Linear and nonlinear degenerate boundary value problems in Besov spaces

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\textbf{A R T I C L E I N F O}

\textbf{Article history:}
Received 15 March 2008
Accepted 23 April 2008

\textbf{Keywords:}
Boundary value problems
Differential-operator equations
Banach-valued Besov spaces
Operator-valued multipliers
Interpolation of Banach spaces

\textbf{A B S T R A C T}

The boundary value problems for linear and nonlinear degenerate differential-operator equations in Banach-valued Besov spaces are studied. Several conditions for the separability of linear elliptic problems are given. Moreover, the positivity and the analytic semigroup properties of associated differential operators are obtained. By using these results, the maximal regularity of degenerate boundary value problems for nonlinear differential-operator equations is derived. As applications, boundary value problems for infinite systems of degenerate equations in Besov spaces are studied.

\textbf{1. Introduction, notations and background}

Boundary value problems (BVPs) for differential-operator equations (DOEs) have been studied extensively by many researchers (see [1–23] and the references therein). The main objective of the present paper is to discuss boundary value problems for linear and nonlinear degenerate differential-operator equations in Banach-valued function spaces. The maximal regularity of the linear and nonlinear degenerate BVPs for DOEs in Banach-valued Besov spaces is established. These results are then applied to boundary value problems for degenerate elliptic, quasi-elliptic partial differential equations, and their finite or infinite systems on cylindrical domains.

Let $E$ be a Banach space and $\gamma = \gamma(x), x = (x_1, x_2, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$. $L_{p, \gamma}(\Omega; E)$ denotes the space of all strongly measurable $E$-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$
\|f\|_{L_{p, \gamma}(\Omega; E)} = \left( \int_{\Omega} \|f(x)\|^p_E \gamma(x) \, dx \right)^{1/p}, \quad 1 \leq p < \infty
$$

$$
\|f\|_{L_{\infty, \gamma}(\Omega; E)} = \operatorname{ess\, sup}_{x \in \Omega} \|f(x)\|_E \gamma(x).
$$

For $\gamma(x) \equiv 1$ we shall denote $L_{p, \gamma}(\Omega; E)$ by $L_p(\Omega; E)$.

The weights $\gamma$ are said to satisfy an $A_p$ condition; i.e., $\gamma \in A_p$, $1 < p < \infty$ if there is a constant $C$ such that

$$
\left( \frac{1}{|Q|} \int_Q \gamma(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \gamma^{-\frac{1}{p-1}}(x) \, dx \right)^{p-1} \leq C,
$$

for all rectangles $Q \subset \mathbb{R}^n$.

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0895-7177/$–$ see front matter. Published by Elsevier Ltd
Let $E$ be a Banach space and $x = (x_1, x_2, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$ and $L_p(\Omega; E)$ denote the space of all strongly measurable $E$-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_{L_p(\Omega; E)} = \left( \int \|f(x)\|_E^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty.$$ 

Let $S (\mathbb{R}^n; E)$ denote a Schwartz class i.e. a space of $E$-valued rapidly decreasing smooth functions $\varphi$ on $\mathbb{R}^n$ and $S' (\mathbb{R}^n; E)$ denote an $E$-valued tempered distribution. Let $y \in R, m \in N$ and $e_i, \ i = 1, 2, \ldots, n$ be standard unique vectors in $\mathbb{R}^n$. Let

$$\Delta_i (y) f(x) = f(x + ye_i) - f(x), \ldots,$$

$$\Delta_i^m (y) f(x) = \Delta_i (y) [\Delta_i^{m-1} (y) f(x)] = \sum_{k=0}^m (-1)^{m+k} C_k^m f(x + kye_i).$$

Let

$$\Delta_i (\Omega, y) = \left\{ \begin{array}{ll}
\Delta_i (y) f(x), & \text{for } [x, x + mye_i] \subset \Omega \\
0, & \text{for } [x, x + mye_i] \notin \Omega
\end{array} \right\}.$$

Let $m$ be an integer, $s$ be a positive number, and

$$m > s, \quad 1 \leq p \leq \infty, \quad 1 \leq q < \infty, \quad y_0 > 0.$$

Let $B^s_{p,q} (\Omega; E)$ be an $E$-valued Besov space, i.e.,

$$B^s_{p,q} (\Omega; E) = \left\{ f : f \in L_p (\Omega; E) \right\},$$

with where

$$\|f\|_{B^s_{p,q}(\Omega; E)} = \|f\|^p_{p,q} = \|f\|_{L_p(\Omega; E)} + \left( \int \sum_{i=1}^n \left( \int_0^{y_0} \|\Delta_i^m (y, \Omega) f(x)\|_{L_{p,q}(\Omega; E)} \, dy \right) \right)^{1/q} < \infty,$$

$$\|f\|_{B^s_{p,q}(\Omega; E)} = \left( \|f\|_p^p + \|A^\theta u\|_p^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad -\infty < \theta < \infty.$$

Let $C$ be the set of complex numbers, and

$$S_\varphi = \{ \lambda ; \lambda \in C, |\arg \lambda| \leq \varphi \} \cup \{0\}, \quad 0 \leq \varphi < \pi.$$

A linear operator $A$ is said to be $\varphi$-positive in a Banach space $E$, with bound $M > 0$ if $D (A)$ is dense on $E$, and

$$\| (A + \lambda I)^{-1} \|_{L(E)} \leq M (1 + |\lambda|)^{-1}$$

with $\lambda \in S_\varphi, \varphi \in [0, \pi), I$ is the identity operator in $E, L (E)$ is the space of bounded linear operators in $E$. Sometimes $A + \lambda I$ will be written as $A + \lambda$ and denoted by $A_\lambda$. It is known [24, Section 1.15.1] that there exist fractional powers $A^\varphi$ of the positive operator $A$. Let $E (A^\varphi)$ denote the space $D (A^\varphi)$ with graphical norm

$$\|u\|_{E(A^\varphi)} = \left( \|u\|_p^p + \|A^\theta u\|_p^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad -\infty < \theta < \infty.$$

Let $E_1$ and $E_2$ be two Banach spaces. By $(E_1, E_2)_{p,q}, 0 < \theta < 1, \ 1 \leq p \leq \infty$ we denote interpolation spaces defined by real method (see e.g. [24, Sections 1.3. and 1.8.], [25]).

A Banach space $E$ is said to be the $UMD$ space (see [26]) if the Hilbert operator

$$(Hf) (x) = \lim_{\varepsilon \to 0} \int_{|x - y| < \varepsilon} \frac{f(y)}{x - y} \, dy$$

is bounded in $L_p (R; E), \ p \in (1, \infty).$ $UMD$ spaces include e.g. $L_p, L_p$ spaces and Lorentz spaces $L_{pq}, p, q \in (1, \infty)$.

Let $E_0$ be the space $E_0$ continuously and densely embedded into $E$ and $l$ be an integer and $(a, b) \subset R = (-\infty, \infty)$. Let $\Omega \subset \mathbb{R}^n, D_k^s = \frac{\partial^s}{\partial x_k^s}, 0 < s < \infty$. Let us introduce a space $B_{p,q}^s (\Omega; E_0)$ (sometimes we call it Sobolev–Besov–Lions type space) of all functions $u \in B_{p,q}^s (\Omega; E_0)$ such that they have generalized derivatives $D_k^s u \in \mathbb{B}_{p,q}^s (\Omega; E), k = 1, 2, \ldots, n$ with norm

$$\|u\|_{B_{p,q}^s (\Omega; E_0)} = \|u\|_{B_{p,q}^s (\Omega; E_0)} + \sum_{k=1}^n \|D_k^s u\|_{S_{p,q}^s (\Omega; E)} < \infty.$$

By $C (\Omega; E)$ and $C^{(m)} (\Omega; E)$ we will denote spaces of $E$-valued bounded all continuous and $m$-times continuously differentiable functions on $\Omega$, respectively. Let $F$ denote the Fourier transform and $S (R^n; E)$ denote a Schwartz class i.e. the
space of all $E$-valued rapidly decreasing smooth functions on $\mathbb{R}^n$. Let the weighted function $\gamma$ be such that $S (\mathbb{R}^n; E)$ is dense in $B_{\infty,0}^s (\mathbb{R}^n; E_1)$. Let $E_1$ and $E_2$ be two Banach spaces. A function $\Psi \in C^0 (\mathbb{R}^n; L(E_1, E_2))$ is called a multiplier from $B_{p,0,\gamma}^s (\mathbb{R}^n; E_1)$ to $B_{p,0,\gamma}^s (\mathbb{R}^n; E_2)$ for $p \in (1, \infty)$ and $q \in [1, \infty]$, $s \in (0, \infty)$ if the map $u \to Fu = F^{-1} \Psi (\xi) Fu$, $u \in S (\mathbb{R}^n; E_1)$ is well defined and extends to a bounded linear operator

$$K : B_{p,0,\gamma}^s (\mathbb{R}^n; E_1) \to B_{p,0,\gamma}^s (\mathbb{R}^n; E_2).$$

The set of all multipliers from $B_{p,0,\gamma}^s (\mathbb{R}^n; E_1)$ to $B_{p,0,\gamma}^s (\mathbb{R}^n; E_2)$ will be denoted by $M_{p,0,\gamma}^q (s, E_1, E_2)$. For $E_1 = E_2 = E$ it will be denoted by $M_{p,0,\gamma}^q (E, E)$. Let

$$H_k = \{ \Psi_h \in M_{p,0,\gamma}^q (E_1, E_2), h \in W \}$$

be a collection of multipliers in $M_{p,0,\gamma}^q (E_1, E_2)$. We say that $\Psi_h = \Psi_h (\xi)$ is a uniform collection of multipliers with respect to $h$ if there exists a constant $C > 0$, independent of $h \in W$, such that

$$\| F^{-1} \Psi_h Fu \|_{B_{p,0,\gamma}^s (\mathbb{R}^n, E_2)} \leq C \| u \|_{B_{p,0,\gamma}^s (\mathbb{R}^n, E_1)}$$

for all $h \in W$ and $u \in S (\mathbb{R}^n; E_1)$.

The exposition the theory of $L_p$-multipliers of the Fourier transformation, and some related references, can be found in [24, Sections 2.2.1–2.2.4]. On the other hand, in vector-valued function spaces, Fourier multipliers have been studied e.g. in [20, 27–31].

Let

$$U = \{ \beta, \beta = (\beta_1, \beta_2, \ldots, \beta_n), \beta_j \in \{0, 1\}, j = 1, 2, \ldots, n \}.$$

**Definition 1.** A Banach space $E$ satisfies a $B$-multiplier condition with respect to $p \in (1, \infty)$, $q \in [1, \infty]$ and $s$, if for any $\Psi \in C^0 (\mathbb{R}^n; L(E))$ the estimate

$$(1 + |\xi|^\beta) \| D^\beta \Psi (\xi) \|_{L(E)} \leq C,$$ 

$\xi \in \mathbb{R}^n \setminus 0, \beta \in U$

implies that $\Psi$ is a Fourier multiplier, i.e. $\Psi \in M_{p,0,\gamma}^q (s, E)$. 

**Remark 1.** By virtue of [1] or [29] all UMD spaces satisfy the $B$-multiplier condition for $\gamma (x) \equiv 1$.

Let $l$ be an integer $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), D^\alpha = D^\alpha_1 D^\alpha_2 \cdots D^\alpha_n, s > 0, \alpha = 1 - \frac{|\alpha|}{T}$. Using similar techniques as in [17] we obtain:

**Theorem A.** Let the following conditions be satisfied:

1. $E$ is a Banach space satisfying the $B$-multiplier condition with respect to $p, q, s$ and weighted function $\gamma$;
2. $0 \leq \mu \leq 1 - \alpha, p \in (1, \infty), q \in [1, \infty]$;
3. $A$ is a $\varphi$-positive operator in $E$ for $0 < \varphi \leq \pi, 0 < h \leq h_0 < \infty$;
4. $\Omega \in \mathbb{R}^n$ is a region such that there exists a bounded linear extension operator from $B_{p,0,\gamma}^{1-s} (\Omega; E (A) , E)$ to $B_{p,0,\gamma}^{1-s} (\mathbb{R}^n; E (A) , E)$.

Then the embedding

$$D^\alpha \| \varphi \|_{B_{p,0,\gamma}^{1-s} (\Omega; E (A) , E)} \subset B_{p,0,\gamma}^{1-s} (\Omega; E (A^{1-\alpha-\mu})$$

is continuous and there exists a positive constant $C_\mu$ such that

$$\| D^\alpha u \|_{p,0,\gamma}^{1-s} (\Omega; E (A^{1-\alpha-\mu})) \leq C_\mu \left[ h^\mu \| u \|_{p,0,\gamma}^{1-s} (\Omega; E (A) , H) + + h^{(1-\mu)} \| u \|_{p,0,\gamma}^{1-s} (\Omega; E) \right]$$

for all $u \in B_{p,0,\gamma}^{1-s} (\Omega; E (A) , E)$.

**Proof.** First the theorem is proved for the case of $\Omega = \mathbb{R}^n$. Really, it is clear to see that the inequality (1) for $\Omega = \mathbb{R}^n$ is equivalent to

$$\| F^{-1} (i\xi)^\alpha A^{1-\alpha-\mu} \hat{u} \|_{p,0,\gamma}^{1-s} (\mathbb{R}^n, E) \leq C_\mu \left[ h^\mu \left( \| F^{-1} \hat{A} \hat{u} \|_{p,0,\gamma}^{1-s} (\mathbb{R}^n, E) + \sum_{k=1}^n \| F^{-1} \left( (i\xi_k)^\alpha \hat{u} \right) \|_{p,0,\gamma}^{1-s} (\mathbb{R}^n, E) \right) + h^{(1-\mu)} \| F^{-1} \hat{u} \|_{p,0,\gamma}^{1-s} (\mathbb{R}^n, E) \right].$$
It is sufficient to show that the operator-function
\[ \Psi (\xi) = \Psi_{h,\mu} (\xi) = (i\xi)^\alpha A^{1-x-\mu} h^{-\mu} \left[ A + \sum_{k=1}^n |\xi_k|^2 + h^{-1} \right]^{-1} \]
is a multiplier in \( B_{p,q,y}^s (\mathbb{R}^n; E) \). Really, by using [17, Lemma 1] we have the uniform estimate
\[ |\xi|^\sigma \| \mathcal{D}^\delta \Psi (\xi) \|_{L(E)} \leq C, \quad \beta \in U, \quad \sigma = 0, 1, 2, \ldots, |\beta| . \]
This estimate implies
\[ (1 + |\xi|)^{\beta} \| \mathcal{D}^\delta \Psi (\xi) \|_{L(E)} \leq C, \quad \beta \in U. \]
Therefore, by Definition 1, \( \Psi (\xi) \) is the multiplier in \( B_{p,q,y}^s (\mathbb{R}^n; E) \). Thus, Theorem A1 for the case of \( \Omega = \mathbb{R}^n \) is proved.

Let \( P \) be a bounded linear extension operator from \( B_{p,q,y}^s (\Omega; E (A)) \) to \( B_{p,q,y}^s (\mathbb{R}^n; E (A)) \). Let \( P_{\Omega} \) be a restriction operator from \( \mathbb{R}^n \) to \( \Omega \). Then for any \( u \in B_{p,q,y}^s (\Omega; E (A)) \) we have
\[ \| \mathcal{D}^\delta u \|_{B_{p,q,y}^s (\Omega; E (A))} \leq C |\mu| \| \mathcal{D}^\delta u \|_{B_{p,q,y}^s (\mathbb{R}^n; E (A))}. \]

**Theorem A2.** Suppose all conditions of the Theorem A1 are satisfied. Then the embedding
\[ D^\delta B_{p,q,y}^s (\Omega; E (A)) \subset B_{p,q,y}^s (\Omega; E (A)). \]
for \( 0 < \mu < 1 - \alpha \) is continuous and there exists a positive constant \( C_\mu \) such that
\[ \| \mathcal{D}^\delta u \|_{B_{p,q,y}^s (\Omega; E (A))} \leq C_\mu \| u \|_{B_{p,q,y}^s (\Omega; E)}. \]

**Proof.** First let us show the theorem for the case \( \Omega = \mathbb{R}^n \). For \( u \in B_{p,q,y}^s (\mathbb{R}^n; E (A)) \) it is sufficient to prove the estimate
\[ \| \mathcal{D}^\delta u \|_{B_{p,q,y}^s (\mathbb{R}^n; E (A))} \leq C_\mu \| u \|_{B_{p,q,y}^s (\mathbb{R}^n; E)}. \]
By definition of interpolation spaces \( E (A) \) \( E (A), E \) (see [24, Section 1.14.5]) the estimate (3) is equivalent to
\[ \| F^{-1} y^{1-x-\mu-\frac{1}{\beta}} \left[ A^{x+\mu} (A + y)^{-1} \right] \xi^\alpha u \|_{B_{p,q,y}^s (\mathbb{R}^n; H^1)} \leq C_\mu C_\mu \| F^{-1} \left[ h^{\alpha} \left( A + \sum_{k=1}^n |\xi_k|^2 + h^{-1} \right) \right] u \|_{B_{p,q,y}^s (\mathbb{R}^n; E)}. \]
The inequality (4) will follow immediately if we can prove that the operator-function
\[ \Psi_{h,\mu} = (i\xi)^\alpha y^{1-x-\mu-\frac{1}{\beta}} \left[ A^{x+\mu} (A + y)^{-1} \right] h^\alpha \left( A + \sum_{k=1}^n |\xi_k|^2 + h^{-1} \right) \]
is the multiplier from \( B_{p,q,y}^s (\mathbb{R}^n; E) \) to \( B_{p,q,y}^s (\mathbb{R}^n; L_p (R_n; E)) \). This fact is proved by the same manner as in Theorem A1.

Therefore, we get (4). Then by using the extension operator as in Theorem A1, we obtain the estimate (3). We note that for \( \gamma \equiv 1 \), \( E (A) = E \) and for \( E = C \) we obtain the results in [32,33], respectively.

From [15] we have:

**Theorem A3.** Let \( E \) be a Banach space, \( A \) be a \( \varphi \)-positive operator in \( E \) with bound \( M \), \( 0 \leq \varphi < \pi \). Let \( m \) be a positive integer, \( 1 < p < \infty \) and \( \alpha \in \left( \frac{1+\varphi}{2p}, \frac{1+\varphi}{2p} + m \right), 0 \leq \varphi < 2p \min |\alpha, m| - 1 \). Then for \( \lambda \in S_\varphi \), an operator \( -A^\frac{1}{\alpha} \) generates a semigroup \( e^{-x A^\frac{1}{\alpha}} \) which is holomorphic for \( x > 0 \). Moreover, there exists a positive constant \( C \) (depending only on \( M, \varphi, m, \alpha \) and \( p \)) such that for every \( u \in E (E (A^m)) \) and \( \lambda \in S_\varphi \),
\[ \int_0^\infty \| A^\frac{1}{\alpha} e^{-x A^\frac{1}{\alpha}} u \|_{L_p (E (A^m))} x^\gamma dx \leq C \left( \| u \|_{L_p (E (A^m))} + \frac{1+\varphi}{2p} \| u \|_{E} \right). \]
Theorem A₁. Let $E$ be a Banach space and $A$ be a positive operator in $E$ of type $\varphi$ with bound $M$. Moreover, let $m$ be a positive integer, $\gamma(x) = x^\alpha$, $0 \leq \gamma < 2p [\min \{\alpha, m\}] - 1$, $1 < p < \infty$, $1 \leq q < \infty$, $s_1 = s_2 = \ldots = s_n = s$, $0 < s < 1$ and

$$\frac{1 + \gamma}{2p} < \alpha < m + \frac{1 + \gamma}{2p}, \quad \eta = \frac{\alpha}{m} - \frac{1 + \gamma}{2pm}.$$  

Then for $\lambda \in S(\varphi)$ the operator $-A_x^\lambda$ generates a semigroup $e^{-A_x^\lambda t}$ which is holomorphic for $x > 0$ and strongly continuous for $x \geq 0$. Moreover, there exists a constant $C > 0$ (depending only on $M$, $\varphi$, $m$, $\alpha$ and $p$) such that

$$\|A_x^\lambda e^{-A_x^\lambda t} u\|_{B^p_q(0, b; E)} \leq C \left( \|u\|_{(E, E(A^m))_{n,p}} + |\lambda|^{mn} \|u\|_E \right)$$  

for every $u \in (E, E(A^m))_{n,p}$ and $\lambda \in S(\varphi)$.

Proof. By the relation

$$\Delta_i (y) f(x) = f(x + ye_i) - f(x), \ldots, \Delta_i^m (y) f(x) = \Delta_i (y) \left[ \Delta_i^{m-1} (y) f(x) \right]$$

we have

$$\Delta_i^m (h) A_x^\lambda e^{-A_x^\lambda t} u \mid_{B^p_q(0, b; E)} \leq \sum_{j=0}^m C_j^m \| \Delta_i (h) A_x^\lambda e^{-A_x^\lambda t} u \|_{B^p_q(0, b; E)}.$$  

(6)

Moreover, by the definition of space $B^p_q(0, b; E)$ we obtain from (6)

$$\| \Delta (h) A_x^\lambda e^{-A_x^\lambda t} u \|_{B^p_q(0, b; E)} = \left( \int_0^{h_0} \left\| h^{-s-\frac{1}{2}} \Delta (h) A_x^\lambda e^{-A_x^\lambda t} u \right\|_{L^p_q(0, b; E)}^q dh \right)^{\frac{1}{q}}.$$  

Now by the definition of $\Delta (h)$ and semigroup properties we obtain from the above

$$\| \Delta (h) A_x^\lambda e^{-A_x^\lambda t} u \|_{B^p_q(0, b; E)} \leq \left( \int_0^{h_0} h^{-sl-1} \left\| e^{-hA_x^\lambda t} - I \right\|_{B(E)}^q A_x^\lambda e^{-A_x^\lambda t} u \|_{L^p_q(0, b; E)}^q dh \right)^{\frac{1}{q}}.$$  

(7)

By virtue of semigroup properties we have

$$e^{-A_x^\lambda t} - I \|_{B(E)} \leq C \| A_x^\lambda e^{-hA_x^\lambda t} \|_{B(E)}, \quad \| A_x^\lambda e^{-hA_x^\lambda t} \|_{B(E)} \leq C.$$  

Therefore we obtain from the above estimates

$$\| A_x^\lambda e^{-A_x^\lambda t} u \|_{B^p_q(0, b; E)} \leq C \left( \int_0^{h_0} h^{-sl-1} A_x^\lambda e^{-A_x^\lambda t} u \|_{L^p_q(0, b; E)}^q dh \right)^{\frac{1}{q}}.$$  

Then by virtue of the estimate (7) and Theorem A₃ we obtain the estimate (5). □

Proposition A₁. Let the following conditions be satisfied:

1. $0 < s < 1$, $l$ is a positive integer and $1 < p < \infty$, $1 \leq q < \infty$, $\gamma(x) = x^\alpha$, $0 \leq \gamma < p(1 + s) - 2$;
\( \omega \) \((\omega_j)\)

By reasoning as in Section 18.10 we obtain:

**Proposition A2.** Let the following conditions be satisfied:

1. \( 0 < s < 1 \), \( I \) is a positive integer and \( 1 < p < \infty \), \( 1 \leq q \leq \infty \), \( \gamma \) (\( x = x' \), \( 0 \leq \gamma < p(1+s) - 2 \));

2. \( \theta_j = \frac{(j+1)^{\gamma}}{p^{1+\gamma}}, \quad \theta_j = \frac{(j+1)^{\gamma}}{p^{1+\gamma}} \), \( 0 \leq j \leq I - 1 \), \( 0 < h_0 < \infty \).

Then, transformations \( u \to u^{(0)}(0) \) are bounded linear from \( B^{s,\gamma}_{p,q} (0, b; E) \) to \( (E_0, E)_{\theta_j,q} \) and for \( u \in B^{s,\gamma}_{p,q} (0, b; E) \) the following inequalities hold

\[ \|u^{(0)}(0)\| \leq C \left( \|u\|_{B^{s,\gamma}_{p,q}(0,b;E)} + \|u\|_{B^{s,\gamma}_{p,q}(0,b;E_0)} \right), \]

\[ \|u^{(0)}(x_0)\| \leq C \left( \|u^{(0)}\|_{B^{s,\gamma}_{p,q}(0,b;E)} + \|u\|_{B^{s,\gamma}_{p,q}(0,b;E_0)} \right), \quad x_0 \neq 0. \]

Let \( A \) be a positive operator in \( E \). Consider the following differential-operator equation

\[ Lu = u^{(m)}(x) + \sum_{k=1}^{m} a_k A^k u^{(m-k)}(x) = f(x), \quad x \in (0, b). \]  

Let \( \omega_1, \omega_2, \ldots, \omega_m \) be the roots of the equation

\[ \omega^m + a_1 \omega^{m-1} + \cdots + a_m = 0 \]  

and

\[ \omega_m = \min \left\{ \arg \omega_j : j = 1, \ldots, v; \ \arg \omega_j + \pi j = v + 1, \ldots, m \right\}, \]

\[ \omega_M = \max \left\{ \arg \omega_j : j = 1, \ldots, v; \ \arg \omega_j + \pi j = v + 1, \ldots, m \right\}. \]

A system of numbers \( \omega_1, \omega_2, \ldots, \omega_m \) is said to be \( v \)-separated if there exists a straight line \( P \) passing through 0 such that no value of the numbers \( \omega_j \) lies on it, and \( \omega_1, \omega_2, \ldots, \omega_v \) are on one side of \( P \) while \( \omega_{v+1}, \ldots, \omega_m \) are on the other.

By reasoning as in [7, Lemma 5.3.2/1] and by using Theorem A1 we obtain:

**Lemma A1.** Let the following conditions be satisfied:

1. \( \gamma = x' \), \( 0 \leq \gamma < 1 - \frac{1}{p} \), \( p \in (1, \infty) \), \( a_m \neq 0 \) and the roots \( \omega_j \) of Eq. (9) are \( v \)-separated;

2. \( E \) is a Banach space satisfying the \( B \)-multiplier condition with respect to \( p, s \) and weighted function \( \gamma \);

3. \( A \) is a closed operator in a Banach space \( E \) with a dense domain \( D(A) \) and

\[ \| (A - \lambda I)^{-1} \| \leq C |\lambda|^{-1}, \quad -\frac{\pi}{2} - \omega_M \leq \arg \lambda \leq \frac{\pi}{2} - \omega_m, \quad |\lambda| \to \infty. \]

Then for \( u(x) \) to be a solution of Eq. (8), belonging to \( B^{s,\gamma}_{p,q} (0, b; E(A), E) \), it is necessary that

\[ u = \sum_{k=1}^{v} e^{-\omega_k A^{\frac{1}{p}}} g_k + \sum_{k=v+1}^{m} e^{-(b-x)\omega_k A^{\frac{1}{p}}} g_k, \]

where

\[ g_k \in (E(A), E)_{\frac{1}{mp-p}}, \quad k = 1, 2, \ldots, v, \quad g_k \in (E(A), E)_{\frac{1}{mp-p}}, \quad k = v + 1, v + 2, \ldots, m. \]
2. Statement of the problem

Let us consider a non-local boundary value problem for a differential-operator equation

\[
Lu = -u^{[2]}(x) + Au(x) + B_1(x)u^{[1]}(x) + B_2(x)u(x) = f(x), \tag{10}
\]

\[
L_1u = \alpha_1u^{[m_1]}(0) + \sum_{j=1}^{N_1} \delta_{ij}u^{[m_1]}(x_j) = f_1, \tag{11}
\]

\[
L_2u = \alpha_2u^{[m_2]}(1) + \sum_{j=1}^{N_2} \delta_{ij}u^{[m_2]}(x_j) = f_2
\]

where

\[
f_1 \in (E(A), E)_{\theta_1, p}, \quad f_2 \in (E(A), E)_{\theta_2, p}, \quad \theta_1 = \frac{m_1}{2} + \frac{1}{2p(1 - \gamma)}, \quad \theta_2 = \frac{m_1}{2} + \frac{1}{2p},
\]

are complex numbers and \(x_j \in (0, 1)\), \(A\) and \(B_k(x)\) for \(x \in [0, 1]\), are possible unbounded operators in \(E\).

The functions belonging to space

\[
B_{p,q,y}^{[2],i}(0, 1; E(A), E) = \left\{ u : u \in B_{p,q}^i(0, 1; E(A)), \ u^{[2]} \in B_{p,q}^i(0, 1; E), \right\}
\]

\[
\|u\|_{B_{p,q,y}^{[2],i}(0, 1; E(A), E)} = \|Au\|_{B_{p,q}^i(0, 1; E)} + \|u^{[2]}\|_{B_{p,q}^i(0, 1; E)} < \infty
\]

and satisfying Eq. (10) a.e. on \((0, 1)\) are said to be the solutions of Eq. (10) on \((0, 1)\).

Let

\[
B_{p,q,y}^{[2],i}(0, 1; E(A), E, L_k) = \left\{ u : u \in B_{p,q,y}^{[2],i}(0, 1; E(A), E), L_ku = 0, k = 1, 2 \right\}.
\]

Remark 2. Under the substitution

\[
y = (1 - \gamma)^{-1}x^{1 - \gamma} \tag{12}
\]

the spaces \(B_{p,q}^i(0, 1; E)\) and \(B_{p,q,y}^{[2],i}(0, 1; E(A), E)\) are mapped isomorphically onto the weighted spaces \(B_{p,q,y_1}^i(0, b; E)\) and \(B_{p,q,y_1}^{[2],i}(0, b; E(A), E)\), respectively, where

\[
b = \frac{1}{1 - \gamma}, \quad y_1 = y_1(y) = (1 - \gamma)^{\frac{1}{1 - \gamma}} y^{\frac{1}{1 - \gamma}}.
\]

Under the substitution (12) the problem (10) and (11) transforms into the following non-degenerate problem

\[
Lu = -u^{[2]}(x) + Au(x) + B_1(x)u^{[1]}(x) + B_2(x)u(x) = f(x), \tag{13}
\]

\[
L_1u = \alpha_1u^{[m_1]}(0) + \sum_{j=1}^{N_1} \delta_{ij}u^{[m_1]}(x_j) = f_1, \tag{14}
\]

\[
L_2u = \alpha_2u^{[m_2]}(1) + \sum_{j=1}^{N_2} \delta_{ij}u^{[m_2]}(x_j) = f_2
\]

in the weighted space \(B_{p,q,y_1}^i(0, b; E)\), where after the substitution the new variable, weighted function \(y_1\) and \(B_k(x(y))\) will be denoted by \(x, \gamma\) and \(B_k(x)\), respectively.

3. Homogeneous equations

Let us first consider the following BVP

\[
(L_0 + \lambda)u = -u^{[2]}(x) + (A + \lambda)u(x) = 0 \tag{15}
\]

\[
L_1u = \alpha_1u^{[m_1]}(0) + \sum_{j=1}^{N_1} \delta_{ij}u^{[m_1]}(x_j) = f_1, \tag{16}
\]

\[
L_2u = \alpha_2u^{[m_2]}(1) + \sum_{j=1}^{N_2} \delta_{ij}u^{[m_2]}(x_j) = f_2
\]
where $\lambda$ is a complex parameter, $m_k \in \{0, 1\}$; $\alpha_k$, $\delta_{kj}$ are complex numbers, $A$ is a possible unbounded operator in a Banach space $E$.

**Theorem 1.** Let $A$ be $\varphi$-positive operator in a Banach space $E$ for $0 < \varphi \leq \pi, 1 < p < \infty, 1 \leq q < \infty$ and let

$0 < s < 1, \quad \alpha_k \neq 0, \quad k = 1, 2, 0 < \gamma < 1 - \frac{1}{p}.$

Then problem (15) and (16) for $f_k \in (E(A), E)_{0k, p}$ and $|\arg \lambda| \leq \varphi$, with sufficiently large $|\lambda|$ has a unique solution that belongs to the space $B_{p, q, \gamma}^2(0, 1; E(A), E)$ and the coercive estimate

$$\sum_{i=0}^{2} |\lambda_i|^{1-\frac{2}{p}} \|u_i\|^p_{B_{p, q}^2(0, 1; E)} + \|Au\|^p_{B_{p, q}^2(0, 1; E)} \leq M \sum_{k=1}^{2} \left( \|f_k\|_{\|E(A), E\|_{0k, p}} + |\lambda_i|^{1-\frac{1}{p}} \|f_k\| \right)$$

holds for the solution of problem (15) and (16).

**Proof.** Under substitution (12) the problem (15) and (16) is transformed into the non-degenerate problem

\[ (L_0 + \lambda) u = -u''(x) + (A + \lambda) u(x) = 0 \]  
\[ L_1 u = \alpha_1 u^{(m_1)}(0) + \sum_{j=1}^{N_1} \delta_{1j} u^{(m_1)}(x_{1j}) = f_1, \]  
\[ L_2 u = \alpha_2 u^{(m_2)}(1) + \sum_{j=1}^{N_2} \delta_{2j} u^{(m_2)}(x_{2j}) = f_2 \]

in the weighted space $B_{p, q, \gamma}^2(0, 1; E)$. By virtue of [6, Lemma 2.6] and by Lemma A1, there exists a holomorphic for $x > 0$ and strongly continuous for $x \geq 0$ semigroup $e^{-\lambda t}$ and arbitrary solution of Eq. (18) for $|\arg \lambda| \leq \varphi$, belonging to space $B_{p, q, \gamma}^2(0, 1; E(A), E)$ has the form

\[ u(x) = e^{-\lambda x} g_1 + e^{-(b-x)\lambda} g_2, \]

where

\[ A_\lambda = A + \lambda, \quad g_1 \in (E(A), E)_{1+\frac{\varphi}{2p}, p}, \quad g_2 \in (E(A), E)_{1+\frac{\varphi}{2p}, p}. \]

Now taking into account boundary conditions (19) we obtain the algebraic linear equations with respect to $g_1, g_2$:

\[ (-1)^{m_k} e^{-b\lambda} A^-_{2\lambda} \left\{ \alpha_1 e^{-b\lambda} g_1 + \sum_{j=1}^{N_1} \delta_{1j} e^{-b\lambda} g_2 \right\} + \sum_{j=1}^{N_2} \delta_{2j} e^{-(b-x)\lambda} g_2 = f_k, \]

using the properties of positive operators and holomorphic semigroups [6, Lemma 2.6] it is clear to see that for $|\arg \lambda| \leq \varphi$, $|\lambda| \rightarrow \infty$ the operator determinant $D(\lambda)$ of (21) is bounded. So, by using a similar way as in [31] we obtain that there exists the solution of the systems of operator-equation (21) and the solution of the BVP (18) and (19) can be expressed in the form

\[ u(x) = D^{-1}(\lambda) \left\{ d_1 \left[ \alpha_2 e^{-(x+b)\lambda} g_1 + \sum_{j=1}^{N_1} \delta_{1j} e^{-(x+b-x_{1j})\lambda} g_2 \right] + d_2 \left[ \alpha_2 e^{-(x+b)\lambda} g_1 + \sum_{j=1}^{N_1} \delta_{1j} e^{-(x+b-x_{1j})\lambda} g_2 \right] \right\} \]

where $d_k$ are some complex numbers. By virtue of properties of holomorphic semigroups and in view of uniform boundedness of the operator $D^{-1}(\lambda)$, for $|\arg \lambda| \leq \varphi$ and $|\lambda|$ sufficiently large, we obtain from (22)

\[ \sum_{i=0}^{2} |\lambda_i|^{1-\frac{2}{p}} \|u_i\|^p_{B_{p, q}^2(0, 1; E)} + \|Au\|^p_{B_{p, q}^2(0, 1; E)} \leq C \|Q(\lambda)\| \]
Then by virtue of Theorem A4 and Remark 2 we obtain from (23) the estimate (17). □

4. Non-homogeneous equations

Now consider the following non-local boundary value problems for the non-homogeneous equations on the region (0, 1):

\[(L_0 + \lambda) u = -u''(x) + (A + \lambda) u(x) = f(x), \quad x \in (0, 1),\]

\[L_1 u = \alpha_1 u^{[m_1]}(0) + \sum_{j=1}^{N_1} \delta_{1j} u^{[m_1]}(x_{1j}) = f_1,\]

\[L_2 u = \alpha_2 u^{[m_2]}(1) + \sum_{j=1}^{N_2} \delta_{2j} u^{[m_2]}(x_{2j}) = f_2,\]

where \(x_{ij} \in (0, 1)\) and \(\alpha_i, \delta_{ij}\) are complex numbers and \(\lambda\) is a complex parameter and \(A\) is a possible unbounded operator in \(E\).

**Theorem 2.** Let all conditions of Theorem 1 be satisfied. Let \(E\) be a Banach space satisfying the \(B\)-multiplier condition with respect to \(p, s\) and weighted function \(\gamma = x^{-\frac{\gamma_1}{\gamma_2}}\). Then the operator \(u \mapsto [(L_0 + \lambda) u, L_1 u, L_2 u]\) for \(|\arg \lambda| \leq \varphi, 0 < \varphi \leq \pi\) and sufficiently large \(|\lambda|\), is an isomorphism from \(B_{p,q,\gamma}^{2,4}(0, 1; E (A), E (A))\) onto \(B_{p,q,\gamma}^{2,3}(0, 1; E (A), E (A))\). Moreover, the coercive estimate

\[
\sum_{j=0}^{2} |\lambda|^{1-\frac{j}{2}} \|u^{[j]}\|_{B_{p,q,\gamma}^{p,q}(0,1)} + \|Au\|_{B_{p,q,\gamma}^{p,q}(0,1)} \leq C \left[ \|f\|_{B_{p,q,\gamma}^{p,q}(0,1)} + \sum_{k=1}^{2} \|f_k\|_{(E(A),E)_{p,q,\gamma}} + |\lambda|^{1-k} \|\tilde{f}\| \right]
\]

holds for the solution of problem (24) and (25).

**Proof.** Under substitution (12) the problem (24) and (25) is transformed into a non-degenerate problem

\[(L_0 + \lambda) u = -u''(x) + (A + \lambda) u(x) = f(x), \quad x \in (0, b),\]

\[L_1 u = \alpha_1 u^{[m_1]}(0) + \sum_{j=1}^{N_1} \delta_{1j} u^{[m_1]}(x_{1j}) = f_1,\]

\[L_2 u = \alpha_2 u^{[m_2]}(1) + \sum_{j=1}^{N_2} \delta_{2j} u^{[m_2]}(x_{2j}) = f_2\]

in the weighted space \(B_{p,q,\gamma}^{2,4}(0, b; E)\). We have proved the uniqueness of the solution of problem (27) and (28) in Theorem 1. Let us define

\[
\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in [0, b] \\ 0 & \text{if } x \not\in [0, b] \end{cases}
\]

We now show that the solution of the problem (27) and (28) which belongs to space \(B_{p,q,\gamma}^{2,4}(0, b; E (A))\) can be represented as a sum \(\upsilon(x) = u_1(x) + u_2(x)\), where \(u_1\) is the restriction on \((0, b)\) of the solution \(\tilde{u}\) of the equation

\[(L_0 + \lambda) u = \tilde{f}(x), \quad x \in R = (-\infty, \infty),\]

\(u_2\) is the solution of the problem

\[(L_0 + \lambda) u = 0, \quad L_k u = f_k - L_0 u_1.\]

The solution of Eq. (29) is given by the formula

\[
u(x) = F^{-1} L_0^{-1} (\lambda, \xi) F \tilde{f}(x) = \frac{1}{2\pi} \int e^{i\xi x} L_0^{-1} (\lambda, \xi) \left( F \tilde{f}(x) \right)(\xi) \, d\xi,
\]

\[(31)\]
where $\hat{F}(\lambda)$ is the Fourier transform of the function $\hat{f}(\lambda)$, and $L_0(\lambda, \xi)$ is an operator-valued symbol of Eq. (29) i.e.

$$L_0(\lambda, \xi) = \xi^2 + \lambda + A.$$

It follows from (31)

$$\sum_{j=0}^{2} |\lambda|^{-\frac{1}{2}} \|u_j\|_{B^\prime_{p,q;\gamma} (R; E)} + \|Au\|_{B^\prime_{p,q;\gamma} (R; E)} = \sum_{j=0}^{2} |\lambda|^{-\frac{1}{2}} \|F^{-1} L_0^{-1}(\lambda, \xi) \hat{F}(\lambda) \|_{B^\prime_{p,q;\gamma} (R; E)}$$

$$+ \|F^{-1} AL_0^{-1}(\lambda, \xi) \hat{F}(\lambda) \|_{B^\prime_{p,q;\gamma} (R; E)}.$$

Let us show that operator-functions

$$\psi_{\lambda}(\xi) = AL_0^{-1}(\lambda, \xi), \quad \psi_{\lambda,j}(\xi) = |\lambda|^{-\frac{1}{2}} \xi^j L_0^{-1}(\lambda, \xi), \quad j = 0, 1, 2$$

are uniform collections of Fourier multipliers in $B^\prime_{p,q;\gamma} (0; b; E)$. For $|\arg \lambda| \leq \varphi$, and $\xi \in (-\infty, \infty)$ we get $(\lambda + \xi^2) \in S(\varphi)$. Therefore, we have

$$\|L_0^{-1}(\lambda, i\xi)\| = \|[A + \lambda + \xi^2]^{-1}\|$$

$$\leq C (1 + |\lambda + \xi^2|)^{-1} \leq C |\xi|^{-2} |\lambda| \|L_0^{-1}(\lambda, i\xi)\|$$

$$= |\lambda| \|[A + \lambda + \xi^2]^{-1}\| \leq C |\lambda| [1 + |\lambda + \xi^2|]^{-1} \leq C$$

$$\|AL_0^{-1}(\lambda, i\xi)\| = \|A[A + \lambda + \xi^2]^{-1}\| \leq C.$$

From (33) for $|\arg \lambda| \leq \varphi$ and $\xi \in (-\infty, \infty)$ we obtain

$$\|\psi_{\lambda}(\xi)\|_{B(E)} \leq C, \quad \|\psi_{\lambda,j}(\xi)\|_{B(E)} \leq C, \quad j = 0, 1, 2.$$

By using (34) we get the following estimates

$$\left| \frac{d^i}{d\xi^i} \psi_{\lambda}(\xi) \right| \leq C |\xi|^{-i}, \quad \left| \frac{d^i}{d\xi^i} \psi_{\lambda,j}(\xi) \right| \leq C |\xi|^{-i}, \quad i = 1, 2, j = 0, 1, 2.$$

Therefore, by virtue of the estimate (35) and Definition 1 we obtain that the operator-functions $\psi_{\lambda}(\xi), \psi_{\lambda,j}(\xi)$ are multipliers in $B^\prime_{p,q;\gamma} (R; E)$ . Then, by using the equality (32) for $|\arg \lambda| \leq \varphi$ we get

$$\sum_{j=0}^{2} |\lambda|^{-\frac{1}{2}} \|u_j^0\|_{B^\prime_{p,q;\gamma} (R; E)} + \|Au\|_{B^\prime_{p,q;\gamma} (R; E)} \leq C \|f\|_{B^\prime_{p,q;\gamma} (R; E)}.$$

It implies that $u_1 \in B^2_{p,q;\gamma} (0; b; E (A), E)$. By virtue of Proposition A1 we get

$$u^{(\alpha k)}(\cdot) \in (E (A); E)_{\alpha_k, p}, \quad k = 1, 2.$$

Hence, $L_0 u_1 \in (E (A), E)_{\alpha_k, p}$. Thus by virtue of Theorem 1, problem (30) has a unique solution $u_2(\chi)$ that belongs to space $B^2_{p,q;\gamma} (0; b; E (A), E)$ and we have

$$\sum_{j=0}^{2} |\lambda|^{-\frac{1}{2}} \|u_2^0\|_{B^\prime_{p,q;\gamma} (R; E)} + \|Au_2\|_{B^\prime_{p,q;\gamma} (R; E)} \leq C \left[ \sum_{k=1}^{2} \left( \|f_k - L_0 u_1\|_{(E (A), E)_{\alpha_k, p}} + |\lambda|^{-\theta_k} \|f_k - L_0 u_1\|_E \right) \right]$$

$$\leq C \left[ \sum_{k=1}^{2} \left( \|f_k\|_{(E (A), E)_{\alpha_k, p}} + |\lambda|^{-\theta_k} \|f_k\|_E + |\lambda|^{-\theta_k} \|L_0 u_1\|_E \right) \right]$$

$$+ \|u_1^{(\alpha k)}\|_{C([0, 1]; (E (A), E)_{\alpha_k, p})} + |\lambda|^{-\theta_k} \|u\|_{C([0, 1]; E)}.$$

From (36) for $|\arg \lambda| \leq \varphi$ we obtain the estimate

$$\sum_{j=0}^{2} |\lambda|^{-\frac{1}{2}} \|u_1^0\|_{B^\prime_{p,q;\gamma} (R; E)} + \|Au_1\|_{B^\prime_{p,q;\gamma} (R; E)} \leq C \|f\|_{B^\prime_{p,q;\gamma} (R; E)}.$$
Therefore, by Proposition A1 and by virtue of estimate (38) we have
\[
\left\| u_1^{(m_k)} (.) \right\|_{(E(A), E)_{b_k, p}} \leq C \left\| u_1 \right\|_{p,q',y}^{2, s} (0, b; E) \leq C \left\| f \right\|_{p,q',y} (0, b; E) , \quad k = 1, 2.
\] (39)

By virtue of Proposition A2 for \( \mu \in C, u \in B^{2, s}_{p,q',y} (0, b; E) \)
\[
\left| \mu \right|^{2-m_k} \left\| u^{(m_k)} (.) \right\|_{E} \leq C \left[ \left| \mu \right|^{\frac{3}{2}} \left\| u \right\|_{p,q',y}^{2, s} (R, E) + \left| \mu \right|^{2+\frac{1}{2}} \left\| u \right\|_{p,q',y} (R, E) \right].
\] (40)

Dividing by \( \left| \mu \right|^{\frac{1}{2}} \) and substituting \( \mu = \mu^2 \) for \( \lambda \in C, u \in B^{2, s}_{p,q',y} (0, b; E) \), we have
\[
\left| \lambda \right|^{1-\theta_k} \left\| u^{(m_k)} (.) \right\|_{E} \leq C \left[ \left\| u \right\|_{p,q',y}^{2, s} (0, b; E) + \left| \lambda \right| \left\| u \right\|_{p,q',y} (0, b; E) \right].
\] (41)

From (39), (40) and (41) we obtain the estimate
\[
\left| \lambda \right|^{1-\theta_k} \left\| u_1^{(m_k)} (.) \right\|_{E} \leq C \left[ \left\| u \right\|_{p,q',y}^{2, s} (0, b; E) + \left| \lambda \right| \left\| u \right\|_{p,q',y} (0, b; E) \right] \leq C \left\| f \right\|_{p,q',y}.
\] (42)

Hence from estimates (42), (38) and (37) for \( |\arg \lambda| \leq \varphi, |\lambda| \to \infty \) we have
\[
\sum_{j=0}^{2} \left| \lambda \right|^{1-\frac{j}{2}} \left\| u_2^{(j)} \right\|_{p,q,y}^{2, s} + \left\| Au_2 \right\|_{p,q,y} \leq C \left( \left\| f \right\|_{p,q,y} + \sum_{k=1}^{2} \left( \left\| f_k \right\|_{(E(A), E)_{b_k, p}} + \left| \lambda \right|^{1-\theta_k} \left\| f_k \right\| \right) \right). \] (43)

Then estimates (38) and (43) and Remark 2 imply (26).

Consider the problem (24) and (25) for \( f_k = 0, k = 1, 2, \) i.e.
\[
(L_0 + \lambda) u = -u^{[2]} (x) + (A + \lambda) u (x) = f (x), \quad x \in (0, 1), \quad \lambda \in (\varphi, \pi) \quad \text{and} \quad \| u^{[2]} \|_{p,q,y} \leq \varphi,
\] (44)
\[
L_1 u = \alpha_1 u^{[1]} (0) + \sum_{j=1}^{N_1} \delta_{1j} u^{[1]} (x_{1j}) = 0,
\] (45)
\[
L_2 u = \alpha_2 u^{[1]} (1) + \sum_{j=1}^{N_2} \delta_{2j} u^{[1]} (x_{2j}) = 0.
\]

Let \( G \) denote the operator in \( F = B^{s}_{p,q} (0, 1; E) \) generated by BVP (44) and (45) i.e.
\[
D (G) = B^{2, s}_{p,q,y} (0, 1; E (A), E, L_k), \quad Gu = -u^{[2]} + Au. \quad \Box
\]

**Result 1.** Theorem 2 implies that the differential operator \( G \) has a resolvent operator \( (G + \lambda)^{-1} \) for \( |\arg \lambda| \leq \varphi, \) and the following estimate holds
\[
\sum_{k=0}^{2} \left| \lambda \right|^{1-\frac{k}{2}} \left\| D^{[k]} (G + \lambda)^{-1} \right\|_{L(F)} + \left\| A (G + \lambda)^{-1} \right\|_{L(F)} \leq C.
\]

**Result 2.** Theorem 2 implies that the operator \( G \) is separable in \( B^{s}_{p,q} (0, 1; E) \), i.e. for all \( f \in B^{s}_{p,q} (0, 1; E) \) all terms of Eq. (44) also are from \( B^{s}_{p,q} (0, 1; E) \) and the operator \( G \) is bounded from \( B^{s}_{p,q} (0, 1; E) \) to \( B^{2, s}_{p,q,y} (0, 1; E (A), E) \). Hence, for all \( u \in B^{2, s}_{p,q,y} (0, 1; E (A), E, L_k) \) there are positive constants \( C_1 \) and \( C_2 \) so that
\[
C_1 \left\| Gu \right\|_{p,q,y} (0, 1; E) \leq \left\| u \right\|_{p,q,y} (0, 1; E (A), E) \leq C_2 \left\| Gu \right\|_{p,q,y} (0, 1; E).
\]

**Remark 3.** Theorem 2 implies that operator \( G \) is positive in \( B^{s}_{p,q} (0, 1; E) \). Then by virtue of [24, Section 1.14.5], for \( \varphi \in \left( \frac{\pi}{2}, \pi \right) \) we obtain that the operator \( G \) is a generator of an analytic semigroup in \( B^{s}_{p,q} (0, 1; E) \).

Consider the problem (10) and (11).
Theorem 3. Assume all conditions of Theorem 2 are satisfied and for any \( \varepsilon > 0 \) there is \( C(\varepsilon) > 0 \) such that for almost all \( x \in [0, 1] \)
\[
\|B_1(x)u\| \leq \varepsilon \|u\|_{(E(A), E)^{1+\frac{1}{2\phi(1-\gamma)}}} + C(\varepsilon) \|u\|, \quad u \in D(A),
\]
\[
\|B_2(x)u\| \leq \varepsilon \|Au\| + C(\varepsilon) \|u\|, \quad u \in D(A),
\]
for \( u \in (E(A), E)^{1+\frac{1}{2\phi(1-\gamma)}} \) the function \( B_1(x)u \) and for \( u \in D(A) \) the function \( B_2(x)u \) are measurable on \([0, 1] \) in \( E \). Then for \( u \in B^{2,4}_{p,q,r} \) \((0, 1; E(A), E) \) the coercive estimate
\[
\sum_{i=0}^{2} |\lambda_i|^{\frac{1}{2}} \|u^{(i)}\|_{B^{p,q}_{r},0(0,1)} + \|Au\|_{B^{p,q}_{r},0(0,1)} \leq C \left[ \|Lu\|_{B^{p,q}_{r},0(0,1)} + \sum_{k=1}^{2} \|L_ku\|_{(E(A), E)_{h_k,p}} + \|u\|_{B^{p,q}_{r},0(0,1)} \right]
\]
holds for the solution of the problem of (10) and (11).

Proof. Consider the problem (13) and (14), where we put \( A + \lambda_0 \), for some sufficiently large \( \lambda_0 > 0 \) instead of operator \( A \), and \( B_2(x) - \lambda_0 \) instead of the operator \( B_2(x) \). Let \( u \in B^{2,4}_{p,q,r} \) \((0, b; E(A), E) \) be a solution of the problem (13) and (14). Then \( u \) is a solution of the problem
\[
-\frac{d^2}{dx^2}u(x) + (A + \lambda_0)u(x) = f(x) + \lambda_0u(x) - B_1(x) \frac{du}{dx}(x) - B_2(x)u(x), \quad L_1u = f_1, \quad L_2u = f_2.
\]
By Theorem 2, for some sufficiently large \( \lambda_0 > 0 \) we have the estimate
\[
\sum_{i=0}^{2} |\lambda_i|^{\frac{1}{2}} \|u^{(i)}\|_{B^{p,q}_{r},0(0,b)} + \|Au\| _{B^{p,q}_{r},0(0,b)} \leq C \left[ \|f + \lambda_0u - B_1u^{(1)} - B_2u\|_{B^{p,q}_{r},0(0,b)} + \sum_{k=1}^{2} \|L_ku\|_{(E(A), E)_{h_k,p}} \right].
\]
By condition of theorem and by Theorem A1, for \( \varepsilon > 0 \) there is \( C(\varepsilon) > 0 \) such that for \( u \in B^{2,4}_{p,q,r} \) \((0, b; E(A), E) \)
\[
\|B_1u^{(1)}\|_{B^{p,q}_{r},0(0,b)} \leq \varepsilon \|u\|_{B^{2,4}_{p,q,r}(0,b;E(A),E)} + C(\varepsilon) \|u\|_{B^{p,q}_{r},0(0,b)},
\]
\[
\|B_2u\|_{B^{p,q}_{r},0(0,b)} \leq \varepsilon \|u\|_{B^{2,4}_{p,q,r}(0,b;E(A),E)} + C(\varepsilon) \|u\|_{B^{p,q}_{r},0(0,b)}.
\]
From (47)-(49) we obtain the coercive estimate
\[
\sum_{i=0}^{2} |\lambda_i|^{\frac{1}{2}} \|u^{(i)}\|_{B^{p,q}_{r},0(0,b)} + \|Au\| _{B^{p,q}_{r},0(0,b)} \leq C \left[ \|Lu\|_{B^{p,q}_{r},0(0,b)} + \sum_{k=1}^{2} \|L_ku\|_{(E(A), E)_{h_k,p}} + \|u\|_{B^{p,q}_{r},0(0,b)} \right]
\]
for the solution \( u \in B^{2,4}_{p,q,r} \) \((0, b; E(A), E) \) of the problem (13) and (14). Then by Remark 2 we obtain from (50) the estimate (46). \( \Box \)

5. Nonlinear BVPs for degenerate DOEs

Consider the following non-local BVP for a nonlinear degenerate DOE
\[
(L + \lambda)u = -u^{(2)}(x) + A(x, u, u^{(1)})u(x) = f(x, u, u^{(1)}), \quad x \in (0, b),
\]
\[
L_0u = \alpha_0u^{|m_1|}(0) + \sum_{j=1}^{N_0} \delta_0 u^{|m_0|}(x_0) = f_0,
\]
\[
L_1u = \alpha_1u^{|m_1|}(b) + \sum_{j=1}^{N_1} \delta_1 u^{|m_1|}(x_1) = f_1,
\]
where \( u^{(i)} = (x^i \frac{dx}{dt})^i u(x), f_k \in E_k = (E(A), E)_{h_k,p}, \theta_0 = \frac{m_0}{2} + \frac{1}{2\phi(1-\gamma)}, \theta_1 = \frac{m_1}{2} + \frac{1}{2p}, p \in (1, \infty), m_k \in \{0, 1\}, \alpha_k, \delta_k \) are complex numbers and \( x_k \in (0, b), k = 0, 1; A \) is generally speaking, an unbounded operator in a Banach space \( E \).

In this section the existence and uniqueness of the maximal regular solution of nonlinear multipoint problem (51) and (52) are established. We denote \( B^{p,q}_{r} \) \((0, b; E) \) and \( B^{2,4}_{p,q,r} \) \((0, b; E(A), E) \) by \( X \) and \( Y \), respectively. Moreover, we let
\[
0 < b \leq b_0 < \infty, \quad E_j = (E(A), E)_{\eta_j,p}, \quad \eta_j = \frac{j + \frac{1}{p(1-\gamma)}}{2}, \quad j = 0, 1,
\]
\[
Y_0 = \left\{ u : u \in B^{2,4}_{p,q,r} \right(0, b; E(A), E), \quad L_0u = 0 \right\}, \quad F_0 = E_0 \times E_1.
Remark 4. By Proposition A₁ and Remark 2, the embedding $D(B^{[2]})^{1,s}_{p,q,Y}$ $(0, b; E(A), E) \in E_j$ is continuous and there are the constants $C_1, C_2$ such that for $w \in Y, W \subseteq \{ w_j \}, w_j = D^i w (.), j = 0, 1$

$$\|D^i w\|_{E_j, \infty} = \sup_{x \in (0,b)} \|D^i w (x)\|_{E_j} \leq C_1 \|w\|_Y,$$

$$\|W\|_{0, \infty} = \sup_{x \in (0,b)} \sum_{k=0}^{n} \|w_k\|_{E_k} \leq C_0 \|w\|_Y.$$  

Condition C. Suppose the boundary conditions (52) such that for all solutions $w(x)$ of BVP (24) and (25) and for all $\epsilon > 0$ there is $\delta > 0$ such that for $0 < b \leq \delta$ and $F = \{ f_k \}, k = 0, 1$

$$\|W - F\|_{0, \infty} < \epsilon.$$  

Remark 5. It should be noted that if the boundary conditions are local then by Remark 4 the Condition C is easy to be examined. For example, by the suitable choice of $\alpha_{kj}$ it can be shown that the following local boundary conditions

$$L_{kj} u = \sum_{i=1}^{m_{kj}} \alpha_{kj} u^{(i)} (0) = f_{kj}, \quad L_{kj} u = \sum_{i=1}^{m_{kj}} \alpha_{kj} u^{(i)} (b) = f_{kj}$$

satisfy the Condition C.

Condition 2. Let the following conditions be satisfied:

1. $0 \leq \gamma < 1 - \frac{1}{p}, 1 < p < \infty, 1 \leq q < \infty, 0 < s < 1, \frac{1}{p} - \frac{1}{q} \leq s, \alpha_k \neq 0, k = 0, 1$;
2. $(A(x, u, v))$ for $x \in [0, b], u = \{ u_k \}, v = \{ v_k \}, v_k \in E_k$ is a positive operator in a Banach space $E$ for $\varphi \in [0, \pi)$, where domain definition $D(A(x, u, v))$ does not depend on $x, u, v$ and $A : [0, b] \times F_0 \rightarrow B(E(A), E)$ is continuous.

Moreover, for each $R > 0$ there is a constant $L(R) > 0$ such that

$$\|A(x, u) - A(x, \tilde{u})\|_E \leq L(R) \|u - \tilde{u}\|_{F_0} \|Ag\|_E \quad \text{for } x \in [0, b], u, \tilde{u} \in F_0, \tilde{u} = \{ \tilde{u}_k \}, g \in E(A)$$

and

$$\|u\|_{F_0} \leq R, \quad \|\tilde{u}\|_{F_0} \leq R;$$

(3) the function $F : [0, b] \times F_0 \rightarrow E$ such that $F(., .)$ is measurable for each $u_k \in E_k$ and $f (t, .)$ is continuous for a.a. $x \in [0, b].$ Moreover, $\|f (x, u) - f (x, \tilde{u})\|_E \leq M \|u - \tilde{u}\|_{F_0}$ for a.a. $x \in [0, b], u_k, \tilde{u}_k \in E_k$ and

$$\|u\|_{F_0} \leq R, \quad \|\tilde{u}\|_{F_0} \leq R, \quad k = 0, 1.$$  

Theorem 4. Let Condition (2) hold. Then there is $b \in (0, b_0]$ such that problem (51) and (52) has a unique solution belonging to space $B^{[2]}_{p,q,Y}$ $(0, b; E(A), E).$

Proof. By virtue of Theorem 2 the linear BVP

$$(L + \lambda) w = \alpha w^1 (x) + A(0, F) w (x) = f (x), \quad x \in (0,b),$$

$$L_0 u = \alpha_0 u^{(m_0)} (0) + \sum_{j=1}^{N_0} \delta_0 u^{(m_0)} (x_0) = f_0,$$

$$L_1 u = \alpha_1 u^{(m_1)} (b) + \sum_{j=1}^{N_1} \delta_1 u^{(m_1)} (x_1) = f_1$$

is maximal regular in $X$ i.e. for all $f \in X$ there is a unique solution $w \in Y$ of the problem (53) and (54) and it has a coercive estimate

$$\|w\|_Y \leq C_0 (\|f\|_X + \|f_1\|_{E_1} + \|f_2\|_{E_2}),$$

where

$$f (x) = f (x, 0, 0).$$

We want to to solve the problem (51) and (52) locally by means of maximal regularity of the linear problem (53) and (54) via the contraction mapping theorem. For this purpose let $w$ be a solution of the linear BVP (53) and (54). Consider a ball $B_r = \{ u \in X, \ u - w \in Y_0, \ \|u - w\|_Y \leq r \}.$
Given $v \in B_r$ solve a linear BVP

$$au^{[2]}(x) + A(0, F)u(x) = F(x, v, v^{[1]}) + \left[ A(0, F) - A(x, v, v^{[1]}) \right]v(x), \quad F = (f_0, f_1),$$

$$L_0u = \alpha_0u^{[m_0]}(0) + \sum_{j=1}^{N_0} \delta_{0j}u^{[m_0]}(x_0) = f_0,$n

$$L_1u = \alpha_1u^{[m_1]}(b) + \sum_{j=1}^{N_1} \delta_{1j}u^{[m_1]}(x_1) = f_1.$$

(56)

Define a map $Q$ on $B_r$ by $Qv = u$, where $u$ is a solution of the problem (55). We want to show that $Q(B_r) \subset B_r$ and that $L$ is a contraction operator in $Y$, provided $b$ is sufficiently small, and $r$ is chosen properly. For this aim by using maximal regularity of the problem (53) and (54) we obtain from (55)

$$\|Qv - w\|_Y = \|u - w\|_Y \leq C_0 \left[ \|f(x, v) - f(x, 0, 0)\|_X + \|A(0, F) - A(x, v)\|_Y \right].$$

(57)

By assumption (2) we have $K_b = \sup_{x \in [0,1]} \|A(0, F) - A(x, F)\|_{L^1(F,b)}$. So for

$$\|A(0, F) - A(x, v, v^{[1]})\|_X \leq \sup_{x \in [0,1]} \left[ \|A(0, F) - A(x, F)\|_{L^1(F,b)} + \|A(x, v, v^{[1]})\|_{L^1(F,b)} \right] \|u\|_Y \leq \left[ K_b + L(R) \right] \|u - w\|_Y \leq M \left[ \|u - w\|_{L^1(F,b)} \right],$$

and by assumption (3) in a similar way we obtain

$$\|f(x, v) - f(x, 0, 0)\|_E \leq \left[ M \|u - w\|_{L^1(F,b)} \right],$$

(58)

where $R = M \left( C_1 + \|w\|_{1,\infty} \right)$ is a fixed number. Since $r \leq 1$ we have from (56) and (57)

$$\|Qv - w\|_Y \leq C_0 \left[ \|K_b + L(R) \right] \|u - w\|_{L^1(F,b)} \leq C_0 \left[ C_0 \right] \|u - w\|_{L^1(F,b)}.$$

In a similar way we obtain

$$\|Qv - Q\bar{v}\|_Y \leq C_0 \left[ \|f(x, v, v^{[1]}) - f(x, \bar{v}, \bar{v}^{[1]})\|_X + \|A(0, F) - A(x, v, v^{[1]})\|_X + \|A(x, v, v^{[1]}) - A(x, \bar{v}, \bar{v}^{[1]})\|_X \right],$$

$$\leq C_0 \left[ C_1 + M + K_b \right] \|\bar{v} - \bar{v}\|_Y \leq C_0 \left[ C_0 \right] \|\bar{v}\|_Y \leq r,$$

(59)

Now fix $r$ such that $r \leq [8C_0C_1L(R)]$ and let $b \in (0, b_0)$ be such that

$$\|(w - F)\|_{L^1(F,b)} \leq \frac{3}{4} \|(w - \bar{v})\|_Y,$$

as well as

$$K_b \leq (8C_0) - 1, \quad M \leq [8C_0C_1]^{-1}.$$

This is possible since $R$ is fixed and $w$ satisfies the solution of the maximal regular BVP (53) and (54). Then we obtain

$$\|Qv - Q\bar{v}\|_Y \leq \frac{3}{4} \|(w - \bar{v})\|_Y,$$

i.e. $Q$ is a contraction operator. If $b$ is chosen so small that $C_0M \|w\|_{1,\infty} \leq \frac{r}{8}$ then $\|Qv - Q\bar{v}\|_Y \leq r$, i.e.

$$Q(B_r) \subset B_r.$$

The contraction mapping principle therefore, implies a unique fixed point of $Q$ in $B_r$, which is the unique strong solution $u \in Y = B^{[2,\infty]}_{\rho, \delta}(0, b; E(A), E).$ □
6. Infinite system of degenerate DOEs

Consider the BVP for the infinite system of degenerate equations

\[-u_m^{(2)}(x) + \sum_{j=1}^{\infty} \left( d_j + \lambda \right) u_j(x) = f_m(x), \quad x \in (0, 1), \quad m = 1, 2, \ldots, \infty.\]  \hspace{1cm} (60)

\[L_1 u = \alpha_1 u_m^{(m)}(0) + \sum_{j=1}^{N_1} \delta_{ij} u_m^{(m)}(x_{ij}) = 0\]

\[L_2 u = \alpha_2 u_m^{(m)}(1) + \sum_{j=1}^{N_2} \delta_{ij} u_m^{(m)}(x_{ij}) = 0, \quad x_{ij} \in (0, 1),\]

where \(\alpha_k, \delta_{ij}\) are complex numbers, \(u_m^{(m)} = (x^m \frac{d}{dx})^i u(x)\).

Let

\[D(x) = \{d_m\}, \quad d_m > 0, \quad u = \{u_m\}, \quad Du = \{d_m u_m\}, \quad m = 1, 2, \ldots, \infty,\]

\[l_q(D) = \left\{ u : u \in l_q, \|u\|_{l_q(D)} = \|Du\|_{l_q} = \left( \sum_{m=1}^{\infty} |d_m u_m|^q \right)^{\frac{1}{q}} < \infty, \quad 1 < q < \infty \right\}.\]

Let \(Q\) be a differential operator in \(B_{p, \theta}^s (R^n; l_q)\) generated by problem (58). Let

\[B = L \left( B_{p, \theta}^s (R^n; l_q) \right) .\]

Theorem 5. Suppose \(0 \leq \gamma < 1 - \frac{1}{p}, 1 < p < \infty, 1 \leq \theta < \infty, 0 < s < 1, \alpha_k \neq 0, k = 1, 2.\) Then:

(a) for all \(f(x) = [f_m(x)]_1^{\infty} \in B_{p, \theta}^s (0, 1; l_q),\) for \(\lambda \in (S(\varphi), \varphi \in [0, \pi),\) the problem (58) has a unique solution \(u = [u_m(x)]_1^{\infty}\) that belongs to space \(B_{p, \theta}^{2, s}(0, 1, l_q(D), l_q)\) and the coercive uniform estimate

\[\sum_{k=0}^{2} |\lambda|^{1 - \frac{k}{2}} \|u^{(k)}\|_{B_{p, \theta}^s (0, 1; l_q)} \leq C \|f\|_{B_{p, \theta}^s (0, 1; l_q)}\]  \hspace{1cm} (61)

holds for the solution of problem (58).

(b) For \(\lambda \in S(\varphi)\) there exists a resolvent \((Q + \lambda)^{-1}\) of operator \(Q\) and

\[\sum_{k=0}^{2} |\lambda|^{1 - \frac{k}{2}} \|[D^{(k)}(Q + \lambda)^{-1}]\|_B \leq C.\]  \hspace{1cm} (62)

Proof. Really, let \(E = l_q\) and \(A\) be infinite matrices, such that

\[A = \left[ d_m \delta_{ij} \right], \quad m, j = 1, 2, \ldots, \infty.\]

Then it is clear that the operator \(A\) is positive in \(l_q\). Therefore, by virtue of Theorem 2 and Result 1 we obtain the assertions. \(\Box\)

Remark 6. There are a lot of positive operators in concrete Banach spaces. Therefore, putting concrete Banach spaces instead of \(E\) and concrete positive differential, pseudo-differential operators, or finite, infinite matrices, etc. instead of operator \(A\), we can obtain the maximal regularity of different classes of linear and nonlinear differential and pseudo-differential equations or system of equations by virtue of Theorems 2–4.

References