

A Monte Carlo Approach to Estimation of G-Renewal Process in Warranty Data Analysis

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1. Introduction

For many years, the most commonly used models for the failure process have been the renewal process (RP) and the nonhomogeneous Poisson process (NHPP). In the framework of the repairable system applications, RP is used to model the situations with restoration to "good-as-new" condition (perfect repair assumption), meanwhile NHPP is applied to the situations with the "same-as-old" restoration (minimal repair assumption). In a sense, these two assumptions can be considered as extreme ones from both theoretical and practical standpoints. In order to avoid this "extremism", several generalizing models have been introduced in recent years. References include Brown & Proschan (1982), Kijima & Sumita (1986), Filkenstein, (1993), Lindqvist (1999). Among these models, the G-Renewal Process (GRP) introduced by Kijima & Sumita (1986) is very attractive, since it covers the intermediate "better-than-old-but-worse-than-new" repair assumption and results in a G-renewal equation, which is a generalization of the well-known ordinary renewal equation. Unfortunately, a closed form solution of the equation is unavailable, which makes the respective statistical estimation challenging.

The objective of this paper is limited to statistical estimation of the parameters of G-Renewal Process, based on typical one-dimensional warranty (grouped failure time) data. The Monte Carlo approach considered below can be also applied to parameter estimation of Kijima's Model I and II (Kijima, 1989).

2. G-Renewal Process

Kijima and Sumita (1986) introduced a *G-Renewal Process*, which can model restoration conditions ranging from "good-as-new" to "same-as-old". The GRP is introduced using the notion of *virtual age*.

Let A_n be the virtual age of a system immediately after the n th repair. If $A_n = y$, then the system has the time to the $(n + 1)$ th failure X_{n+1} , which is distributed according to the following cumulative distribution function (CDF):

$$F(X | A_n = y) = \frac{F(X + y) - F(y)}{1 - F(y)}, \quad (1)$$

where $F(X)$ is the CDF of the time-to-first-failure (TTFF) distribution of a new system. The sum

$$S_n = \sum_{i=1}^n X_i,$$

with $S_0 = 0$, is called the *real age* of the system.

In the framework of the GRP it is assumed that the n th repair can remove the damage incurred only during the time between the $(n - 1)$ th and the n th failures, so that the respective virtual age after the n th repair is

$$A_n = A_{n-1} + q X_n = q S_n, \quad n = 1, 2, \dots$$

where q is the *parameter of rejuvenation* (or *repair effectiveness parameter*) and the virtual age of a new system $A_0 = 0$, so that the TTFF is distributed according to $F(t|0) \equiv F(t)$.

The time between the first and second failures is distributed according to (1) with $A_1 = qX_1$. Respectively, the time between the second and third failure is distributed according to (1) with $A_2 = q(X_1 + X_2)$, and so on.

It is clear that for $q = 0$, the considered process coincides with an ordinary RP, thus, modeling the "good-as-new" repair assumption. With $q = 1$, a system is restored to the "same-as-old" condition, which is similar to NHPP. The case of $0 < q < 1$ corresponds to the intermediate "better-than-old-but-worse-than-new" repair assumption. Finally, with $q > 1$, the virtual age $A_n > S_n$, i.e., the repair damages the system to a higher degree than it was just before the respective failure, which corresponds to the "worse-than-old" repair assumption.

The expected number of failures in $(0, t]$, which is called a *cumulative intensity function*, is given by a solution of G-renewal equation (Kijima, et al., 1988):

$$H(t) = \int_0^t \left(g(\tau|0) + \int_0^{\tau} h(x)g(\tau-x|x)dx \right) d\tau, \quad (2)$$

where

$$g(t|x) = \frac{f(t+qx)}{1-F(qx)}, \quad t, x \geq 0,$$

is such that $h(t) = d(H(t))/dt$, $g(t|0) = f(t)$, and $F(t)$ and $f(t)$ are the CDF and PDF of the TTF distribution.

The closed form solution of (2) is not available, and even numerical solutions are difficult to obtain, since the equation contains a recurrent infinite system (Finkelstein, 1997). A Monte Carlo based solution is, however, possible and was discussed by Kaminskiy and Krivtsov (1998).

3. Warranty Data

Typical one-dimensional warranty data are collected as results of observations on a large population of identical repairable units. The population size, N_0 , is known and it can be assumed as constant in time (the number of lost units, if any, is negligible with respect to N_0). The following table provides an example of real warranty data, which are analyzed in the next section.

Table 1. Example of Warranty Data for a Repairable System. Population Size, $N_0 = 100000$.

Month in Service, t	3	6	9	12	15	18	21	24	27
Cumulative Number of Failures Per System, $H_{emp}(t)$	0.03	0.09	0.14	0.24	0.38	0.54	0.70	0.90	1.17

4. Estimation Procedure

Based on the warranty data, the *empirical cumulative intensity function*, $H_{emp}(t)$, is calculated as

$$H_{emp}(t_i) = \frac{N(t_i)}{N_0}, \quad t_i < t_{i+1}, i = 1, 2, \dots, n \quad (3)$$

where $N(t)$ is the cumulative number of failures in $(0, t]$.

Denote a solution of G-renewal equation (2) obtained by Monte Carlo simulations by

$$H_{mc}(t) = f(F(\tau|\alpha), q, t), \quad (4)$$

where $F(\tau|\alpha)$ is a given time-to-first-failure CDF with unknown vector of parameters, α .

Using (3) and (4), the least squares estimates of GRP parameters α and q can be obtained as a solution of GRP parameters α and q can be obtained as a solution of the following optimization problem:

$$\min_{\alpha, q} \left(\sum_{i=1}^n (H_{emp}(t_i) - H_{mc}(F(\tau|\alpha), q, t_i))^2 \right)$$

5. Examples

5.1 Simulated Data

The empirical cumulative intensity function, $H_{emp}(t)$, was obtained by simulating a GRP with a Weibull distributed TTFF (shape parameter, $\beta = 1.5$ and scale parameter, $\Theta = 1$) and the GRP rejuvenation parameter, $q = 0.5$ using $N_0 = 100$ realizations over the observation interval, $T = 5\Theta$.

Estimates of $\hat{\beta}$, $\hat{\Theta}$, and \hat{q} were obtained based on $n_0 = 1000$ realizations of GRP as follows:
 $\hat{\beta} = 1.48$, $\hat{\theta} = 1.00$, $\hat{q} = 0.49$. Tables 2 - 3 show the sample correlation and covariance matrices for the obtained estimates of GRP parameters for 30 simulated empirical cumulative intensity functions, $H_{emp}(t)$.

Table 2. Sample Correlation Matrix

	β	Θ	q
β	1.000		
Θ	0.702	1.000	
q	0.079	-0.523	1.000

Table 3. Sample Covariance Matrix

	β	Θ	q
β	$2.6 \cdot 10^{-3}$		
Θ	$1.9 \cdot 10^{-3}$	$2.8 \cdot 10^{-3}$	
q	$2.3 \cdot 10^{-4}$	$-1.6 \cdot 10^{-3}$	$3.2 \cdot 10^{-3}$

With the empirical cumulative intensity function simulated for $N_0 = 100000$ realizations (which is a more typical case for a company concerned with mass production) and using $n_0 = 1000000$, the estimation procedure returns the original GRP parameters with close to zero variance.

5.2 Real Data

The warranty data collected on a system during first 18 months (see Table 1) were used for estimation of GRP parameters. The Weibull distribution with the shape parameter, β , and the scale parameter, Θ , was assumed as the underlying TTFF distribution. The solid line in Figure 1 represents the least squares fit from a family of G-renewal functions simulated in the following parameter domain: $\{1 < \beta < 2, 10 < \Theta < 50, 0 < q < 1\}$.

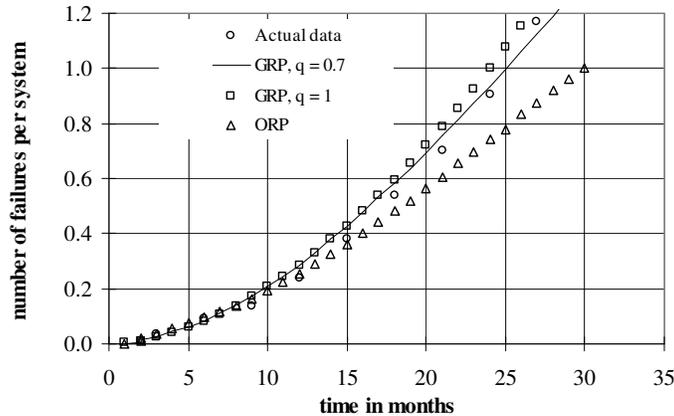


Figure 1. A Least Square Fit of G-Renewal Function

The obtained estimates of GRP parameters are $\hat{\beta} = 1.8$, $\hat{\theta} = 24$, $\hat{q} = 0.70$. The estimated G-renewal function shows a good fit to the data not only in the interval (0, 18] months (used for estimation) but also in the remaining interval (18, 30] months (obtained by prediction), see Figure 1. The figure also shows the extreme repair conditions modeled by the RP ($q = 0$) and the GRP ($q = 1$).

It is reasonable to conclude that the approach considered above is not only practically applicable for estimation of the GRP parameters, but also for prediction of the G-renewal function, which is often essential in warranty data analysis.

References

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