Review of the methodology for the inversion of surface-wave phase velocities in a slightly anisotropic medium

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A B S T R A C T

In this paper, a new inversion scheme for surface-wave velocities in a slightly anisotropic medium will be presented, revisiting the inversion method performed by Corchete and Badal (2004). An extension of the Smith–Dahlen formulation will be examined and revised. The surface-wave propagation in a slightly anisotropic medium will be considered, and the inversion of the azimuthal dependence of the surface-wave dispersion will be performed. The new inversion scheme proposed in this paper will be verified by numerical matrix inversion in two examples: hexagonal symmetry and 13 nonzero canonical harmonic components. Also, two additional experiments with observed data will be performed. The new method will be shown as more efficient than the method previously used by Corchete and Badal. This new method should be a useful tool for studying the slightly anisotropic structure of wide areas of the Earth, if sufficient high-quality dispersion data are collected and the isotropic properties of the medium are well known.

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1. Introduction

The anisotropy evidenced for the upper mantle is principally slight anisotropy. This slight anisotropy exists over wide areas in the Earth and it has a clear effect on the propagation of surface waves. For this reason, the study of surface-wave propagation in slightly anisotropic media is important in seismology. The study of anisotropy is in general very complex but it can be simplified strictly when slight anisotropy is considered (Smith and Dahlen, 1973). Following this hypothesis, the inversion of surface-wave phase velocities in a slightly anisotropic medium was performed by Corchete and Badal (2004), who calculated the inverse matrix by means of the least-squares inverse. This inversion process is revised in this paper to computing the inverse matrix by means of the s-squares inverse. This inversion process is revised in this paper to computing the inverse matrix by means of the s-squares inverse. This inversion process is revised in this paper to computing the inverse matrix by means of the s-squares inverse. This inversion process is revised in this paper to computing the inverse matrix by means of the s-squares inverse. This inversion process is revised in this paper to computing the inverse matrix by means of the s-squares inverse.

2. Methodology and background

2.1. Azimuthal variation of the surface-wave dispersion

The azimuthal dependence of the surface-wave phase velocity \( c(\omega, \theta) \), in a slightly anisotropic structure is of the form (Smith and Dahlen, 1973)

\[
\begin{align*}
c(\omega, \theta) &= A_1(\omega) + A_2(\omega) \cos 2\theta + A_3(\omega) \sin 2\theta \\
&+ A_4(\omega) \cos 4\theta + A_5(\omega) \sin 4\theta
\end{align*}
\]

where \( \omega \) is the angular frequency, \( \theta \) is the azimuth of the wavenumber vector, and \( A_i(\omega) \) are coefficients dependent on the frequency \( \omega \). This equation also can be rewritten as

\[
c(\omega, \theta) = c(\omega) + \delta c(\omega, \theta)
\]

where \( c(\omega) \) is the isotropic phase velocity of the surface wave. For the Love wave the anisotropy term \( \delta c(\omega, \theta) \) is of the form (Smith and Dahlen, 1973)

\[
\delta c(\omega, \theta) = \frac{1}{2 G_1(\omega)} \left[ L_1(\omega) + L_2(\omega) \cos 2\theta \\
+ L_3(\omega) \sin 2\theta + L_4(\omega) \cos 4\theta + L_5(\omega) \sin 4\theta \right]
\]

and for the Rayleigh wave

\[
\delta c(\omega, \theta) = \frac{1}{2 G_2(\omega)} \left[ R_1(\omega) + R_2(\omega) \cos 2\theta + R_3(\omega) \sin 2\theta \\
+ R_4(\omega) \cos 4\theta + R_5(\omega) \sin 4\theta \right]
\]

where \( G_1(\omega) \) and \( G_2(\omega) \) denote the isotropic Love-wave and Rayleigh-wave group velocities, respectively. \( L_\ell(\omega) \) and \( R_\ell(\omega) \)
are coefficients dependent on the frequency \( o \) that can be evaluated by means of the relationships given by Smith and Dahlen (1973).

In Eqs. (1) and (2), the effect of the slight anisotropy on the dispersion of the surface waves, is associated with the canonical harmonic components \( \gamma_{ijkl}^{\omega m}(z) \) through the coefficients \( L_{ij}(o) \) and \( R_{ij}(o) \), because these coefficients are written in terms of these components (Smith and Dahlen, 1973). Moreover, the components \( \gamma_{ijkl}^{\omega m}(z) \) are related to the elements of the elastic tensor \( \gamma_{ijkl}(z) \), \( \gamma_{ijkl}(z) \) being a small perturbation of the isotropic elastic tensor \( \gamma_{ijkl}(z) \) (Backus, 1970; Smith and Dahlen, 1973); i.e., the elastic tensor \( \gamma_{ijkl}(z) \) of the medium is perturbed from \( \gamma_{ijkl}^{0}(z) \) to \( \gamma_{ijkl}^{0}(z)+\gamma(z) \), where \( \gamma(z) \) is a small perturbation compared with \( \gamma_{ijkl}^{0}(z) \) (i.e., \( \gamma_{ijkl}^{0}(z)+\gamma(z) \leq \gamma_{ijkl}^{0}(z)E_{ijkl}(z) \)). Thus, Eqs. (1) and (2) give the explicit dependence of \( L_{ij}(o) \) and \( R_{ij}(o) \) on the anisotropic elastic properties \( \gamma_{ijkl}^{0}(z) \) of the half-space, being precisely the equations that are required for the inversion process (Corchete and Badal, 2004).

### 2.2. Matrix formulation

Corchete and Badal (2004) provided an inversion scheme to determine the anisotropic structure, given by the elastic tensor \( \gamma(z) = \gamma^{0}(z) + \gamma(z) \), from the \( L_{ij}(o) \) and \( R_{ij}(o) \) coefficients. This inversion scheme starts with the formulation given by Smith and Dahlen (1973), which relates \( L_{ij}(o) \) and \( R_{ij}(o) \) to the canonical harmonic components \( \gamma_{ijkl}^{0}(z) \). For instance, \( L_{ij}(o) \) and \( R_{ij}(o) \) depend explicitly only on \( \gamma_{ijkl}^{0}(z) \) through the integral expressions

\[
L_{ij}(o) = \frac{1}{(2\pi)^{2}} \int_{0}^{\infty} \int_{-\infty}^{\infty} \gamma_{ijkl}^{0}(z)W_{i}(z)dz,
\]

\[
R_{ij}(o) = \frac{1}{(2\pi)^{2}} \int_{0}^{\infty} \int_{-\infty}^{\infty} \gamma_{ijkl}^{0}(z)V_{j}(z)dz,
\]

where \( W(z) \) and \( V(z) \) are scalar functions of depth corresponding to the Love-wave and Rayleigh-wave (horizontal) displacement field (Smith and Dahlen, 1973). In a multilayered anisotropic medium, \( \gamma_{ijkl}^{0}(z) \) is constant for each layer, having then

\[
L_{ij}(o) = \frac{1}{L_{ij}^{0}} \sum_{n=1}^{n} \int_{z_{n}}^{z_{n+1}} W_{i}(z)dz,
\]

\[
R_{ij}(o) = \frac{1}{R_{ij}^{0}} \sum_{n=1}^{n} \int_{z_{n}}^{z_{n+1}} V_{j}(z)dz,
\]

where \( n \) is now the number of layers of the earth model considered, \( d_{n} \) is the thickness of the \( n \)th layer, and \( \gamma_{ijkl}^{0}(z) \) is the canonical harmonic component for the \( n \)th layer. Obviously, for \( R_{ij}(o) \) a similar relation can be written. It should be noted that it is possible to introduce a matrix formulation in the form

\[
L_{ij}(o)L_{ij}(o) = \sum_{n=1}^{n} \int_{z_{n}}^{z_{n+1}} W_{i}(z)S_{i}(o)dz,
\]

\[
S_{i}(o) = - \int_{z_{n}}^{z_{n+1}} W^{o}(o)dz,
\]

where \( o \) is a fixed angular frequency. This matrix relation can be written for \( L_{ij}(o) \) in the form

\[
L = AX
\]

where (using the summation convention for repeated subscripts)

\[
L_{ij}(o) = (L_{ij}^{0})_{ij} (1, j = 1, \ldots, m), \quad L_{ij}^{0} = L_{ij}(o)A_{ij}X_{j}
\]

\[
X = (X_{i}) \quad (i = 1, j = 1, \ldots, n), \quad X_{i} = \gamma_{ijkl}^{0}(z)
\]

\[
A = (A_{ij}), \quad A_{ij} = S_{i}(o)
\]

A similar matrix formulation could be written for \( R_{ij}(o) \). Thus, in general, for \( L_{ij}(o) \) and \( R_{ij}(o) \), matrix equations can be written that relate these coefficients explicitly to the canonical harmonic components \( \gamma_{ijkl}^{0}(z) \). These matrix equations will be of the form

\[
L_{ij}X = A_{ij}X
\]

\[
R_{ij}X = A_{ij}X (n = 1, 2, \ldots, 5),
\]

where the matrices \( A_{ij}, A_{ij}, \) and \( X \) will be written in a form similar to that presented above for \( L_{ij}(o) \). The elements of these matrices will be integrals of scalar functions of depth, corresponding to the surface-wave displacement field for each layer. Eq. (3) are five linear relations of \( L_{ij}(o) \) and \( R_{ij}(o) \) to the canonical harmonic components \( \gamma_{ijkl}^{0}(z) \), which are constants for each layer. An inversion process to obtain \( \gamma_{ijkl}^{0}(z) \) from \( L_{ij}(o) \) and \( R_{ij}(o) \) can be performed by linear inversion (Aki and Richards, 1980; Tarantola, 1987).

### 2.3. Inversion process

Once the forward problem has been defined by (3), the inversion of (3) or equivalently the inversion of the matrix relation

\[
y = Ax\]

must be performed, where \( y = L_{ij} \) and \( A = A_{ij} \) (for a Love wave), \( y = R_{ij} \) and \( A = A_{ij} \) (for a Rayleigh wave), and \( X = X \) (for both Love and Rayleigh waves).

The inverse of the matrix \( A \) in relation (4) was calculated by Corchete and Badal (2004) using the inversion process called least-squares inversion. Thus, the inverse matrix \( B \) is obtained by

\[
B = (A^{T}A)^{-1}A^{T},
\]

but the matrix \( A^{T}A \) can be singular, or nearly singular, for small or zero eigenvalues (Tarantola, 1987). For this reason, a parameter \( \varepsilon \) is introduced to stabilize the solution of the inverse problem. The inverse matrix is written in the form

\[
B = (A^{T}A + \varepsilon I)^{-1}A^{T},
\]

where \( \varepsilon \) is the damping parameter or damping factor and \( I \) is the identity matrix. The addition of this factor in the main diagonal of the matrix \( A^{T}A \) increases the magnitude of its eigenvalues, avoiding the problems caused by small or zero eigenvalues that can make the matrix \( A^{T}A \) singular or nearly singular. Substituting the relation called the singular value decomposition (Dimri, 1992) of \( A \),

\[
A = U_{p}A_{p}V_{p}^{T},
\]

in formula (6), \( B \) can be written as

\[
b = V_{p}A_{p}(A_{p}^{2} + \varepsilon I)^{-1}U_{p},
\]

where the matrices \( U_{p}, A_{p}, V_{p} \), are described in detail by Aki and Richards (1980) and Dimri (1992). The matrix \( B \) defined by formula (7) is called the damped generalized inverse (Dimri, 1992). Then the linear equations given by (4) are inverted to obtain \( x \) by means of (7), having

\[
x = By,
\]

where it should be noted that

\[
x = BAx.
\]

The product of matrices \( B \) and \( A \) is called the resolution matrix and it is obtained through

\[
BA = V_{p}A_{p}^{2}(A_{p}^{2} + \varepsilon I)^{-1}V_{p}^{T},
\]

If the resolution matrix given by (9) is the identity matrix, the solution obtained, \( x \), is unique and the resolution is perfect. Nevertheless, it should be noted that the addition of the factor \( \varepsilon \) in the main diagonal means that the equality \( BA = I \) is never reached when the damping factor \( \varepsilon \) is not zero. Therefore, the particular solution obtained, \( X \), is never the true solution. The solution obtained is a smoothed solution, because each model parameter is a weighted mean of the parameters \( X \) that define the true model. For this reason, the factor \( \varepsilon \) must be as small as possible to obtain the best solution possible, i.e., the particular solution as near as possible to the true solution.

The error in the model parameters is given by the corresponding variance as (assuming that the data are statistically independent and
the same behavior for the model parameters)

\[ \sigma_i^2 = \sigma_0^2 V_p A_p^2 (\lambda_0^2 + 2I)^{-2} V_p^T \]  

(10)

where \( \sigma_0^2 \) and \( \sigma_i^2 \) are the variances in the model parameters and data (Dimri, 1992), respectively.

The inversion scheme proposed in this paper is given by (7) and it introduces the damping parameter, which degrades resolution but stabilizes the solution obtained by reduction of the variance, as can be seen in (9) and (10). Thus, an increase in the value of \( \alpha \) reduces the error in the determination of the model parameters, at the expense of resolution. The inversion scheme proposed by Corchete and Badal (2004) computes the inverse matrix using Eq. (5); then the particular solution obtained, \( \mathbf{X} \), is the true solution because \( \alpha \) is zero and \( \mathbf{B} \mathbf{A}^{-1} \mathbf{I} \), but the error can have very high values when small or zero eigenvalues are present in the matrix \( \mathbf{A}^T \mathbf{A} \), because this matrix can be singular or nearly singular. In practice, the inversion scheme given by (7) is more efficient than the scheme given by (5), because the introduction of the damping parameter stabilizes the inverse and reduces the variance, with a reasonable loss of resolution. On the other hand, the inverse given by (7) uses the method of singular value decomposition (SVD) instead of the least-squares method used in (5), the SVD method being preferable to have high computation accuracy (Keilis-Borok, 1989).

3. Computation process

3.1. Tensor definitions

The dominant isotropic properties of the medium will be given by the elastic isotropic tensor \( E_{ij}^0 (\rho) \), written as

\[ E_{ij}^0 (\rho) = \lambda (\rho) (\delta_{ij} - \frac{2}{3} \delta_{ij}) + \mu (\rho) (\delta_{ij} - \frac{2}{3} \delta_{ij}), \]

where \( \lambda \) and \( \mu \) are the Lamé constants

\[ \mu (\rho) = \rho^2 (\rho^2 - 2 \mu /\rho), \quad \lambda (\rho) = \rho^2 (\rho^2 - 2 \mu /\rho) - 2 \mu (\rho) \]

obtained through the density \( \rho \), the P-wave velocity \( v_p \), and the S-wave velocity \( v_s \) from an isotropic earth model as listed in Table 1.

The anisotropic properties of the medium will be given by the elastic anisotropic tensor \( E_{ijkl}^0 (\rho) \), written in the usual form of the matrix \( E_{ij}^0 \) given by (Voigt, 1928; Babuska and Cara, 1991)

\[ (E_{ij}) = \begin{pmatrix}
E_{1111} & E_{1112} & E_{1113} & E_{1112} & E_{1112} \\
E_{1222} & E_{1222} & E_{1223} & E_{1223} & E_{1223} \\
E_{2333} & E_{2333} & E_{2333} & E_{2333} & E_{2333} \\
E_{3333} & E_{3333} & E_{3333} & E_{3333} & E_{3333} \\
E_{4444} & E_{4444} & E_{4444} & E_{4444} & E_{4444} \\
\end{pmatrix},
\]

in which the indices \( i \) and \( j \) vary from 1 to 6. Table 2 gives \( E_{ijkl} (\rho) \) for the first and second examples considered in this study, respectively.

### Table 1

Isotropic earth model considered in this study for numerical computation.

<table>
<thead>
<tr>
<th>Thickness (km)</th>
<th>( \alpha ) (km/s)</th>
<th>( \beta ) (km/s)</th>
<th>( \rho ) (g/cm³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5.80</td>
<td>3.40</td>
<td>2.70</td>
</tr>
<tr>
<td>20</td>
<td>6.59</td>
<td>3.81</td>
<td>2.90</td>
</tr>
<tr>
<td>80</td>
<td>8.135</td>
<td>4.670</td>
<td>3.32</td>
</tr>
<tr>
<td>∞</td>
<td>9.00</td>
<td>5.40</td>
<td>3.60</td>
</tr>
</tbody>
</table>

Note: \( \alpha \): compressional seismic velocity; \( \beta \): shear velocity; \( \rho \): mass density.

### Table 2

Stiffness tensor components in the matrix notation \( E_{ij} \) for all cases considered in this study.

<table>
<thead>
<tr>
<th>Index</th>
<th>Case 1 (layer 2)</th>
<th>Case 1 (layer 3)</th>
<th>Case 2 (layer 2)</th>
<th>Case 2 (layer 3)</th>
<th>Case 3 (layer 2)</th>
<th>Case 3 (layer 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1111</td>
<td>5.1 ± 0.4</td>
<td>10.0 ± 0.6</td>
<td>5.0 ± 0.4</td>
<td>4.00 ± 0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1122</td>
<td>1.25 ± 0.21</td>
<td>5.0 ± 0.4</td>
<td>1.05 ± 0.11</td>
<td>12.0 ± 0.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1133</td>
<td>8.25 ± 0.19</td>
<td>10.0 ± 0.3</td>
<td>8.05 ± 0.05</td>
<td>6.0 ± 0.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2222</td>
<td>5.1 ± 0.4</td>
<td>10.0 ± 0.6</td>
<td>5.0 ± 0.1</td>
<td>12.0 ± 0.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2223</td>
<td>8.25 ± 0.19</td>
<td>10.0 ± 0.3</td>
<td>8.05 ± 0.05</td>
<td>10.0 ± 0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3333</td>
<td>–4.9 ± 0.6</td>
<td>–10 ± 1</td>
<td>–5.0 ± 0.1</td>
<td>–2.0 ± 0.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3333</td>
<td>–2.10 ± 0.14</td>
<td>–2.49 ± 0.23</td>
<td>–2.02 ± 0.04</td>
<td>–7.5 ± 0.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1212</td>
<td>–1.90 ± 0.15</td>
<td>2.51 ± 0.26</td>
<td>–2.01 ± 0.05</td>
<td>3.0 ± 0.4</td>
<td></td>
<td></td>
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<tr>
<td>2313</td>
<td>0</td>
<td>0</td>
<td>–3</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1112</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10.0 ± 0.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2212</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>0</td>
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<table>
<thead>
<tr>
<th>Index</th>
<th>Case 4 (layer 2)</th>
<th>Case 4 (layer 3)</th>
<th>Case 5 (layer 2)</th>
<th>Case 5 (layer 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1111</td>
<td>5.1 ± 0.4</td>
<td>10.0 ± 0.6</td>
<td>5.0 ± 0.4</td>
<td>4.00 ± 0.3</td>
</tr>
<tr>
<td>1122</td>
<td>1.25 ± 0.21</td>
<td>5.0 ± 0.4</td>
<td>1.05 ± 0.11</td>
<td>12.0 ± 0.4</td>
</tr>
<tr>
<td>1133</td>
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<td>10.0 ± 0.3</td>
<td>8.05 ± 0.05</td>
<td>6.0 ± 0.2</td>
</tr>
<tr>
<td>2222</td>
<td>5.1 ± 0.4</td>
<td>10.0 ± 0.6</td>
<td>5.0 ± 0.1</td>
<td>12.0 ± 0.4</td>
</tr>
<tr>
<td>2223</td>
<td>8.25 ± 0.19</td>
<td>10.0 ± 0.3</td>
<td>8.05 ± 0.05</td>
<td>10.0 ± 0.3</td>
</tr>
<tr>
<td>3333</td>
<td>–4.9 ± 0.6</td>
<td>–10 ± 1</td>
<td>–5.0 ± 0.1</td>
<td>–2.0 ± 0.1</td>
</tr>
<tr>
<td>3333</td>
<td>–2.10 ± 0.14</td>
<td>–2.49 ± 0.23</td>
<td>–2.02 ± 0.04</td>
<td>–7.5 ± 0.4</td>
</tr>
<tr>
<td>1212</td>
<td>–1.90 ± 0.15</td>
<td>2.51 ± 0.26</td>
<td>–2.01 ± 0.05</td>
<td>3.0 ± 0.4</td>
</tr>
<tr>
<td>2313</td>
<td>0</td>
<td>0</td>
<td>–3</td>
<td>0</td>
</tr>
<tr>
<td>1112</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10.0 ± 0.4</td>
</tr>
<tr>
<td>2212</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Index</th>
<th>Case 6 (0–4 Myr)</th>
<th>Case 6 (4–20 Myr)</th>
<th>Case 6 (20–52 Myr)</th>
<th>Case 6 (52–110 Myr)</th>
<th>Case 6 (110–Myr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1111</td>
<td>–19.2 ± 1.3</td>
<td>–19.7 ± 1.8</td>
<td>–29.2 ± 1.7</td>
<td>19.9 ± 1.8</td>
<td>9.9 ± 1.1</td>
</tr>
<tr>
<td>1122</td>
<td>–24.3 ± 1.6</td>
<td>–30.1 ± 1.7</td>
<td>–35.3 ± 1.5</td>
<td>–10.1 ± 1.3</td>
<td>–15.1 ± 1.6</td>
</tr>
<tr>
<td>1133</td>
<td>–2.5 ± 1.5</td>
<td>9.5 ± 1.3</td>
<td>–2.8 ± 1.3</td>
<td>29.5 ± 1.1</td>
<td>20.5 ± 1.2</td>
</tr>
<tr>
<td>2222</td>
<td>–19.2 ± 1.3</td>
<td>–19.7 ± 1.8</td>
<td>–29.2 ± 1.7</td>
<td>19.9 ± 1.8</td>
<td>9.9 ± 1.1</td>
</tr>
</tbody>
</table>
The canonical harmonic components $\gamma_{ijkl}(z)$ will be calculated by the inversion of (3). This inversion will be performed by (7) and (8); their $1/C_0$ errors $D_{ij}$ are calculated by (10).

### 3.2. Phase velocity computations

The azimuthal variation of surface-wave phase velocity will be obtained by means of the equation $c(\alpha,\beta) = c(\alpha) + \delta c(\alpha,\beta)$, where $c(\alpha)$ is computed from the elastic isotropic properties related to the values given in Table 1 (Abo-Zena, 1979; Kennett and Clarke, 1983). The term $\delta c(\alpha,\beta)$ will be computed from Eqs. (1) and (2), as described in the previous section. The effect of a slight anisotropy is associated with $\gamma_{ijkl}(z)$ or equivalently with $\gamma_{ijkl}(z)$, where the tensor $\gamma_{ijkl}(z)$ is computed by the difference $E_{ijkl}(\beta) - E_{ijkl}(\alpha)$.

### 3.3. Tensor computations

The canonical harmonic components $\gamma_{ijkl}(z)$ will be calculated by the inversion of (3). This inversion will be performed by (7) and the error will be calculated by (10). The elements of the elastic tensor $\gamma_{ijkl}(z)$ will be calculated from $\gamma_{ijkl}(z)$ through the relations existing between them (Backus, 1970; Smith and Dahlen, 1973). Finally, the elastic tensor $E(\alpha)$ of the anisotropic medium will be calculated from $\gamma_{ijkl}(z)$ and $E_{ijkl}(\omega)$, computing the sum $E_{ijkl}(\omega) = E_{ijkl}(\alpha) + \gamma_{ijkl}(z)$.

### 4. Testing the inversion method

In order to show the efficiency of the new inversion method proposed in this paper, this method is tested on four examples: hexagonal symmetry, 13 nonzero canonical harmonic components, and two experiments with observed data obtained in previous studies of the Pacific (Nishimura and Forsyth, 1985, 1988, 1989). The inversion of the azimuthal dependence of Love and Rayleigh wave dispersion will be performed.

4.1. Hexagonal symmetry

This kind of symmetry is considered for this example because several studies (Cara et al., 1984; Mindevall and Mitchell, 1989; Nishimura and Forsyth, 1989) have shown that realistic models are obtained assuming transverse isotropy (hexagonal symmetry), with the axis of symmetry oriented vertically. In this example, the earth model is obtained from the parameters given in Table 1 for $E_{ijkl}(\omega)$ and the parameters given in Table 2 for $E_{ijkl}(\omega)$. The surface-wave phase velocity $c(\alpha)$ and $\delta c(\alpha,\beta)$ are computed from $E_{ijkl}(\omega)$ and $E_{ijkl}(\beta)$, as described in the previous section. An uncertainty in $L_{ijkl}(\omega)$ and $R_{ijkl}(\omega)$ of approximately 1% is taken into account as error in the observed data, because this amount of error usually remains in the estimation of surface-wave dispersion data. This error in the $L_{ijkl}(\omega)$ and $R_{ijkl}(\omega)$ data is used to compute the error in the $\gamma_{ijkl}(z)$ parameters by means of Eq. (10). The value of $\gamma_{ijkl}(z)$ parameters is determined from the $L_{ijkl}(\omega)$ and $R_{ijkl}(\omega)$ data by means of (7) and (8). Finally, the elements of the elastic tensors $\gamma_{ijkl}(z)$ and $E_{ijkl}(\omega)$ are calculated from $\gamma_{ijkl}(z)$ and $E_{ijkl}(\omega)$, as described in the previous section. The values obtained for $\gamma_{ijkl}(z)$ in this example are shown in

<table>
<thead>
<tr>
<th>Index</th>
<th>Case 8 (0–80 Myr)</th>
<th>Case 8 (80+ Myr)</th>
<th>Case 9 (0–80 Myr)</th>
<th>Case 9 (80+ Myr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1111</td>
<td>−69 ± 1</td>
<td>8 ± 1</td>
<td>−68 ± 1</td>
<td>7 ± 1</td>
</tr>
<tr>
<td>1122</td>
<td>−81 ± 2</td>
<td>−25 ± 7</td>
<td>−80 ± 12</td>
<td>−25 ± 7</td>
</tr>
<tr>
<td>1133</td>
<td>−21 ± 3</td>
<td>9 ± 2</td>
<td>−20 ± 13</td>
<td>9 ± 1</td>
</tr>
<tr>
<td>2222</td>
<td>1 ± 1</td>
<td>−65 ± 11</td>
<td>0 ± 7</td>
<td>0 ± 7</td>
</tr>
<tr>
<td>2233</td>
<td>−19 ± 4</td>
<td>7 ± 1</td>
<td>−19 ± 13</td>
<td>7 ± 7</td>
</tr>
<tr>
<td>3333</td>
<td>3 ± 3</td>
<td>19 ± 2</td>
<td>2 ± 19</td>
<td>20 ± 8</td>
</tr>
<tr>
<td>3334</td>
<td>40 ± 1</td>
<td>3 ± 1</td>
<td>40 ± 8</td>
<td>1 ± 4</td>
</tr>
<tr>
<td>3334</td>
<td>41 ± 3</td>
<td>−3 ± 1</td>
<td>42 ± 8</td>
<td>−3 ± 4</td>
</tr>
<tr>
<td>1212</td>
<td>8 ± 2</td>
<td>15 ± 2</td>
<td>7 ± 7</td>
<td>14 ± 4</td>
</tr>
<tr>
<td>2313</td>
<td>1.2 ± 0.1</td>
<td>−0.9 ± 0.1</td>
<td>1.1 ± 0.6</td>
<td>−1.0 ± 0.3</td>
</tr>
<tr>
<td>1112</td>
<td>−1.2 ± 0.1</td>
<td>1.1 ± 0.1</td>
<td>−1.0 ± 0.4</td>
<td>1.1 ± 0.3</td>
</tr>
<tr>
<td>2222</td>
<td>−1.2 ± 0.1</td>
<td>1.1 ± 0.1</td>
<td>−1.0 ± 0.4</td>
<td>1.1 ± 0.3</td>
</tr>
<tr>
<td>3312</td>
<td>−1.8 ± 0.5</td>
<td>1.2 ± 0.3</td>
<td>−0.8 ± 0.8</td>
<td>1.1 ± 0.5</td>
</tr>
</tbody>
</table>

Notes: Units are in GPa. Case 1: first example; case 2: second example; case 3: perturbations of the stiffness tensor components $\gamma_{ij}$ (first example) and $1 − \sigma$ errors $\Delta \gamma_{ij}$; case 4: perturbations of the stiffness tensor components $\gamma_{ij}$ (second example) and $1 − \sigma$ errors $\Delta \gamma_{ij}$; case 5: perturbations of the stiffness tensor components $\gamma_{ij}$ (third example) and $1 − \sigma$ errors $\Delta \gamma_{ij}$; case 6: four experiments with observed data obtained for each age region by Corchete and Badal (2004); case 7 (layer 2): perturbations of the stiffness tensor components $\gamma_{ij}$ (second experiment) and $1 − \sigma$ errors $\Delta \gamma_{ij}$ obtained for each age region 0–80 and 80–200 Myr by Corchete and Badal (2004). In the cases 3, 5, 6, 8, the perturbations of the stiffness tensor components $\gamma_{ij}$ are obtained from the canonical harmonic components computed by (7) and (8); their $1 − \sigma$ errors $\Delta \gamma_{ij}$ are calculated by (10).
Table 2. In this table, it should be noted that $\gamma_{ijkl}(z)$ is obtained with a small error. In Table 2 are shown the values obtained by Corchete and Badal (2004) for the same example. Comparing Table 2, cases 3 and 4, it can be seen that the new method is more accurate than the previous one developed by Corchete and Badal (2004).

4.2. Thirteen nonzero canonical harmonic components

In this second example, the earth model is obtained from the parameters given in Table 1 for $E_{ijkl}(z)$ and the parameters given in Table 3 for $E_{ijkl}(z)$. The surface-wave phase velocity $c(o)$ and $\delta c(o,\theta)$ are computed from $E_{ijkl}(z)$ and $E_{ijkl}(z)$, as described in the previous section. Again, an uncertainty in $L_o(\omega)$ and $R_o(\omega)$ of approximately 1% is taken into account, as error in the observed data. This error in the $L_o(\omega)$ and $R_o(\omega)$ data is used to compute the error in the $\gamma_{ijkl}(z)$ parameters by means of (10). The value of $\gamma_{ijkl}(z)$ is determined from the $L_o(\omega)$ and $R_o(\omega)$ data by means of (7) and (8). Finally, the elements of the elastic tensors $\gamma_{ijkl}(z)$ and $E_{ijkl}(z)$ are calculated from $\gamma_{ijkl}(z)$ and $E_{ijkl}(z)$, as described in the previous section. The values obtained for $\gamma_{ijkl}(z)$ in this second example are shown in Table 2. Table 2, cases 3 and 5, shows the new method proposed in this paper as a very efficient method to invert the azimuthal dependence of the surface-wave dispersion, in order to calculate the slightly anisotropic structure of Earth. This method should be very useful, if sufficient high-quality dispersion data are collected and the isotropic properties of the medium are well known.

4.3. First experiment

The observed Love and Rayleigh wave dispersions determined in previous studies for the Pacific region (Nishimura and Forsyth, 1985, 1988, 1989) will be inverted in this example, as an additional reliability test of the new inversion method proposed in this paper. In this first experimental test, an anisotropic structure with hexagonal symmetry will be considered, for the inversion of the regionalized dispersion curves shown in Fig. 1, because no azimuthal dependence of surface-wave propagation is given in these dispersion data. Prior to the inversion process, a starting isotropic earth model is proposed for each age region considered, as listed in Table 3. The Love-wave theoretical dispersion curves obtained from this isotropic model are in clear discrepancy from the respective observed curves (Fig. 1). This fact is well known in regions of Earth, such as the Pacific, in which anisotropy is present. This phenomenon is called Love-Rayleigh discrepancy (Nishimura and Forsyth, 1983, 1988, 1989), because an isotropic earth model, which provides Rayleigh-wave theoretical dispersion with good agreement with the respective observed dispersion, is not able to fit the Love-wave observed dispersion. Then, Love and Rayleigh wave dispersion curves are not compatible with a unique isotropic model, and thus an anisotropic model must be considered. When an anisotropic model is taken into account, good agreement in both Love and Rayleigh theoretical dispersion curves and the respective observed curves is achieved (Fig. 1). This fact confirms that the final anisotropic model obtained is a valid earth model for each age region considered. The inversion process performed to achieve this fit is the same as described in the first example (hexagonal symmetry), but now the effect of a slight anisotropy is associated with the second elastic layer of the earth model. The values obtained for $\gamma_{ijkl}(z)$ in this example are shown in Table 2. It should be noted that the $\gamma_{ijkl}(z)$ parameters are obtained with an error smaller than that computed by Corchete and Badal (2004), which is listed in Table 2. Comparing Table 2, cases 6 and 7, it can be seen that the new method is more accurate than the previous one developed by Corchete and Badal (2004). Moreover, the results obtained by Nishimura and Forsyth (1989) agree better with the ones presented in this paper, than with the results obtained by Corchete and Badal (2004), as shown in Table 4.

<table>
<thead>
<tr>
<th>Layer (no.)</th>
<th>Thickness (km)</th>
<th>$c_o$ (km/s)</th>
<th>$\beta_o$ (km/s)</th>
<th>$\rho$ (g/cm$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>6.80</td>
<td>3.90</td>
<td>2.90</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>7.15</td>
<td>4.07</td>
<td>2.95</td>
</tr>
<tr>
<td>3</td>
<td>50</td>
<td>7.75</td>
<td>4.50</td>
<td>3.35</td>
</tr>
<tr>
<td>4</td>
<td>50</td>
<td>8.90</td>
<td>4.68</td>
<td>3.52</td>
</tr>
<tr>
<td>5</td>
<td>50</td>
<td>9.15</td>
<td>4.85</td>
<td>3.65</td>
</tr>
<tr>
<td>6</td>
<td>50</td>
<td>9.40</td>
<td>5.05</td>
<td>3.80</td>
</tr>
<tr>
<td>7</td>
<td>50</td>
<td>9.68</td>
<td>5.19</td>
<td>3.89</td>
</tr>
<tr>
<td>8</td>
<td>50</td>
<td>9.94</td>
<td>5.40</td>
<td>3.99</td>
</tr>
<tr>
<td>9</td>
<td>100</td>
<td>10.30</td>
<td>5.64</td>
<td>4.12</td>
</tr>
<tr>
<td>10</td>
<td>Infinite</td>
<td>10.79</td>
<td>5.95</td>
<td>4.32</td>
</tr>
</tbody>
</table>

**Table 3** Starting isotropic earth models proposed for each age region considered in this study.

- $c_o$: compressional seismic velocity; $\beta_o$: shear velocity; $\rho$: mass density.

Note: $\gamma_{ijkl}(z)$ parameters are obtained from the $\gamma_{ijkl}(z)$ parameters in Table 2.
4.4. Second experiment

In this second data test, the azimuthal variation of the surface-wave dispersion (Nishimura and Forsyth, 1989) will be inverted to determine an anisotropic model, which satisfies the Love and Rayleigh dispersion data jointly. The starting isotropic earth model, for each region considered, is listed in Table 5. In Fig. 2, good agreement between theoretical Rayleigh wave dispersion curves (obtained by forward modeling from the isotropic models listed in Table 5) and the corresponding observed curves is shown for both regions. Again, the Love-wave observed dispersion curves cannot be satisfied by an isotropic model, but good agreement in both Love and Rayleigh theoretical dispersion curves and the respective observed curves is achieved when an anisotropic model is taken into account. In this second experiment, an anisotropic

Table 4

Comparison of the $\beta_v$ parameter obtained from the new inversion method proposed in this paper with the values obtained by Corchete and Badal (2004) and Nishimura and Forsyth (1989).

<table>
<thead>
<tr>
<th>Age region (Myr)</th>
<th>Depth range (km)</th>
<th>$\beta_v^a$ (km/s)</th>
<th>$\beta_v^b$ (km/s)</th>
<th>$\beta_v^c$ (km/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–4</td>
<td>15–145</td>
<td>4.376 ± 0.058</td>
<td>4.472 ± 0.152</td>
<td>4.02–4.43</td>
</tr>
<tr>
<td>4–20</td>
<td>5–95</td>
<td>4.448 ± 0.041</td>
<td>4.743 ± 0.132</td>
<td>4.13–4.58</td>
</tr>
<tr>
<td>20–52</td>
<td>5–95</td>
<td>4.447 ± 0.083</td>
<td>5.040 ± 0.178</td>
<td>4.21–4.61</td>
</tr>
<tr>
<td>52–110</td>
<td>5–145</td>
<td>4.591 ± 0.035</td>
<td>4.192 ± 0.071</td>
<td>4.20–4.63</td>
</tr>
<tr>
<td>110+</td>
<td>20–95</td>
<td>4.590 ± 0.033</td>
<td>4.699 ± 0.123</td>
<td>4.43–4.63</td>
</tr>
</tbody>
</table>

*a* This paper.

*b* Corchete and Badal (2004).

c Nishimura and Forsyth (1989).
inversion of the azimuthal variation of the Rayleigh-wave dispersion, jointly to the Love-wave dispersion, will be performed for each age region considered. For it, 13 canonical harmonic components γ_{ijkl}^{(s)}(z) nonzero will be obtained, as a result of the computation for each age region. The stiffness tensor perturbations γ_{ijkl}^{(s)}(z) listed in Table 2 are calculated from γ_{ijkl}^{(s)}(z) and β_{ijkl}^{(s)}(z), as described in the previous section. It should be noted that γ_{ijkl}^{(s)}(z) parameters are obtained with a smaller error than that computed by Corchete and Badal (2004), which are listed in Table 2. Comparing Table 2, cases 8 and 9, it can be seen that the new method is more accurate than the previous one developed by Corchete and Badal (2004). Again, good agreement of both Love and Rayleigh theoretical dispersion curves with the respective observed curves is shown in Fig. 2. On the other hand, the theoretical values predicted by the anisotropic model obtained by Corchete and Badal (2004), are shown also in the lower part (dashed line).

\[ \gamma_{ijkl}^{(s)}(z) = \frac{1}{z^2} \times \text{stiffness tensor perturbations} \]

The theoretical values predicted by the anisotropic model obtained by Corchete and Badal (2004), are shown also in the lower part (dashed line).
other hand, the Rayleigh-wave azimuthal anisotropy coefficients are also satisfied by the anisotropic models obtained for each region, as shown in Fig. 3. In Figs. 2 and 3, it can be seen that the new inversion method, proposed in this paper gives a better fit to the observed data than the one developed by Corchete and Badal (2004).

5. Conclusions

The existence of a slight anisotropy over wide areas of the Earth justifies the inversion method of Corchete and Badal (2004), developed to study the surface-wave propagation in slightly anisotropic structures, being revisited to be improved. The inversion of surface-wave phase velocities in a slightly anisotropic medium performed by Corchete and Badal (2004) calculates the inverse matrix by means of the least-squares inverse. The new inversion method proposed in this paper computes the inverse matrix by means of the damped generalized inverse, because it is more efficient than the method previously performed by Corchete and Badal (2004). This efficiency is shown by computing four examples concerning different anisotropic structures that could be present in practice for some regions of Earth. Thus, much more realistic earth models could be obtained if the new method proposed in this paper is used.

Acknowledgments

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References


