalent node is

\[
E[W_{eq}^{(14)}] = E[R_{eq}^{(14)}] - \frac{1}{\mu_{eq}^{(14)}}, \quad (85)
\]

where \(E[R_{eq}^{(14)}]\) and \(E[W_{eq}^{(14)}]\) are shown in (84) and (83), respectively.

(15) Nodes (Q_{eq1}, A_{eq1}), (Q_{eq2}, A_{eq2}): The average queue length of the equivalent node is

\[
E[N_{eq}] = \frac{\rho_{sys}}{1 - \rho_{sys}} = E[N_{eq}^{(13)}] + E[N_{eq}^{(14)}] = \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_2 p_3 + \lambda_4 p_7}{\mu_{eq}^{(13)} - (\lambda_1 p_1 + \lambda_4 p_7)} + \frac{\lambda_2 p_4 + \lambda_4 p_8}{\mu_{eq}^{(13)} - (\lambda_1 p_2 + \lambda_4 p_8)} + \frac{2(\lambda_1 p_1 + \lambda_4 p_7)}{\mu_1 - (\lambda_1 p_1 + \lambda_4 p_7)} + \frac{2(\lambda_1 p_2 + \lambda_4 p_8)}{\mu_1 - (\lambda_1 p_2 + \lambda_4 p_8)}
\]

But, \(\rho_{sys} = \frac{\lambda_1}{\mu_1}.\) So, the equivalent service rate of the equivalent single queue-single server network is

\[
\mu_{sys} = \lambda + \delta + \frac{1}{D15}, \quad (87)
\]

where

- **D15** = \(\lambda_1 D15a + (\lambda_1 p_1 + \lambda_4 p_7) D15b + (\lambda_2 p_3 + \lambda_4 p_7) D15c + \lambda_3 D15d + \lambda_4 D15e + (\lambda_2 p_4 + \lambda_4 p_8) D15f + (\lambda_2 p_4 + \lambda_4 p_8) D15g + \lambda_4 D15h,\)
- **D15a** = \(\frac{1}{\mu_{eq}^{(13)}},\)
- **D15b** = \(\frac{1}{\mu_{eq}^{(13)} - (\lambda_1 p_1 + \lambda_4 p_7)}\),
- **D15c** = \(\frac{2}{\mu_{eq}^{(13)} - (\lambda_1 p_1 + \lambda_4 p_7)} + \frac{2(\lambda_1 p_1 + \lambda_4 p_7)}{\mu_{eq}^{(13)} - (\lambda_1 p_1 + \lambda_4 p_7)}\),
- **D15d** = \(\frac{1}{\mu_{eq}^{(13)}},\)
- **D15e** = \(\frac{1}{\mu_{eq}^{(13)}},\)
- **D15f** = \(\frac{1}{\mu_{eq}^{(13)}},\)
- **D15g** = \(\frac{1}{\mu_{eq}^{(13)}},\)
- **D15h** = \(\frac{1}{\mu_{eq}^{(13)}},\)
4. Numerical results

4.1. Response time and Queue length of the most optimal path in the four-input network

(1) No weights: Let \((\lambda + \delta)\) be the total number of arrivals in the four-input queueing network. In the example considered in this section, the arrival rate, \((\lambda + \delta) = 2, 4, \ldots, 30\). The other specifications include

- Probability of entering \(\lambda\) or \(\delta\) networks from the original source, \((\lambda + \delta)\), are \(s_1\) and \(s_2\), respectively.
- Probability of arrivals at queues \(Q_4\) and \(Q_5\) are \((q_1, q_2) = (0.5, 0.5)\), respectively.
- Probability of arrivals at queues \(Q_{21}\) and \(Q_{20}\) are \((r_1, r_2) = (0.5, 0.5)\), respectively.
- The service rate specifications of different servers in the network are \(\mu_1 = \mu_1', \mu_2 = \mu_2' = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = \mu_{10} = 12, \mu_2 = 12.5, \mu_2' = 12, \mu_3' = 12.5, \mu_5' = 12.5, \mu_8 = 11.5, \mu_{10} = 11.5, \mu_{11} = 12, \mu_{11} = 12.5, \mu_{12} = 12, \mu_{13} = 12.5, \mu_{14} = 13.\)
- The probabilities, \((s_1, s_2) = (0.5, 0.5)\), \((p_1, p_2) = (p_5, p_6) = (0.4, 0.6)\), and \((p_3, p_4) = (p_7, p_8) = (0.6, 0.4)\).

For each value of \((\lambda + \delta)\), the utilizations, average queue lengths, average response times, and average waiting times in all the nodes of the four-input queueing network are computed. The average queue lengths in paths \(X_1, X_2, \ldots, X_8\) are computed from (31) and (21), respectively. The average response times in paths \(X_1, X_2, \ldots, X_8\) are computed from (22)–(29), respectively.

The minimum of the average response times is computed. It is found that for all arrival rates, the minimum response time corresponds to path \(X_1\). The nodes in the optimal path \(X_1\) are \((Q_4, A_4), (Q_8, A_8)\) and \((Q_{11}, A_{11})\). The average queue length corresponding to path \(X_1\) is noted. The minimum response time and its corresponding queue length are plotted in Fig. 4.

From Fig. 4, it is found that as the arrival rate increases, the average queue length in the optimum path increases much more than the response time. This shows that the queues with the given service rates in the optimum path \(X_1\) are able to serve more customers with not much increase in the response time.

(2) Including weights: When weights are incorporated, the service rates of \(A_4, A_5, A_8, A_9, A_7\) and \(A_{10}\) are halved from their original values given in part (1) [no weights section]. All other specifications remain unchanged. The arrival rate, \((\lambda + \delta) = 2, 4, \ldots, 22\). For each value of \((\lambda + \delta)\), the utilizations, average queue lengths, average response times and average waiting times in all the nodes of the four-input queueing network are computed.

The average queue lengths in paths \(X_1, X_2, \ldots, X_8\) are computed from (14)–(21), respectively. The average response times in paths \(X_1, X_2, \ldots, X_8\) are computed from (22)–(29), respectively. The minimum of all average response times is computed. It is found that for arrival rates, \((\lambda + \delta) = 2, 4, \ldots, 10\), the minimum response time corresponds to path \(X_1\), and for arrival rates, \((\lambda + \delta) = 12, 14, \ldots, 22\), the minimum response time corresponds to path \(X_5\). The nodes in path \(X_1\) are \((Q_4, A_4), (Q_8, A_8)\) and \((Q_{11}, A_{11})\). The nodes in path \(X_5\) are \((Q_{21}, A_{21}), (Q_3, A_3), (Q_{17}, A_{17}), (Q_9, A_9)\), and \((Q_{12}, A_{12})\). The average queue lengths corresponding to paths \(X_1\) and \(X_5\) for the appropriate arrival rates is noted. The minimum response time and its corresponding queue length are plotted in Fig. 5. From Fig. 5, it is found that as the arrival rate increases, the average queue length
creases much more than the average response time. Comparing Figs. 4 and 5, we find that we are able to serve more customers for the same arrival rate when weights are included as compared to the no weight case.

(3) Performance comparison of four-input queuing network with and without weights: Increase in response time (during high load) is caused only because of queuing of the requests. An increase in the user request arrival pattern contributes to an increase in response time. In Fig. 4, when the arrival rate \( \lambda \geq 12 \), there is a steady increase in the response time of all the nodes in the path \( X_1 \) of the four-input network. In Fig. 5, when \( \lambda \geq 14 \), there is a gradual increase in response time of all the nodes in the paths \( X_1 \) and \( X_5 \) of the four-input network. For \( 20 \leq \lambda \leq 22 \), the rise in response time is relatively higher as compared to the other arrival rates.

Comparing Figs. 4 and 5, it can be observed that for the same arrival rate, the response time obtained in Fig. 5 is higher than that obtained in Fig. 4. This can be attributed to the fact that the service rates of some servers in the queuing network are halved for the case including weights (Fig. 5) as compared to the case without weights (Fig. 4).

Due to adhoc user request arrival pattern, long queues are formed in various service centers which leads to high response time and large queue lengths. Comparing Figs. 4 and 5, it can be observed that the queue lengths generated in the case incorporating weights is much larger than that for the case without weights.
4.2. Response time and queue length of the equivalent queue and server of the four-input network

(1) **No weights:** Let \((\lambda + \delta)\) be the total number of arrivals at the equivalent queue and server system as shown in Fig. 3. In the example considered in this section, the arrival rate \((\lambda + \delta) = 1, 2, \ldots, 20\).

- The service rate specifications of different servers in the network are 
  \[ \mu_1 = \mu'_1 = 12, \mu_2 = 12.5, \mu'_2 = 12, \mu_5 = 12, \mu'_5 = 11.5, \]
  \[ \mu'_2 = 12.5, \mu_8 = 11.5, \mu_{12} = 12, \mu_{11} = 12.5, \mu_{13} = 13, \mu_{14} = 13, \mu_9 = \mu_6 = \mu_7 = \mu_8 = \mu_9 = 12. \]

- The probabilities, \((s_1, s_2) = (r_1, r_2) = (q_1, q_2) = (0.5, 0.5), (p_1, p_2) = (p_5, p_6) = (0.4, 0.6), (p_7, p_8) = (0.6, 0.4). \]

For each value of \((\lambda + \delta)\), the average queue length of the equivalent queue, and the average response time of equivalent queue are computed from (86) and (88), respectively. The minimum response time and average queue length are plotted in Fig. 6. From Fig. 6, it is found that the queue length of the equivalent queue increases much more than the response time. This shows that the equivalent server with the equivalent service rate as given by (87) is able to serve more customers with not much increase in the response time.

(2) **Including weights:** When weights are incorporated, the service rates of A4, A5, A8, A9, A7 and A10 are halved from their original values in part (1) [no weights section]. All other specifications remain unchanged. The arrival rate,

![Fig. 6. Response times and queue lengths for equivalent system for four-input network.](image)

![Fig. 7. Response times and queue lengths for equivalent system for four-input network including weights.](image)
For each value of \((\lambda + \delta)\), the average queue length and average response time of the equivalent queue are computed from (86) and (88), respectively. The minimum response time and the average queue length are plotted in Fig. 7. From Fig. 7, it is found that the queue length of the equivalent queue increases much more than the response time.

Comparing Figs. 6 and 7 for the equivalent system, we find that we are able to serve more customers for the same arrival rate when weights are included as compared to the no weight case.

Hence, for the case incorporating weights, the service rate of the equivalent server is lower than that for the case without weights.

For the case incorporating weights in Fig. 7, when \(\lambda > 6\), there is a very small increase in response time. This is because the service rates of some servers in the queuing network are halved, thus doubling their service times. The service rate of the equivalent server in the single queue-single server system decreases as the service rates of the individual servers decreases.

Hence, for the case incorporating weights, the service rate of the equivalent server is lower than that for the case without weights.

By making a comparison of Figs. 6 and 7, for a specific arrival rate, the queue length computed for the case including weights is much larger than the case without weights.

5. Conclusions

In this paper, the most optimal path for routing the items is (i) path \(X_1\) for the no weight case, and (ii) both paths \(X_1\) and \(X_5\) for the case including weights, because these paths produce the least response time for the given set of specifications (probability of entering a new path, arrival rates and service rates). The nodes in the optimal path, \(X_1\) are \((Q_4, A4), (Q_8, A8),\) and \((Q_1, A11)\), and the nodes in the optimal path, \(X_5\) are \((Q_{21}, A21), (Q_3, A3), (Q_{17}, A17), (Q_9, A9),\) and \((Q_{12}, A12)\). The choice of the optimal path depends on the specifications used in numerically evaluating the response time of the queuing network model. The total number of items in the corresponding nodes of the most optimal path constitutes the capacity of the four-input network. Decision for routing is made at the last node in each stage of the network as to which path to choose for obtaining the least response time. Performance measures such as average queue lengths and average response times are derived and plotted. Performance measures such as average waiting times and steady-state probabilities are also derived.

The industrial system is modeled as an equivalent queue-server system. Performance measures such as average queue lengths, average response times and average waiting times are derived and plotted. The service rate of the equivalent server is computed. For both the systems, we find that we are able to serve more customers for the same arrival rate when weights are included as compared to the no weight case.

The whole analysis carried out in this paper can be extended to M/G/1 queues and M/G/1 queues with multiple vacations, as a future work.

Appendix

In this section, the expressions for the steady-state probabilities of having a certain number of jobs in the system for each of the models is presented. The steady-state probability of having \(k_i\) jobs at node \(i\) is \(\Pi_i(k_i) = (1 - \rho_i)\rho_i^{k_i}\) [15].

The steady-state probabilities of jobs in Stage I are:

\[
\begin{align*}
\Pi_4(k_4) &= (1 - \rho_4)\rho_4^{k_4} = \left(1 - \frac{\lambda_1}{\mu_1}\right)\left(\frac{\lambda_1}{\mu_1}\right)^{k_4}, \\
\Pi_5(k_5) &= (1 - \rho_5)\rho_5^{k_5} = \left(1 - \frac{\lambda_2}{\mu_1}\right)\left(\frac{\lambda_2}{\mu_1}\right)^{k_5}, \\
\Pi_{21}(k_{21}) &= (1 - \rho_{21})\rho_{21}^{k_{21}} = \left(1 - \frac{\lambda_3}{\mu_2}\right)\left(\frac{\lambda_3}{\mu_2}\right)^{k_{21}}, \\
\Pi_3(k_3) &= (1 - \rho_3)\rho_3^{k_3} = \left(1 - \frac{\lambda_4}{\mu_5}\right)\left(\frac{\lambda_4}{\mu_5}\right)^{k_3}, \\
\Pi_{20}(k_{20}) &= (1 - \rho_{20})\rho_{20}^{k_{20}} = \left(1 - \frac{\lambda_4}{\mu_2}\right)\left(\frac{\lambda_4}{\mu_2}\right)^{k_{20}}, \\
\Pi_6(k_6) &= (1 - \rho_6)\rho_6^{k_6} = \left(1 - \frac{\lambda_4}{\mu_6}\right)\left(\frac{\lambda_4}{\mu_6}\right)^{k_6}, \\
\Pi_{22}(k_{22}) &= (1 - \rho_{22})\rho_{22}^{k_{22}} = \left(1 - \frac{\lambda_4}{\mu_{22}}\right)\left(\frac{\lambda_4}{\mu_{22}}\right)^{k_{22}}.
\end{align*}
\]
The steady-state probabilities of jobs in Stage II are:

\[
\Pi_b(k_3) = (1 - \rho_b)\rho_b^k_b = \left(1 - \frac{\lambda_1 p_1 + \lambda_4 p_2}{\mu_8}\right)\left(\frac{\lambda_1 p_1 + \lambda_4 p_2}{\mu_8}\right)^k_b,
\]

\[
\Pi_{17}(k_{17}) = (1 - \rho_{17})\rho_{17}^k_{17} = \left(1 - \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_{17}}\right)\left(\frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_{17}}\right)^k_{17},
\]

\[
\Pi_b(k_9) = (1 - \rho_b)\rho_b^k_b = \left(1 - \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_8}\right)\left(\frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_8}\right)^k_b,
\]

\[
\Pi_{15}(k_{15}) = (1 - \rho_{15})\rho_{15}^k_{15} = \left(1 - \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{15}}\right)\left(\frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{15}}\right)^k_{15},
\]

\[
\Pi_{18}(k_{18}) = (1 - \rho_{18})\rho_{18}^k_{18} = \left(1 - \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{18}}\right)\left(\frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{18}}\right)^k_{18},
\]

\[
\Pi_{10}(k_{10}) = (1 - \rho_{10})\rho_{10}^k_{10} = \left(1 - \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_8}\right)\left(\frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_8}\right)^k_{10}.
\]

The steady-state probabilities of jobs in Stage III are:

\[
\Pi_{11}(k_{11}) = (1 - \rho_{11})\rho_{11}^k_{11} = \left(1 - \frac{\lambda_2 p_3 + \lambda_4 p_7}{\mu_{11}}\right)\left(\frac{\lambda_2 p_3 + \lambda_4 p_7}{\mu_{11}}\right)^k_{11},
\]

\[
\Pi_{12}(k_{12}) = (1 - \rho_{12})\rho_{12}^k_{12} = \left(1 - \frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_{12}}\right)\left(\frac{\lambda_2 p_3 + \lambda_3 p_5}{\mu_{12}}\right)^k_{12},
\]

\[
\Pi_{13}(k_{13}) = (1 - \rho_{13})\rho_{13}^k_{13} = \left(1 - \frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{13}}\right)\left(\frac{\lambda_1 p_2 + \lambda_4 p_8}{\mu_{13}}\right)^k_{13},
\]

\[
\Pi_{14}(k_{14}) = (1 - \rho_{14})\rho_{14}^k_{14} = \left(1 - \frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{14}}\right)\left(\frac{\lambda_2 p_4 + \lambda_3 p_6}{\mu_{14}}\right)^k_{14}.
\]

References