Definability in the Homomorphic Quasiorder of Finite Labeled Forests

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Abstract. We prove that for any \( k \geq 3 \) each element of the homomorphic quasiorder of finite \( k \)-labeled forests is definable, provided that the minimal non-smallest elements are allowed as parameters. As corollaries, we show that the structure is atomic and characterize the automorphism group of the structure. Similar results hold true for two other relevant structures: the homomorphic quasiorder of finite \( k \)-labeled trees, and of finite \( k \)-labeled trees with a fixed label of the root element.

Keywords. Labeled tree, forest, homomorphic quasiorder, definability, atomic structure, automorphism.

1 Introduction

In [Se04,Se06,KS06] the structure \((\mathcal{F}_k;\leq)\), \( k \geq 2 \), of finite \( k \)-labeled forests with the homomorphic quasiorder was studied. The structure is interesting in its own right since the homomorphic quasiorder is one in a series of relations on words, trees and forests relevant to computer science. The original interest to this structure [Se04] was motivated by its close relationship to the Boolean hierarchy of \( k \)-partitions. See [H96,Ko00,KW00,Se04,Se06,KS06,Ku06] for more motivation and background. Throughout this paper, \( k \) denotes an arbitrary integer, \( k \geq 2 \), which is identified with the set \( \{0,\ldots,k-1\} \). The cardinality of a set \( F \) is denoted by \(|F|\).

We use some standard notation and terminology on posets which may be found in any book on the subject, see e.g. [DP94]. We will not be very cautious when applying notions about posets also to quasiorders (known also as preorder); in such cases we mean the corresponding quotient-poset of the quasiorder.

* Supported by RFBR Grant 05-01-00819a and by DFG-RFBR Grant 06-01-04002.
** Supported by RFBR Grant 4413.2006.1, by DFG Mercator program and by DFG-RFBR Grant 06-01-04002.
A poset \((P; \leq)\) will be often shorter denoted just by \(P\) (this applies also to structures of other signatures in place of \(\{\leq\}\)). Any subset of \(P\) may be considered as a poset with the induced partial ordering. In particular, this applies to the “cones” \(\tilde{x} = \{y \in P | x \leq y\}\) and \(\tilde{x} = \{y \in P | y \leq x\}\) defined by any \(x \in P\).

By a forest we mean a finite poset in which every lower cone \(\tilde{x}\) is a chain. A tree is a forest having a unique minimal element (called the root of the tree). Note that any forest is uniquely representable as a disjoint union of trees, the roots of the trees being the minimal elements of the forest. A proper forest is a forest which is not a tree. Notice that our trees and forests “grow bottom up”, like the natural ones while trees in \([Se04,Se06]\) grow in the opposite direction.

A \(k\)-labeled forest (or just a \(k\)-poset) is an object \((P; \leq, c)\) consisting of a poset \((P; \leq)\) and a labeling \(c : P \to k\). Sometimes we simplify notation of a \(k\)-poset to \((P, c)\) or even to \(P\). A homomorphism \(f : (P; \leq, c) \to (P'; \leq', c')\) between \(k\)-posets is a monotone function \(f : (P; \leq) \to (P'; \leq')\) respecting the labelings, i.e. satisfying \(c = c' \circ f\).

Let \(F_k\) and \(T_k\) be the classes of all finite \(k\)-forests and finite \(k\)-trees, respectively. Define \([Ko00,KW00,Se04]\) a quasiorder \(\leq\) on \(F_k\) as follows: \((P, c) \leq (P', c')\), if there is a homomorphism from \((P, c)\) to \((P', c')\). By \(\equiv\) we denote the equivalence relation on \(F_k\) induced by \(\leq\). For technical reasons we consider also the empty \(k\)-forest \(\emptyset\) (which is assumed to be a tree) assuming that \(\emptyset \leq P\) for each \(P \in F_k\). Note that in this paper (contrary to notation in [Se04]) we assume that \(\emptyset \in F_k\).

For arbitrary finite \(k\)-trees \(T_0, \ldots, T_n\), let \(F = T_0 \sqcup \cdots \sqcup T_n\) be their join, i.e. the disjoint union. Then \(F\) is a \(k\)-forest whose trees are exactly \(T_0, \ldots, T_n\). Of course, every \(k\)-forest is (equivalent to) the join of its trees. Note that the join operation applies also to \(k\)-forests, and the join of any two \(k\)-forests is clearly their supremum under \(\leq\). Hence, \((F_k; \leq)\) is an upper semilattice.

A natural subset of \(k\)-trees is formed by \(k\)-chains, i.e. by words over the alphabet \(k = \{0, \ldots, k-1\}\). We will denote such words in the usual way, as strings of symbols. E.g., \(01221\) and \(011022\) denote some words over the alphabet \(\{0, 1, 2\}\). Note that any word is equivalent modulo \(\equiv\) to a unique repetition-free word. E.g., the words above are equivalent to \(0121\) and \(0102\), respectively.

For every finite \(k\)-forest \(F\) and every \(i < k\), let \(p_i(F)\) be the \(k\)-tree obtained from \(F\) by joining a new smallest element and assigning the label \(i\) to this element. In particular, \(p_i(\emptyset)\) will be the singleton tree carrying the label \(i\). In [Se06] some properties of the operations \(p_0, \ldots, p_{k-1}\) were established. It is clear that any \(k\)-forest is equivalent to a term of signature \(\{\sqcup, p_0, \ldots, p_{k-1}, \emptyset\}\) without variables. E.g., the words from the preceding paragraph are equivalent to \(p_0(p_1(p_2(p_1(\emptyset))))\) and \(p_0(p_1(p_0(p_2(\emptyset))))\), respectively. Below we omit parenthesis whenever they are clear from the context, e.g. we could write the last term as \(p_0p_1p_0p_2(\emptyset)\).

For each \(i < k\), let \(T^i_k\) be the set of finite \(k\)-trees the roots of which carry the label \(i\). Our interest to the sets \(F_k, T_k\) and \(T^i_k\) is explained by the above-mentioned relation to the Boolean hierarchy of \(k\)-partitions. Namely, the sets \(T^i_k\)
and $\mathcal{F}_k \setminus \mathcal{T}_k$ generalize respectively the non-self-dual and self-dual levels levels of the Boolean hierarchy of sets (i.e., of 2-partitions).

The main result of this paper is now formulated as follows.

**Theorem 1.** For any $k \geq 3$, each element of the quotient structure of $(\mathcal{F}_k; \leq, 0, \ldots, k - 1)$ is definable. The same is true for the quotient structures of $(\mathcal{T}^0_k; \leq, 01, \ldots, 0(k - 1))$ and $(\mathcal{T}_k; \leq, 0, \ldots, k - 1)$.

Recall that a relation in a structure is *definable* if there is a first-order formula of signature of the structure true exactly on the tuples that satisfy the relation. An element is definable if the corresponding singleton set is definable.

Because of space constraints, we give in this conference paper the proof of the main theorem only for the first structure; the omitted proofs for the other two are rather similar to the proof we present below, if one takes into account the corresponding results and techniques from [Se04,Se06,KS06].

Then we deduce some corollaries. The first main corollary states that the quotient structures of $(\mathcal{F}_k; \leq)$, $(\mathcal{T}^0_k; \leq)$ and $(\mathcal{T}_k; \leq)$ are atomic for all $k \geq 2$ and $i < k$. The second main corollary completely characterizes the automorphism groups of the three structures.

In Section 2 we formulate some necessary auxiliary facts, in Section 3 we prove the main theorem, in Section 4 we present some corollaries of the main theorem and in Section 5 we conclude with some additional remarks and open questions.

## 2 Auxiliary Facts

In this section we formulate without proofs several necessary facts established in [Se04,Se06]. Most of the corresponding proofs are short and straightforward and hence maybe hopefully easily recovered by the reader. Otherwise, please consult the source papers.

Recall that a quasiorder is called a *well quasiorder* (wqo) if it has neither infinite descending chains nor infinite antichains. Any wqo $P$ has a rank $\text{rk}(P)$ which is the greatest ordinal isomorphically embeddable into $P$. For any $x \in P$, $\text{rk}(x) = \text{rk}(\hat{x})$ is the rank of $x$. With any quasiorder we associate also its *width* $w(P)$ defined as follows: if $P$ has antichains with any finite number of elements, then $w(P) = \omega$, otherwise $w(P)$ is the greatest natural number $n$ for which $P$ has an antichain with $n$ elements.

The first result cites some facts established in [Se04]. It implies in particular that any element $x$ of $\mathcal{F}_k$ is uniquely representable as a finite union of join-irreducible elements; we call this the *canonical lattice representation* of $x$.

**Proposition 1.** (i) For any $k \geq 2$, $(\mathcal{F}_k; \leq)$ is a wqo with $\text{rk}(\mathcal{F}_k) = \omega$.

(ii) $w(\mathcal{F}_2) = 2$ and $w(\mathcal{F}_k) = \omega$ for $k > 2$.

(iii) For any $k \geq 2$, the quotient structure of $(\mathcal{F}_k; \leq)$ is a distributive lattice.

(iv) The finite $k$-trees define exactly the non-empty join-irreducible elements of the lattice $(\mathcal{F}_k; \leq)$.
The next easy fact was observed in [Se06].

**Proposition 2.** (i) \((T_0^k, \ldots, T_{k-1}^k)\) is a partition of \(T_k\) modulo \(\equiv\).

(ii) For any bijection \(f : k \to k\), the map \((F, c) \mapsto (F, f \circ c)\) defines an automorphism of \((F_k; \leq)\).

(iii) For all \(i, j < k\), \((T_i^j; \leq)\) is isomorphic to \((T_k^j; \leq)\). Moreover, there is an automorphism of \((F_k; \leq)\) sending \(T_k^i\) onto \(T_i^k\).

Next we formulate an interesting property of the lattice \(F_k\) enriched by the unary operations \(p_i\) from Introduction. For this we recall the following notion introduced in [Se82].

**Definition 1.** By a semilattice with discrete closures of rank \(k\) (a dc-semilattice for short) we mean a structure \((S; \cup, p_0, \ldots, p_{k-1})\) satisfying the following axioms:

1) \((S; \cup)\) is an upper semilattice, i.e. it satisfies \((x \cup y) \cup z = x \cup (y \cup z)\), \(x \cup y = y \cup x\) and \(x \cup x = x\); as usual, by \(\leq\) we denote the induced partial order on \(S\) defined by \(x \leq y\) iff \(x \cup y = y\).

2) Every \(p_i\), \(i < k\), is a closure operation on \((S; \leq)\), i.e. it satisfies \(x \leq p_i(x)\), \(x \leq y \rightarrow p_i(x) \leq p_i(y)\) and \(p_i(p_i(x)) = p_i(x)\).

3) The operations \(p_i\) have the following discreteness property: for all distinct \(i, j < k\), \(p_i(x) \leq p_j(x) \rightarrow p_i(x) \leq y\).

4) Every \(p_i(x)\) is join-irreducible, i.e. \(p_i(x) \leq y \cup z \rightarrow (p_i(x) \leq y \lor p_i(x) \leq z)\).

The next fact was observed in [Se06].

**Proposition 3.** The quotient structure of \((F_k; \cup, p_0, \ldots, p_{k-1})\) is a dc-semilattice, and \(p_i(F_k) = T_i^k\) for any \(i < k\).

We recall also a result on the minimal forests, i.e. \(k\)-forests not equivalent (under \(\equiv\)) to \(k\)-forests of lesser cardinality. For a finite poset \(P\), \(h(P)\) will denote the height of \(P\), i.e. the number of elements of the longest chain in \(P\). For any \(i, 1 \leq i \leq h(P)\), let \(P(i) = \{x \in P | h(x) = i\}\). Then \(P(1), \ldots, P(h(P))\) is a partition of \(P\) on "levels"; note that \(P(1)\) is the set of minimal elements of \(P\).

The next assertion is Theorem 1.4 from [Se06] giving a kind of inductive description (by induction on the cardinality) of the minimal \(k\)-forests.

**Proposition 4.** (i) Any trivial (i.e., empty or singleton) \(k\)-forest is minimal.

(ii) A nontrivial \(k\)-tree \((T, c)\) is minimal iff \(\forall x \in T(1) \forall y \in T(2) (c(x) \neq c(y))\) and the \(k\)-forest \((T \setminus T(1), c)\) is minimal.

(iii) A proper \(k\)-forest is minimal iff all its \(k\)-trees are minimal and pairwise incomparable under \(\leq\).

This provides the canonical term representations of the elements of \(F_k\) by variable-free terms of signature \(\{\cup, p_0, \ldots, p_{k-1}, \emptyset\}\). The canonical terms are obtained from minimal trees by the usual procedure relating terms and trees.
3 Definability

Here we present a proof of the first assertion of Theorem 1. We start with a series of lemmas. For any \( x \in \mathcal{T}_k \), let \( x' = \bigcup \{ y \in \mathcal{T}_k : y < x \} \). Since \( x \) is join-irreducible, \( x' < x \). We need some properties of the introduced function \( x \mapsto x' \) from \( \mathcal{T}_k \) to \( \mathcal{T}_k \). We call a partial function \( f : \mathcal{F}_k \rightarrow \mathcal{F}_k \) definable if its graph is definable (note that this implies that both domain and range of \( f \) are definable). The first lemma is obvious.

**Lemma 1.**

(i) For any \( x \in \mathcal{T}_k \), \( x' \) is the biggest element \( y \) of \( \mathcal{F}_k \) with \( y < x \).

(ii) The function \( x \mapsto x' \) is definable.

For \( x \in \mathcal{F}_k \), let \( \text{pred}(x) \) be the set of maximal elements in \( \{ y \in \mathcal{F}_k : y < x \} \).

**Lemma 2.**

(i) For any \( x \in \mathcal{T}_k \), \( \text{pred}(x) = \{ x' \} \).

(ii) For all \( n > 0 \) and pairwise incomparable \( x_0, \ldots, x_n \in \mathcal{T}_k \), \( \text{pred}(x_0 \sqcup \cdots \sqcup x_n) = \{ y_0, \ldots, y_n \} \), where \( y_j = x'_j \sqcup (\bigcup_{i \neq j} x_i) \).

(iii) Let \( n > 0 \), \( i < k \) and \( x_0, \ldots, x_n \) be pairwise incomparable elements in \( \mathcal{T}_k \setminus \mathcal{T}_k^i \). Then \( p_i(y_0), \ldots, p_i(y_n) \) (where \( y_i \) are as in (ii)) are pairwise incomparable.

(iv) For all \( x, \tilde{x} \in \mathcal{F}_k \setminus \mathcal{T}_k \), \( \text{pred}(x) = \text{pred}(\tilde{x}) \) implies \( x = \tilde{x} \).

**Proof.**

(i) is obvious.

(ii) Let \( x = x_0 \sqcup \cdots \sqcup x_n \), then obviously \( y_j \leq x \) for any \( j \leq n \). Since \( x_j \not\leq y_j \), \( y_j < x \). Elements \( y_0, \ldots, y_n \) are pairwise incomparable because \( x_i \not\leq y_j \) and \( x_l \not\leq y_l \) for all \( j \neq l \). It remains to show that if \( z < x \) then \( z \leq y_j \) for some \( j \leq n \). By distributivity (see Proposition 1), \( z = \bar{x}_0 \sqcup \cdots \sqcup \bar{x}_n \) for some \( \bar{x}_0 \leq x_0, \ldots, \bar{x}_n \leq x_n \). Since \( z < x \), \( \bar{x}_j < x_j \), hence \( \bar{x}_j \leq x'_j \) for some \( j \leq n \). Therefore, \( z \leq y_j \).

(iii) Follows from the fact that \( x_i \leq p_i(y_j) \) and \( x_l \leq p_i(y_l) \) for all \( l \neq j \).

(iv) Let \( x = x_0 \sqcup \cdots \sqcup x_n \) and \( \tilde{x} = \bar{x}_0 \sqcup \cdots \bar{x}_n \) be canonical lattice representations, so \( n, m > 0 \). From \( \text{pred}(x) = \text{pred}(\tilde{x}) \) and (ii) we get \( n = m \) and w.l.o.g. \( y_0 = y_0, \ldots, y_n = y_n \). For any \( j \leq n \) we have \( x_j < x \), hence \( x_j \leq \tilde{y}_j \) for some \( l \leq n \). Therefore, \( x \leq \tilde{x} \). Since \( j \) was arbitrary, \( x \leq \tilde{x} \). By symmetry, \( \tilde{x} \leq x \). This completes the proof of the lemma.

The next lemma implies that the function \( x \mapsto x' \) may be computed by induction on the canonical term representation from Section 2.

**Lemma 3.**

(i) For any \( i < k \), \( (p_i(\emptyset))' = \emptyset \).

(ii) For all \( n > 0 \), \( i < k \), and pairwise incomparable elements \( x_0, \ldots, x_n \in \mathcal{T}_k \setminus \mathcal{T}_k^i \), \( (p_i(x_0 \sqcup \cdots \sqcup x_n))' = p_i(y_0) \sqcup \cdots \sqcup p_i(y_n) \), where \( y_j = x'_j \sqcup (\bigcup_{i \neq j} x_i) \), and the elements \( p_i(y_0), \ldots, p_i(y_n) \) are pairwise incomparable.

(iii) For all \( x \in \mathcal{F}_k \) and distinct \( i, j < k \), \( (p_ip_j(x))' = p_j(x) \sqcup p_i((p_j(x))') \) and the elements \( p_j(x), p_i((p_j(x))') \) are incomparable.

**Proof.**

(i) is obvious.

(ii) It suffices to check that if \( z \in \mathcal{T}_k \) and \( z < p_i(x_0 \sqcup \cdots \sqcup x_n) \) then \( z \leq p_i(y_0) \sqcup \cdots \sqcup p_i(y_n) \), because other properties follow from Lemma 2. If \( z \notin \mathcal{T}_k^i \) then
$z \leq x_0 \sqcup \cdots \sqcup x_n$, hence $z \leq x_j$ for some $j \leq n$. Therefore, $z \leq p_i(y_0) \sqcup \cdots \sqcup p_i(y_n)$.

If $z \in T_k^i$ then $z = p_i(z_0)$ for some canonical term $p_i(z_0)$. Then $z_0 < x_0 \sqcup \cdots \sqcup x_n$, hence $z_0 \leq y_j$ for some $j \leq n$. Therefore, $z \leq p_i(y_0) \sqcup \cdots \sqcup p_i(y_n)$.

(iii) First we check incomparability. The relation $p_j(x) \not\leq p_i((p_j(x))')$ is clear. Suppose that $p_i((p_j(x))') \leq p_j(x)$. Since $i \not= j$, $p_i((p_j(x))') < p_j(x)$ and therefore $p_i((p_j(x))') \leq p_j(x)$.

Clearly, $p_j(x) \cup p_i((p_j(x))') < p_jp_j(x)$. It remains to show that if $y \in T_k$ and $y < p_jp_j(x)$ then $y \leq p_j(x) \cup p_i((p_j(x))')$. If $y \not\in T_k$ then $y \leq p_j(x)$ and we are done. Otherwise, $y \equiv p_i(z)$ for some $z$. Assuming w.l.o.g. that terms in the expression $p_i(z) < p_jp_j(x)$ are canonical (see end of Section 2), by definition of $\leq$ we get $z < p_j(x)$. Therefore, $z \leq (p_j(x))'$ and hence $y \leq p_i((p_j(x))')$. This completes the proof.

For any $x \in F_k$, let $l(x) = \{i < k : i \leq x\}$, i.e. $l(x)$ is the set of labels assigned to nodes of some (or, equivalently, of any) forest representing $x$. The next assertion is an immediate corollary of the preceding one.

Lemma 4. For any $x \in T_k \setminus k$, $l(x) = l(x')$.

The function $x \mapsto x'$ is in general not injective. E.g., if $x, y \in T_k$ are of the same rank, $l(x) = l(y)$ and $|l(x)| < 3$ then $x' = y'$ is the unique element of rank $rk(x) - 1$ below $x$. The next assertion implies that the function $x \mapsto x'$ is injective in all other cases. Let $G_k = \{x \in T_k : |l(x)| \geq 3\}$.

Lemma 5. If at least one of elements $x, y \in T_k$ is in $G_k$ then $x' = y'$ implies $x = y$.

Proof. In case when exactly one of $x, y$ is in $G_k$ the assertion follows from Lemma 4. So assume that $x, y \in G_k$, then $x', y' \in G_k$ by Lemma 4. Let $x = p_i(t)$ and $y = p_j(s)$ be the canonical term representations.

First we check that $i = j$. Suppose not, and consider several cases. If $t \not\in T_k$ and $s \in T_k$ then, by Lemma 3, all components in the canonical lattice representation of $x'$ are in $T_k^i$ while one of the components of $y'$ is in $T_k^j$, a contradiction with $y' = x'$. The case when $t \in T_k$ and $s \not\in T_k$ follows by symmetry.

Now let $s, t \in T_k$. By Lemma 3(iii), $t = p_j(s')$ and $s = p_i(t')$. Since $s' \not\in T_k$ by Lemma 3, $s' < t$. By Lemma 1, $s' \leq t'$. A symmetric argument shows that $t' \leq s'$, hence $s' = t'$. If $t' \in G_k$ then, by induction on $(rk(x), rk(y))$, $s = t$ and therefore $p_j(s') = p_i(t')$, a contradiction. Finally, assume that $t' \not\in G_k$, i.e. $|l(t')| \leq 2$. By Lemma 4, $l(t') = l(t)$ and $l(s') = l(s)$. Since $t = p_j(s')$ and $s = p_i(t')$, $i \in l(s)$ and $j \in l(t)$. Therefore, $l(t) = l(s) = \{i, j\}$ and consequently $l(x) = l(y) = \{i, j\}$, a contradiction.

We have checked that $i = j$, so $x = p_i(t)$ and $y = p_i(s)$. The case when exactly one of $s, t$ is in $T_k$ is impossible by an above-used argument. If $s, t \not\in T_k$ then, by Lemma 3, $\{p_i(z) : z \in \text{pred}(t)\} = x' = y'$ = $\{p_i(z) : z \in \text{pred}(s)\}$. From Lemma 2 it follows that $\text{pred}(t) = \text{pred}(s)$ and hence $t = s$. Therefore, $x = y$. If $s, t \in T_k$ then, by Lemma 3, $t \cup p_i(t') = s \cup p_i(s')$ and the components are incomparable. Hence, $s = t$ and $x = y$. This completes the proof of the lemma.

Now we start to define some elements in our structures.
Lemma 6. Elements 01, 10, 02, 20, 12, 21 are definable in \((F_3; \leq, 0, 1, 2)\).

Proof. By symmetry, it suffices to show that 01 and 10 are definable. It is easy to see that \(x \in \{01, 10\}\) iff \(x\) is a minimal join-irreducible element above 0, 1 with \(2 \not\leq x\). The following formula \(\psi(x)\) is true on 01 and false on 10:

\[
\exists z \exists y (\text{ir}(z) \land \text{ir}(y) \land \forall t (t < z \leftrightarrow t \leq x \land y) \land \forall t (t < y \leftrightarrow t \leq 0 \cup 2))
\]

where \text{ir} is a formula of signature \(\{\leq\}\) that defines in every lattice exactly the non-zero join-irreducible elements. (Such a formula is written easily in signature \(\{0, \leq, \lor\}\), namely \(x \neq 0 \land \forall y \forall z (x \leq y \lor z \rightarrow (x \leq y \lor x \leq z))\). Since 0 and \(\lor\) are first-order definable in signature \(\{\leq\}\) the last formula may be rewritten as an equivalent formula of \(\{\leq\}\).

Indeed, for the case \(x = 01\) we put \(z = p_0(1 \cup 2)\), \(y = 02\) and immediately obtain \(\psi(01)\). Now let \(x = 10\), and assume, towards a contradiction, that \(\psi(10)\) is true and let \(z, y\) be some satisfying values. Then, by the second equivalence, \(y \in \{02, 20\}\), let e.g. \(y = 02\). By the first equivalence and Proposition 3, \(z\) is above at least one of \(p_0(10 \cup 02)\), \(p_1(10 \cup 02)\), \(p_2(10 \cup 02)\). Taking in the first case \(t = 010\), in the second case \(t = 10\) and in the third case \(t = 210\) we obtain a contradiction with the first equivalence. This completes the proof of the lemma.

Lemma 7. The set \(T_2^0\) is definable in \((F_3; \leq, 0, 1, 2)\).

Proof. Let \(L\) be the set of 3-trees which have labels 0 and 2 and carry label 2 only on the leaves (i.e., maximal elements). Let \(T_{2,0}\) be the set of trees in \(T_2\) having a label 0. Let \(\text{del} : L \rightarrow T_{2,0}\) be the function that deletes all 2-labeled leaves. It is easy to see that the function \(\text{del}\) respects the homomorphic quasiorder. Then we have:

\[
\begin{align*}
x \in T_2 & \rightarrow \text{ir}(x) \land 2 \not\leq x, \\
x \in T_{2,0} & \rightarrow \text{ir}(x) \land 0 \leq x \land 2 \not\leq x, \\
x \in L & \rightarrow \text{ir}(x) \land 0 \leq x \land 2 \leq x \land 20 \not\leq x \land 21 \not\leq x,
\end{align*}
\]

and \(\text{del}(x) = \bigcup \{z \in T_2 : z \leq x\}\). It is easy to check that for all \(y \in F_3\) there holds

\[
y \in T_2^0 \leftrightarrow y \in T_{2,0} \land \forall x \in L (y = \text{del}(x) \rightarrow 02 \leq x).
\]

By Lemma 6 and definitions above, \(T_2^0\) is definable. This completes the proof of the lemma.

Now we can prove a strengthening of Lemma 6.

Lemma 8. For any \(k \geq 3\), each element \(a \in T_k \setminus G_k\) is definable in \((F_k; \leq, 0, \ldots, k - 1)\).

Proof. If \(|l(a)| \leq 1\) i.e. \(a \in \{0, 0, \ldots, k - 1\}\) then the assertion is obvious. Now let \(l(a) = \{i, j\}\) for some distinct \(i, j < k\). By Proposition 2(ii), there is an automorphism of \((F_k; \leq)\) sending \(i\) to 0 and \(j\) to 1, hence we can w.l.o.g. assume that \(i = 0\) and \(j = 1\), i.e. \(a\) is (represented by) a repetition-free binary word \(w\).
We use induction on the length \(|w|\) of \(w\). Suppose \(|w| > 1\) and \(w\) starts with 0 (the last assumption may be made by symmetry), so \(w = 0v\) for the unique repetition-free word of length \(|w| - 1\) starting with 1. By induction, \(v\) is definable by a formula \(\phi_v(x)\). Then \(w\) is the least (under \(\leq\)) element \(y\) of \(\mathcal{T}^0_2\) such that \(\exists x (\phi_v(x) \land x \leq y)\). Since \(\mathcal{T}^0_2\) is definable by the previous lemma, the last informal definition of \(w\) may be rewritten as a formula of signature \(\{\leq, 0, \ldots, k - 1\}\). This completes the proof.

**Proof of Theorem 1 for \(\mathcal{F}_k\).** Let \(k \geq 3\) and \(a \in \mathcal{F}_k\). We check by induction on \(\text{rk}(a)\) that \(a\) is definable in \((\mathcal{F}_k, \leq, 0, \ldots, k - 1)\). The case \(\text{rk}(a) \leq 1\), i.e. \(a \in \{\emptyset, 0, \ldots, k - 1\}\), is obvious. Let \(a \not\in \mathcal{I}_k \cup \{\emptyset\}\) and \(a = a_0 \cup \cdots \cup a_n, n > 0\), be the canonical lattice representation of \(a\). By induction, \(a_i\) is definable by a formula \(\phi_{a_i}(x)\) for all \(i \leq n\). Then the formula

\[
\phi_a(x) = \exists x_0 \cdots \exists x_n ((\bigwedge_{i \leq n} \phi_{a_i}(x_i)) \land x = x_0 \cup \cdots \cup x_n)
\]

defines \(a\). Now let \(a \in \mathcal{I}_k\) and \(\text{rk}(a) > 1\). If \(a \not\in \mathcal{I}_k \setminus \mathcal{G}_k\) the assertion holds by the previous lemma. Finally, let \(a \in \mathcal{G}_k\). By induction, there is a formula \(\phi_{a'}(x)\) that defines \(a'\). By Lemmas 5 and 1, the formula \(\phi_a(x) = \text{ir}(x) \land \exists y (\phi_{a'}(y) \land y = x')\) defines \(a\). This completes the proof.

### 4 Atomicity and Automorphisms

In this section we establish some corollaries of the main theorem. First we show that all three our structures are atomic, i.e. realize only the principal types (for definition of these well-known notions from model theory see any standard text on this subject, e.g. [CK73]).

Since each element of the structure \((\mathcal{F}_k; \sqcup, p_0, \ldots, p_{k-1})\) is represented by a term without variables, this structure is obviously atomic for every \(k \geq 2\). Informally this means that the structure is the smallest in a model-theoretic sense. In [Se06] it was shown that this structure is also the smallest in an algebraic sense.

From the main theorem it follows that the structure \((\mathcal{T}_k; \leq, 0, \ldots, k - 1)\) is atomic for every \(k \geq 2\). The next result is a slight strengthening of this fact.

**Theorem 2.** For all \(k \geq 2\) and \(i < k\), the quotient structures of \((\mathcal{F}_k; \leq), (\mathcal{T}_k^i; \leq)\) and \((\mathcal{T}_k; \leq)\) are atomic.

**Proof.** We give a proof only for the first structure but it works for the other two structures as well. As is well-known, atomicity is equivalent to definability of orbits of all tuples of elements. So we have to show that for any tuple \(\bar{a} = (a_0, \ldots, a_n), n \geq 0\), of elements of \(\mathcal{F}_k\) its orbit \(\text{Orb}(\bar{a}) = \{f(\bar{a}) : f \in \text{Aut}(\mathcal{F}_k; \leq)\}\) is definable, where \(f(\bar{a}) = (f(a_0), \ldots, f(a_n))\). By the main theorem, for any \(i \leq n\) there is a formula \(\phi_{a_i}(x)\) of signature \(\{\leq, 0, \ldots, k - 1\}\) that defines \(a_i\). Then the formula \(\phi_{\bar{a}}(\bar{x}) = \phi_{a_0}(x_0) \land \cdots \land \phi_{a_n}(x_n)\) of signature \(\{\leq, 0, \ldots, k - 1\}\) defines \(\bar{a}\).

Since any automorphism of \((\mathcal{F}_k; \leq)\) preserves ranks of elements and \((\mathcal{F}_k; \leq)\) is a wqo, the orbit \(\text{Orb}(\bar{a})\) is finite, i.e. \(\text{Orb}(\bar{a}) = \{\bar{b}_0, \ldots, \bar{b}_m\}\) for some \(m <
\[ \omega \text{ and tuples } \bar{b}_0, \ldots, \bar{b}_m. \text{ Then the formula } \phi(\bar{x}) = \phi_{b_0}(\bar{x}) \lor \cdots \lor \phi_{b_m}(\bar{x}) \text{ of signature } \{\leq, 0, \ldots, k-1\} \text{ defines the orbit } \text{Orb}(\bar{a}) \text{ in } (\mathcal{F}_k; \leq, 0, \ldots, k-1). \] The set \{0, \ldots, k-1\} is definable in \((\mathcal{F}_k; \leq)\) by a formula \(\mu(\bar{x})\) stating that \(x\) is minimal among all non-smallest elements. Let \(\psi(\bar{x})\) be the formula

\[ \exists u_0 \cdots \exists u_{k-1} ((\bigwedge_{i \neq k} u_i \neq u_j) \land (\bigwedge_{i < k} \mu(u_i)) \land \phi'(\bar{x})) \]

where \(\phi'(\bar{x})\) is obtained from \(\phi(\bar{x})\) by substituting variables \(u_0, \ldots, u_{k-1}\) in place of constant symbols \(0, \ldots, k-1\), respectively. Then \(\psi(\bar{x})\) defines \(\text{Orb}(\bar{a})\) in \((\mathcal{F}_k; \leq)\), completing the proof of the theorem.

Since each element of the structure \((\mathcal{F}_k; \leq, p_0, \ldots, p_{k-1})\) is represented by a term without variables, this structure is rigid, i.e. it has only the trivial identity automorphism. From the main theorem it follows that the structure \((\mathcal{F}_k; \leq, 0, \ldots, k-1)\), as well as the other two structures with the constants, is also rigid. It is easy to obtain also a complete description of the automorphism groups of the structures without the constants. Let \(S_k\) be the symmetric group on \(k\) elements, i.e. the group of permutations of elements \(0, \ldots, k-1\). Let \(\text{Aut}(A)\) denote the automorphism group of a structure \(A\). By \(\simeq\) we denote the isomorphism relation.

**Theorem 3.**

(i) For any \(k \geq 2\), \(\text{Aut}(\mathcal{F}_k; \leq) \simeq \text{Aut}(\mathcal{T}_k; \leq)\).

(ii) \(\text{Aut}(\mathcal{T}_2; \leq) \simeq S^2_k\).

(iii) For any \(k \geq 3\), \(\text{Aut}(\mathcal{F}_k; \leq) \simeq S_k\).

(iv) For all \(k \geq 2\) and \(i < k\), \(\text{Aut}(\mathcal{T}_k; \leq) \simeq S_{k-1}\).

**Proof.**

(i) For any \(f \in \text{Aut}(\mathcal{F}_k; \leq)\), let \(f^*\) be the restriction of \(f\) to \(\mathcal{T}_k\). Since \(\mathcal{T}_k\) is definable in \((\mathcal{F}_k; \leq)\), \(f^* \in \text{Aut}(\mathcal{T}_k; \leq)\). By Proposition 1, the map \(f \mapsto f^*\) is a surjective group homomorphism with the trivial kernel. Hence, \(\text{Aut}(\mathcal{F}_k; \leq) \simeq \text{Aut}(\mathcal{T}_k; \leq)\).

(ii) It is well known and obvious that \((\mathcal{T}_2; \leq)\) is isomorphic to the partial order obtained from \((\omega; \leq)\) by substituting a pair of incomparable points \((a_n^0, a_n^1)\) in place of any \(n \in \omega\). The group \(S^2_k\) is isomorphic to the \(\omega\)-cartesian power of the cyclic group \(\{0, 1\}\) with two elements, hence it consists of functions \(h : \omega \to \{0, 1\}\). Relate to any such \(h\) the function \(h^* : \mathcal{T}_2 \to \mathcal{T}_2\) defined by: for any \(n < \omega\), if \(h(n) = 0\) then \(h^*(a_n^0) = a_n^0\) and \(h^*(a_n^1) = a_n^1\); otherwise, \(h^*(a_n^0) = a_n^1\) and \(h^*(a_n^1) = a_n^0\). Then \(h \mapsto h^*\) is a desired automorphism.

(iii) For any \(f \in \text{Aut}(\mathcal{F}_k; \leq)\), let \(f^*\) be the restriction of \(f\) to \(\{0, \ldots, k-1\}\). By Proposition 2(ii), the map \(f \mapsto f^*\) is a group homomorphism from \(\text{Aut}(\mathcal{F}_k; \leq)\) onto \(S_k\). Since \((\mathcal{F}_k; \leq, 0, \ldots, k-1)\) is rigid, the kernel of this homomorphism is trivial. Hence, \(\text{Aut}(\mathcal{F}_k; \leq) \simeq S_k\).

The assertion (iv) is proved similarly to (iii). This completes the proof.

**5 Conclusion**

In this paper definability in our structures was used to establish important algebraic and model-theoretic properties of the structures. In the journal version
of the definability was used to show that for every $k \geq 3$ the elementary theory of any of the three structures is computably isomorphic to the elementary theory of the structure $(\omega; +, \cdot)$. A small addition to that proof shows that actually the structure $(\omega; +, \cdot)$ is definable in $(F_k; \leq)$ (as well as in the other two structures) without parameters. At the same time, some natural definability questions remain open. E.g., we do not currently know whether the set $C_k$ of finite $k$-chains or the relation “$y = f(x)$ for some $f \in \text{Aut}(F_k; \leq)$” are definable in $(F_k; \leq)$.

References


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