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Improved Perturbation Bounds for the Matrix Exponential

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Abstract. In this paper we give asymptotic series expansions in $\varepsilon = \|E\|$ for the bound of the perturbation $\|\exp(t(A+E)) - \exp(tA)\|$ in the matrix exponential $\exp(tA)$.

1 Introduction

Bounds and perturbation bounds for the matrix exponential $\exp(tA)$, where A is a (complex) $n \times n$ matrix, have been proposed by B. Kagström [5] and C. Van Loan [11] in 1977, see also [4] and [2]. However, as shown in [6] and [10] these bounds give rather pessimistic results for some defective matrices. In particular an overestimation of the real perturbation of hundreds of orders of magnitude was observed for low order and well behaved systems $x' = Ax$.

In this paper we give improved perturbation bounds of the form

$$\Delta(t) = \|\exp(t(A+E)) - \exp(tA)\| \leq f(t,\varepsilon),\ \varepsilon = \|E\|$$

where E is a perturbation in A. For this purpose we use bounds for $\|\exp(tA)\|$ based on Schur and Jordan decompositions of A. After that a linear r-th order differential equation for f is derived, where r is the dimension of the dominant Jordan block of A. A study of this equation allows to obtain improved perturbation bounds which are often better than the known in the literature.

Asymptotic series expansions in ε (treated as a small parameter) are also given.

The above results are applicable to the development of condition and error estimates for the solution of linear and nonlinear differential equations.

We denote by $\|.\|$ the matrix 2-norm in $\mathcal{F}^{n.n}$, where $\mathcal{F}$ is the set of real numbers $\mathcal{R}$ or the set of complex numbers $\mathcal{C}$. The unit $n \times n$ matrix is denoted I_n and N_n is the nilpotent $n \times n$ matrix with unit elements at positions $(i, i+1)$ and zeros otherwise.

Throughout the paper A is a fixed $n \times n$ real or complex matrix with spectral abscissa $\alpha = \max\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{spect}(A)\}$, where $\mathrm{spect}(A)$ is the spectrum of A.

2 Problem Statement

The matrix exponential $\exp : \mathcal{F}^{n.n} \to \mathcal{F}^{n.n}$ defined by the power series

$$\exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!}$$

appears in the solution of linear differential equations, e.g.

$$\begin{aligned} Y'(t) &= AY(t) + Y(t)B + C(t) \\ Y(0) &= Y_0 \end{aligned}$$

where $Y(t) \in \mathcal{F}^{n.n}$:

$$Y(t) = \exp(tA)Y_0 \exp(tB) + \int_0^t \exp((t-s)A)C\exp((t-s)B)\mathrm{d}s \ .$$

In practice, the mathematical model of a real phenomenon is always contaminated with measurement errors. Also, when solving a numerical problem by a numerically stable method, the computed solution is near to the exact solution of a slightly perturbed problem. In all these situations one has to deal not with the "exact" value $\exp(tA)$ but rather with the perturbed matrix $\exp(t(A+E))$, where $E \in \mathcal{F}^{n.n}$ is the perturbation in the matrix A. Usually the inequality $||E||/||A|| \ll 1$ is fulfilled reflecting the fact that the perturbation is relatively small. Hence the problem arises to estimate the norm of the matrix

$$H(t,E) := \exp(t(A+E)) - \exp(tA)$$

as a function of the current time t and the quantity $||E||$. It is easy to show that

$$H(t,E) = \int_0^t \exp((t-s)A)E\exp(s(A+E))\mathrm{d}s \ .$$

Let

$$h(t,\varepsilon) := \max\{||H(t,E)|| : ||E|| \le \varepsilon\} \ .$$

Then our aim is to find an asymptotic bound of the form

$$h(t,\varepsilon) \le \sum_{i=1}^{\infty} \varepsilon^i h_i(t) \ .$$

The expression for $H(t,E)$ may be represented as a sum of terms $H_m(t,E)$ of order m in E:

$$H(t,E) = \sum_{m=1}^{\infty} H_m(t,E)$$

where

$$||H_m(t,E)|| = \mathrm{O}(||E||^m), \ ||E|| \to 0; \ m = 1,2,\ldots$$

We have

$$H_m(t,E) = \int_0^t \exp((t-s)A)G_m(s,E)\mathrm{d}s$$

where

$$G_m(s,E) := \sum_{r=m-1}^{\infty} \frac{s^r}{r!} \sum_{i_1+\cdots+i_m=r-m+1} \prod_{k=1}^{m} (EA^{i_k}) \quad .$$

In particular, we have

$$G_1(s,E) = E\sum_{r=0}^{\infty}\frac{s^r}{r!}A^r = E\exp(sA)$$

$$G_2(s,E) = E\sum_{r=1}^{\infty}\frac{s^r}{r!}\sum_{i+j=r-1} A^iEA^j$$

$$G_3(s,E) = E\sum_{r=2}^{\infty}\frac{s^r}{r!}\sum_{i+j+k=r-2} A^iEA^jEA^k \quad .$$

Note that $H_1(1,.)$,

$$H_1(1,E) = \int_0^1 \exp((1-s)A)E\exp(sA)\mathrm{d}s,$$

is the Frechet derivative of the function $X \mapsto \exp(X)$ at the point $X = A$, see also [9].

3 Estimates for the Matrix Exponential

When finding perturbation bounds for the matrix exponential, some bounds for the norm of the exponential $\|\exp(tA)\|$ itself are usually used [5, 11, 8].

Several estimates of the form

$$\|\exp(tA)\| \leq C(\beta)\exp(t(\alpha+\beta))$$

are known, where β may be chosen arbitrarily from certain interval $(0,b)$, and $C(\beta)$ is a certain expression such that $C(\beta) \to \infty$ as $\beta \to 0$. These estimates lead to immediate perturbation bounds for the matrix exponential; the latter, however, are often too pessimistic if A is defective. That is why we shall use the more sophisticated bounds based on the Schur and Jordan canonical forms of A. To make all the results comparable, we assume that A has a single $n \times n$ Jordan block J with an eigenvalue λ with $\alpha = \mathrm{Re}(\lambda)$. This is not a restrictive assumption since the general case may be reduced to this particular case if we consider only the dominant Jordan block of A corresponding to the eigenvalue λ of A with $\mathrm{Re}(\lambda) = \alpha$.

Denote by c_A the minimum condition number of the transformation matrix $T \in \mathcal{F}^{n.n}$ reducing A into its Jordan normal form $J = T^{-1}AT = \lambda I_n + N_n$:

$$c_A = \min\{\|T\|\,\|T^{-1}\| : T^{-1}AT = J\}$$

(note that such c_A exists, see [7].) Then

$$\|\exp(tA)\| = \|T\exp(tJ)T^{-1}\| \le c_A\|\exp(tJ)\|$$

and since

$$\|\exp(tJ)\| = \left\|\exp(\lambda t)\sum_{k=0}^{n-1} N_n^k \frac{t^k}{k!}\right\| \le \exp(\alpha t)\sum_{k=0}^{n-1}\frac{t^k}{k!}$$

we have

$$\|\exp(tA)\| \le c_A\exp(\alpha t)\sum_{k=0}^{n-1}\frac{t^k}{k!} \quad . \tag{1}$$

Consider now the Schur form $S = U^H AU = \lambda I_n + N$ of A, where $U \in \mathcal{F}^{n.n}$ is an unitary matrix and N is a strictly upper triangular matrix. Denote

$$\nu_A = \min\left\{\|N\| : U^H U = I_n,\ U^H AU = \lambda I_n + N\right\} \quad .$$

Then we have

$$\begin{aligned}\|\exp(tA)\| = \|U\exp(tS)U^H\| = \|\exp(tS)\| = \|\exp(\lambda t)\exp(Nt)\| = \\ = |\exp(\lambda t)|\|\exp(Nt)\| = \exp(\alpha t)\|\exp(Nt)\| \quad .\end{aligned}$$

Hence

$$\|\exp(tA)\| = \exp(\alpha t)\left\|\sum_{k=0}^{n-1}\frac{(Nt)^k}{k!}\right\| \le \exp(\alpha t)\sum_{k=0}^{n-1}\frac{(\nu_A t)^k}{k!} \quad . \tag{2}$$

Relations (1) and (2) may be written in an unified manner as

$$\|\exp(tA)\| \le e(t) = b\exp(\alpha t)\omega(t),\ \omega(t) := \sum_{k=0}^{n-1}\frac{(\beta t)^k}{k!}$$

where

Table 1.

	b	β
Jordan	c_A	1
Schur	1	ν_A

Denote $F = E/\varepsilon$. Since

$$H'(t,E) = AH(t,E) + \varepsilon F(H(t,E) + \exp(tA))$$
$$H(0,E) = 0$$

we may express H as

$$H(t,E) = \varepsilon \int_0^t \exp((t-s)A)F(H(s,E) + \exp(sA))\mathrm{d}s \ .$$

Hence

$$h(t,\varepsilon) \le \varepsilon \int_0^t e(t-s)(h(s,\varepsilon) + e(s))\mathrm{d}s \ .$$

Thus

$$h(t,\varepsilon) \le u(t)$$

where u is the solution to the majorant Volterra integral equation

$$u(t) = \varepsilon \int_0^t e(t-s)(u(s) + e(s))\mathrm{d}s \ .$$

Setting

$$u(t) = b\exp(\alpha t)z(t) - e(t) = b\exp(\alpha t)(z(t) - \omega(t))$$

we get

$$z(t) = \omega(t) + \mu \int_0^t \omega(t-s)z(s)\mathrm{d}s, \ \mu := \varepsilon b \ . \tag{3}$$

The solution of (3) may be represented as a convergent power series

$$z(t) = \sum_{r=0}^{\infty} \mu^r z_r(t)$$

where

$$z_0(t) = \omega(t)$$
$$z_r(t) = \int_0^t \omega(t-s)z_{r-1}(s)\mathrm{d}s, \ r \ge 1 \ .$$

In particular, for $r = 1$ we have

$$z_1(t) = \int_0^t \omega(t-s)\omega(s)\mathrm{d}s$$

and the norm of the Frechet derivative is estimated from

$$\|H(1,E)\| \le \mu b \exp(\alpha) \left(\sum_{k=0}^{n-1} \frac{\beta^k}{k!} + \sum_{k=n}^{2n-2} \frac{2n-1-k}{(k+1)!}\beta^k \right) \ .$$

Another way to solve (3) is via a reduction to an n-th order linear differential equation. Indeed, differentiating both sides of (3) n times we get the initial value problem

$$z^{(n)}(t) = \mu \sum_{i=0}^{n-1} \beta^{n-1-i} z^{(i)}(t)$$
$$z^{(k)}(0) = b(\beta + \varepsilon b)^k;\ k = 0, 1, \ldots, n-1\ .$$

Setting $\tau = \beta t$, $\nu = \mu/\beta$ and $z(\tau/\beta) = y(\tau)$ we obtain (the differentiation is now in τ)

$$y^{(n)}(\tau) = \nu \sum_{k=0}^{n-1} y^{(k)}(\tau) \tag{4}$$
$$y^{(k)}(0) = (1+\nu)^k;\ k = 0, 1, \ldots, n-1\ .$$

The solution of the initial value problem (4) may be represented as

$$y = y_n(\tau) = \sum_{s=0}^{\infty} \nu^s y_{n,s}(\tau)$$

where

$$y_{n,s}(\tau) = \sum_{k=s}^{(s+1)n-1} c_k(n,s) \frac{t^k}{k!}\ .$$

Here

$$c_k(n,s) = \binom{s+1}{k-s}_n$$

and $\binom{s}{i}_n$ are the so called n-nomial coefficients defined from

$$\left(1 + x + \cdots + x^{n-1}\right)^s = \sum_{i=0}^{(n-1)s} \binom{s}{i}_n x^i\ .$$

The coefficients $c_k(n,s)$ satisfy the recurrence relation [1, 3]

$$c_k(n,s) = \sum_{i=0}^{n-1} c_{k-i}(n, s-1),\ c_s(n,s) = 1\ .$$

For $n = 2$ the coefficients $\binom{s}{i}_2$ are equal to the binomial coefficients $\binom{s}{i}$.

4 Examples

Example 1. To illustrate the effectiveness of the estimate proposed, consider the problem of estimating the perturbation in the matrix exponential, where we choose the matrices $A,\ E \in \mathcal{R}^{2.2}$ as

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, E = \begin{bmatrix} 0 & 0 \\ 10^{-4} & 0 \end{bmatrix}\ .$$

The results are shown at the table bellow, where the second column contains the exact perturbed quantity, est1 is the estimate based on the exact solution of the differential equation for z, est2 is the asymptotic bound using the dominant term in the solution for z and est3 is the bound proposed in [4].

Table 2.

t	$\frac{\Vert\exp(t(A+E))-\exp(tA)\Vert}{\Vert\exp(tA)\Vert}$	est1	est2	est3
1	0.763×10^{-4}	$0.606 \times 10^{+2}$	$0.300 \times 10^{+2}$	0.400×10^{-3}
10	0.180×10^{-2}	$0.878 \times 10^{+1}$	$0.438 \times 10^{+1}$	0.122
20	0.681×10^{-2}	$0.406 \times 10^{+1}$	$0.200 \times 10^{+1}$	0.920
30	0.152×10^{-1}	$0.246 \times 10^{+1}$	$0.122 \times 10^{+1}$	$0.316 \times 10^{+1}$
40	0.270×10^{-1}	$0.169 \times 10^{+1}$	0.839	$0.792 \times 10^{+1}$
50	0.423×10^{-1}	$0.125 \times 10^{+1}$	0.628	$0.168 \times 10^{+2}$
100	0.175	0.553	0.349	$0.280 \times 10^{+3}$

Example 2. This example is similar to Example 1 where A is a 3×3 Jordan block with an eigenvalue -1 and E has a single nonzero entry 10^{-4} in position (3,1). The results are as follows.

Table 3.

t	$\frac{\Vert\exp(t(A+E))-\exp(tA)\Vert}{\Vert\exp(tA)\Vert}$	est1	est2	est3
1	0.684×10^{-4}	$0.341 \times 10^{+3}$	$0.120 \times 10^{+3}$	0.625×10^{-3}
10	0.204×10^{-2}	$0.135 \times 10^{+2}$	$0.593 \times 10^{+1}$	$0.396 \times 10^{+1}$
20	0.141×10^{-1}	$0.312 \times 10^{+1}$	$0.181 \times 10^{+1}$	$0.152 \times 10^{+3}$
30	0.464×10^{-1}	$0.120 \times 10^{+1}$	$0.100 \times 10^{+1}$	$0.294 \times 10^{+4}$
40	0.110	0.700	0.804	$0.818 \times 10^{+5}$
50	0.218	0.681	0.849	$0.566 \times 10^{+7}$
100	$0.222 \times 10^{+1}$	$0.417 \times 10^{+1}$	$0.379 \times 10^{+1}$	$0.370 \times 10^{+27}$

The proposed estimates are asymptotically better than this from [4]. In fact, we have est3/est1 $\to \infty$ as $t \to \infty$. The examples show that our estimates are better even for moderate values of t.

The proposed perturbation bounds require more computational effort compared to those in [4]. However, both approaches involve the preliminary computation (or estimation) of either Jordan or Schur form of A. The extra amount of computations required by our approach is due to the need to find the coefficients $c_k(n,s)$ which, for a given order n of A, may be done in advance.

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