On continuous dependence on coefficients of the Brinkman–Forchheimer equations

A.O. Çelebi\textsuperscript{a}, V.K. Kalantarov\textsuperscript{b}, D. Uğurlu\textsuperscript{c,*}

\textsuperscript{a}Department of Mathematics, Middle East Technical University, Ankara, Turkey
\textsuperscript{b}Department of Mathematics, Koç University, Istanbul, Turkey
\textsuperscript{c}Department of Mathematics, Abant İzzet Baysal University, Bolu, Turkey

Received 10 March 2004; received in revised form 27 October 2005; accepted 4 November 2005

Abstract

We prove continuous dependence of solutions of the Brinkman–Forchheimer equations on the Brinkman and Forchheimer coefficients in $H^1$ norm.

© 2005 Elsevier Ltd. All rights reserved.

Keywords: Structural stability; Porous media; Continuous dependence on the coefficients

1. Introduction

In this work, we study the following initial-boundary value problem for the Brinkman–Forchheimer equations:

\[ \begin{align*}
    u_t &= \gamma \Delta u - au - b|u|^{\alpha\gamma}u - \nabla p, \quad \nabla \cdot u = 0, \quad x \in \Omega, \quad t > 0, \\
    u(x, 0) &= u_0(x), \quad x \in \Omega, \\
    u &= 0, \quad x \in \partial \Omega, \quad t > 0.
\end{align*} \tag{1.1} \tag{1.2} \tag{1.3} \]

Here $u = (u_1, u_2, u_3)$ is the fluid velocity vector, $\gamma \geq 0$ is the Brinkman coefficient, $a > 0$ is the Darcy coefficient, $b > 0$ is the Forchheimer coefficient, $p$ is the pressure, $\alpha \in [1, 2]$ is a given number, $\Omega$ is a bounded domain of $\mathbb{R}^3$ whose boundary $\partial \Omega$ is assumed to be of class $C^2$.

We study the problem of continuous dependence of solutions to the problem (1.1)–(1.3) on coefficients $b$ and $\gamma$.

Continuous dependence of solutions on coefficients of equations is a type of structural stability, which reflects the effect of small changes in coefficients of equations on the solutions. Many results of this type can be found in the monograph of Ames and Straughan [1]. Structural stability in flows of fluid in porous media represented by the Darcy and Brinkman systems are investigated in the articles of Ames and Payne [2], Payne and Straughan [4–7]. In [7], Payne and Straughan considered the initial-boundary value problem (1.1)–(1.3) with $\alpha = 1$ which describes the flow of fluid in a saturated porous medium. They proved continuous dependence of solutions of the problem on the

* Corresponding author.
E-mail address: uguulu_d@ibu.edu.tr (D. Uğurlu).

0893-9659/S - see front matter © 2005 Elsevier Ltd. All rights reserved.
coefficients \( b \) and \( \gamma \) in \( L^2 \) norm. Our aim is to show continuous dependence on these coefficients in a stronger norm, that is, in \( H^1 \) norm.

In the following we will use the function spaces \( \tilde{H}_0^1(\Omega, \mathbb{R}^3) = \{ u \in H_0^1(\Omega, \mathbb{R}^3) : \nabla \cdot u = 0 \} \) and \( \tilde{L}^2(\Omega, \mathbb{R}^3) \), where the latter space is the closure of \( \tilde{H}_0^1(\Omega, \mathbb{R}^3) \) in \( L^2(\Omega, \mathbb{R}^3) \). For simplicity we shall write \( \tilde{L}^2(\Omega, \mathbb{R}^3) = \tilde{L}^2(\Omega) \) and \( \tilde{H}_0^1(\Omega, \mathbb{R}^3) = \tilde{H}_0^1(\Omega) \). We use the notation \( \| \cdot \|_p \) for the norm in \( L^p(\Omega) \). We denote by \( \| \cdot \| \) the norm and by \( \langle \cdot , \cdot \rangle \) the inner product in \( L^2(\Omega) \).

2. Continuous dependence on the Forchheimer coefficient

In this section, we prove that the solution of the problem (1.1)–(1.3) depends continuously on the Forchheimer coefficient \( b \) in \( H^1(\Omega) \) norm.

Using the Faedo–Galerkin method we can prove the following existence and uniqueness theorem; see, for instance, [3, Theorem 9.3 and Theorem 10.2].

**Theorem 1.** Assume that \( 1 \leq \alpha \leq 2 \). Then for any \( u_0 \in H_0^1(\Omega) \), there exists a unique solution \( u \in C([0, T]; \tilde{H}_0^1(\Omega)) \) of the problem (1.1)–(1.3). Furthermore, we have

\[
\sup_{0 \leq t \leq T} \| \nabla u(t) \| \leq D \quad \text{and} \quad \int_0^T \| u_t(t) \|^2 dt \leq D
\]

(2.1)

for any \( T > 0 \), where \( D \) is a generic positive constant depending on the initial data and the parameters of (1.1).

Let us show just how to get the estimates (2.1). First we multiply (1.1) by \( u_t + u \) in \( L^2(\Omega) \):

\[
2\| u_t(t) \|^2 + \frac{d}{dt} \left[ \gamma \| \nabla u(t) \|^2 + (a + 1)\| u(t) \|^2 + \frac{2b}{\alpha + 2} \int_\Omega |u(x, t)|^{\alpha + 2} dx \right] \\
+ 2\gamma \| \nabla u(t) \|^2 + 2a\| u(t) \|^2 + 2b \int_\Omega |u(x, t)|^{\alpha + 2} dx = 0.
\]

(\( I_1 \))

It follows from this inequality that the function

\[
\Phi(t) = \gamma \| \nabla u(t) \|^2 + (a + 1)\| u(t) \|^2 + \frac{2b}{\alpha + 2} \int_\Omega |u(x, t)|^{\alpha + 2} dx
\]

satisfies the inequality \( \frac{d}{dt} \Phi(t) + \frac{2\alpha}{a + 1} \Phi(t) \leq 0 \). The latter implies the estimate

\[
\gamma \| \nabla u(t) \|^2 + (a + 1)\| u(t) \|^2 + \frac{2b}{\alpha + 2} \int_\Omega |u(x, t)|^{\alpha + 2} dx \leq D_1e^{-\frac{2b}{a + 1}t},
\]

(\( I_2 \))

where

\[
D_1 = \gamma \| \nabla u_0 \|^2 + (a + 1)\| u_0 \|^2 + \frac{2b}{\alpha + 2} \int_\Omega |u_0(x)|^{\alpha + 2} dx.
\]

The boundedness of \( \Phi(t) \) and hence the first of the estimates (2.1) follows from (\( I_2 \)); the second of the estimates (2.1) follows if we integrate (\( I_1 \)) over time and exploit the boundedness of \( \Phi(t) \).

Now assume that \((u, p)\) is the solution of the problem

\[
u_t = \gamma \Delta u - au - b_1|u|^\alpha u - \nabla p, \quad \nabla \cdot u = 0, \quad x \in \Omega, \quad t > 0, \\
u(x, 0) = u_0(x), \quad x \in \Omega, \\
u = 0, \quad x \in \partial \Omega, \quad t > 0,
\]

and \((v, q)\) is the solution of

\[
v_t = \gamma \Delta v - av - b_2|v|^\alpha v - \nabla q, \quad \nabla \cdot v = 0, \quad x \in \Omega, \quad t > 0, \\
v(x, 0) = u_0(x), \quad x \in \Omega, \\
v = 0, \quad x \in \partial \Omega, \quad t > 0.
\]
Let \( w = u - v \) and \( \pi = p - q \). Then \((w, \pi)\) is a solution of the problem
\[
\begin{align*}
    w_t &= \gamma \Delta w - a w - b_1 |u|^{\alpha} u + b_2 |v|^{\alpha} v - \nabla \pi, & \nabla \cdot w &= 0, & x \in \Omega, & t > 0, \\
    w(x, 0) &= 0, & x \in \Omega, \\
    w &= 0, & x \in \partial \Omega, & t > 0.
\end{align*}
\] (2.2)

The main result of this section is the following theorem:

**Theorem 2.** Let \( w \) be the solution of the problem (2.2)–(2.4). Then \( w \) satisfies the estimate
\[
\| \nabla w(t) \|^2 + \| w(t) \|^2 \leq K(b_1 - b_2)^2, \quad \forall t > 0, \tag{E1}
\]
where \( K \) is a positive constant depending on the parameters of (1.1).

**Proof.** Multiplying (2.2) by \( w \) in \( L^2(\Omega) \) we get
\[
\begin{align*}
    \frac{1}{2} \frac{d}{dt} \| w(t) \|^2 + \gamma \| \nabla w(t) \|^2 + a \| w(t) \|^2 = -\hat{b} \langle |u(t)|^{\alpha} u(t), w(t) \rangle - b_2 \langle |u(t)|^{\alpha} u(t) - |v(t)|^{\alpha} v(t), w(t) \rangle,
\end{align*}
\] (2.5)

where \( \hat{b} = b_1 - b_2 \). Since the operator \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by \( T(u) = |u|^{\alpha} u \) is monotone, we have
\[
\langle |u(t)|^{\alpha} u(t) - |v(t)|^{\alpha} v(t), w(t) \rangle \geq 0. \tag{2.6}
\]

Thus, from (2.5) we get
\[
\begin{align*}
    \frac{1}{2} \frac{d}{dt} \| w(t) \|^2 + \gamma \| \nabla w(t) \|^2 + a \| w(t) \|^2 \leq \left| \hat{b} \langle |u(t)|^{\alpha} u(t), w(t) \rangle \right|.
\end{align*}
\] (2.7)

Using the Hölder and Sobolev inequalities we obtain the estimate
\[
\begin{align*}
    \left| \hat{b} \langle |u(t)|^{\alpha} u(t), w(t) \rangle \right| &\leq \| \hat{b} \|_{2,6} \| |u(t)|^{\alpha} u(t) \|_{5,6} \| w(t) \|_6 \\
    &\leq \| \hat{b} \|_{2,6} \| \nabla u(t) \|_{6} \| \nabla w(t) \| \leq \frac{\| \hat{b} \|_{2,6}^2}{2^\alpha} \| \nabla u(t) \|_{2(1+\alpha)} + \frac{\gamma}{2} \| \nabla w(t) \|^2, \tag{2.8}
\end{align*}
\]

where \( d_0 \) is the constant in the Sobolev inequality
\[
\| v \|_p \leq d_0 \| \nabla v \|, \quad 1 \leq p \leq 6 \tag{S}
\]
which is valid for each \( v \in H^1_0(\Omega, \mathbb{R}^3) \). By using (2.8) and (2.1) in (2.7) we get
\[
\frac{d}{dt} \| w(t) \|^2 + \gamma \| \nabla w(t) \|^2 + 2a \| w(t) \|^2 \leq K_0 \hat{b}^2, \tag{2.9}
\]

where \( K_0 = D^{2\alpha+2} d_0^{2\alpha+4} \gamma^{-1} \).

Multiplication of (2.2) with \( w_t \) in \( L^2(\Omega) \) gives
\[
\begin{align*}
    \| w_t(t) \|^2 + \frac{1}{2} \frac{d}{dt} \left\{ \gamma \| \nabla w(t) \|^2 + a \| w(t) \|^2 \right\} = -b_2 \left| \langle |u(t)|^{\alpha} u(t) - |v(t)|^{\alpha} v(t), w_t(t) \rangle \right| - \hat{b} \left\langle |u(t)|^{\alpha} u(t), w_t(t) \right\rangle.
\end{align*}
\] (2.10)

Let us estimate the expressions in the right-hand side of (2.10). Using the mean value theorem and then Hölder’s inequality we get
\[
\left| \langle |u(t)|^{\alpha} u(t) - |v(t)|^{\alpha} v(t), w_t(t) \rangle \right| \leq 3(\alpha + 1) \int_{\Omega} \left( |u(t)|^{\alpha} + |v(t)|^{\alpha} \right) |w(t)||w_t(t)| dx \\
\leq 3(\alpha + 1) \left( \| |u(t)|^{\alpha} \|_{3,\alpha} + \| v(t) \|_{3,\alpha} \right) \| w(t) \|_6 \| w_t(t) \|. \tag{2.11}
\]

Since \( 1 \leq \alpha \leq 2 \), we use the Sobolev inequality and obtain
\[
\left| \langle |u(t)|^{\alpha} u(t) - |v(t)|^{\alpha} v(t), w_t(t) \rangle \right| \leq 6(\alpha + 1) d_0^{\alpha+1} D^{\alpha} \| \nabla w(t) \| \| w_t(t) \| \\
\leq 18b_2(\alpha + 1)^2 d_0^{2\alpha+2} D^{2\alpha} \| \nabla w(t) \|^2 + (2b_2)^{-1} \| w_t(t) \|^2. \tag{2.12}
\]
Similarly we estimate the last term in the right hand side of (2.10):
\[ |(u(t)^a u(t), w_1(t))| \leq 2^{-1} |b| a_0^{2(\alpha+1)} D^{2(\alpha+1)} + (2|b|)^{-1} \| w_1(t) \|^2. \]
(2.13)
Therefore, using (2.12) and (2.13) in (2.10) we obtain
\[ \frac{d}{dt} \left\{ \gamma \| \nabla w(t) \|^2 + a \| w(t) \|^2 \right\} \leq K_1 \dot{b}^2 + K_2 \| \nabla w(t) \|^2, \]
(2.14)
where \( K_1 = (d_0 D)^{2\alpha+2} \) and \( K_2 = 36b_2^2(\alpha + 1)^2 a_0^{2\alpha+2} D^{2\alpha} \). At this point, we multiply (2.14) by \( \frac{\gamma}{2K_2} \) and add this to (2.9):
\[ \frac{d}{dt} \left[ \frac{\gamma^2}{2K_2} \| \nabla w(t) \|^2 + \frac{a \gamma + 2K_2}{2K_2} \| w(t) \|^2 \right] + \frac{\gamma}{2} \| \nabla w(t) \|^2 + 2a\| w(t) \|^2 \leq \left( K_0 + \frac{\gamma K_1}{2K_2} \right) \dot{b}^2. \]
(2.15)
Inequality (2.15) implies
\[ Y'(t) + \beta Y(t) \leq K_3 \dot{b}^2, \]
(2.16)
where \( K_3 = K_1 + 2K_2K_0 \gamma^{-1}, \quad \beta = K_2 \min\{\gamma^{-1}, 4a(a \gamma + 2K_2)^{-1}\} \) and
\[ Y(t) = \gamma^2 \| \nabla w(t) \|^2 + (a \gamma + 2K_2) \| w(t) \|^2. \]

It follows from (2.16) that
\[ \gamma^2 \| \nabla w(t) \|^2 + (a \gamma + 2K_2) \| w(t) \|^2 \leq K_3 \beta^{-1}(1 - e^{-\beta t}) \dot{b}^2. \]

Hence the statement of the theorem holds, that is, \((E_1)\) is satisfied and we have
\[ \| \nabla w(t) \| \to 0 \quad \text{as} \quad \dot{b} \to 0, \quad t > 0. \]

**Remark 1.** It is not difficult to see that \((E_1)\) remains true also when \( a > -\lambda_1 \), where \( \lambda_1 > 0 \) is the first eigenvalue of the problem
\[ -\gamma \Delta \psi + \nabla p = \lambda \psi, \quad \nabla \cdot \psi = 0, \quad x \in \Omega; \quad \psi = 0, \quad x \in \partial \Omega. \]

**Remark 2.** In our calculations above we essentially used the estimate \( \| \nabla u(t) \| \leq D, \forall t > 0 \), from (2.1). Actually this estimate is valid for each solution of the problem (1.1)–(1.3) with \( \gamma > 0, b > 0 \) and \( a \in \mathbb{R} \). But it is no longer true when \( b < 0 \). To show this, let us consider energetic equalities for solutions of the problem (1.1)–(1.3):
\[ \frac{d}{dt} \| u(t) \|^2 = -2\gamma \| \nabla u(t) \|^2 - 2a \| u(t) \|^2 + 2|b| \| u(t) \|^{\alpha+2}_{\alpha+2}, \]
(R1)
\[ \frac{d}{dt} \left[ -\frac{\gamma}{2} \| \nabla u(t) \|^2 - \frac{a}{2} \| u(t) \|^2 + \frac{|b|}{\alpha+2} \| u(t) \|^{\alpha+2}_{\alpha+2} \right] = \| u_1(t) \|^2. \]
(R2)
Integrating \((R_2)\) with respect to \( t \) we get
\[ -\frac{\gamma}{2} \| \nabla u(t) \|^2 - \frac{a}{2} \| u(t) \|^2 + \frac{|b|}{\alpha+2} \| u(t) \|^{\alpha+2}_{\alpha+2} \geq E_0, \]
(R3)
where
\[ E_0 = -\frac{\gamma}{2} \| \nabla u_0 \|^2 - \frac{a}{2} \| u_0 \|^2 + \frac{|b|}{\alpha+2} \| u_0 \|^{\alpha+2}_{\alpha+2}. \]

Employing \((R_3)\) in \((R_1)\) we obtain
\[ \frac{d}{dt} \| u(t) \|^2 \geq \left( 2|b| - \frac{4|b|}{\alpha+2} \right) \| u(t) \|^{\alpha+2}_{\alpha+2} + 4E_0 \geq 2\frac{|b|a}{\alpha+2} \| \Omega \|^{-\alpha/2} \| u(t) \|^2(\alpha+2)/2 + 4E_0. \]
From the last inequality it follows that if \( \| u_0 \| > 0 \) and \( E_0 \geq 0 \), then \( \| u(t) \| \) tends to infinity in a finite time.
3. Continuous dependence on the Brinkman coefficient

In this section we show that the solution of the problem (1.1)–(1.3) depends continuously on the Brinkman coefficient $\gamma$ in $H^1(\Omega)$ norm. We let $(u, p)$ be the solution of

$$
\begin{align*}
  u_t &= \gamma_1 \Delta u - au - b|u|^\alpha u - \nabla p, & x \in \Omega, & t > 0, \\
  u(x, 0) &= u_0(x), & x \in \Omega, \\
  u &= 0, & x \in \partial \Omega, & t > 0,
\end{align*}
$$

and $(v, q)$ be the solution of

$$
\begin{align*}
  v_t &= \gamma_2 \Delta v - av - b|v|^\alpha v - \nabla q, & x \in \Omega, & t > 0, \\
  v(x, 0) &= v_0(x), & x \in \Omega, \\
  v &= 0, & x \in \partial \Omega, & t > 0.
\end{align*}
$$

Let $w = u - v$ and $\pi = p - q$ and $\hat{\gamma} = \gamma_1 - \gamma_2$. Then $(w, \pi)$ satisfies the problem

$$
\begin{align*}
  w_t &= \gamma_1 \Delta w + \hat{\gamma} \Delta v - aw - b(u^\alpha u - v^\alpha v) - \nabla \pi, & x \in \Omega, & t > 0, \\
  w(x, 0) &= 0, & x \in \Omega, \\
  w &= 0, & x \in \partial \Omega, & t > 0.
\end{align*}
$$

The following theorem establishes continuous dependence of the solution of (1.1)–(1.3) on the coefficient $\gamma$ in $H^1(\Omega)$ norm.

**Theorem 3.** If $(w, \pi)$ is the solution of (3.7)–(3.9), then the following estimate holds:

$$
\|\nabla w(t)\|^2 + \|w(t)\|^2 \leq L(\gamma_1 - \gamma_2)^2,
$$

(E2)

where $L$ is a positive constant, which depends on the parameters of (1.1).

**Proof.** We start the proof by taking the inner product of (3.7) by $w$ in $L^2(\Omega)$:

$$
\begin{align*}
  \frac{1}{2} \frac{d}{dt} \|w(t)\|^2 + \gamma_1 \|\nabla w(t)\|^2 &+ a \|w(t)\|^2 \\
  &= -\hat{\gamma} \langle \nabla v(t), \nabla w(t) \rangle - b(\|u^\alpha u - v^\alpha v\|) \langle w(t), w(t) \rangle.
\end{align*}
$$

By using (2.6) and the inequality

$$
|\hat{\gamma} \langle \nabla v(t), \nabla w(t) \rangle| \leq \frac{\gamma_1}{2} \|\nabla w(t)\|^2 + \frac{\hat{\gamma}^2}{2\gamma_1} \|v(t)\|^2
$$

we obtain from (3.10)

$$
\frac{d}{dt} \|w(t)\|^2 + \gamma_1 \|\nabla w(t)\|^2 + 2a \|w(t)\|^2 \leq \hat{\gamma}^2 \gamma_1^{-1} \|\nabla v(t)\|^2.
$$

On the other hand, taking the inner product of (3.7) and $w_t$ in $L^2(\Omega)$ we obtain

$$
\begin{align*}
  \frac{1}{2} \frac{d}{dt} \left[ \gamma_1 \|\nabla w(t)\|^2 + a \|w(t)\|^2 \right] &+ \|w_t(t)\|^2 \\
  &= \hat{\gamma} \langle \Delta v(t), w_t(t) \rangle - b(\|u^\alpha u - v^\alpha v\|) \langle w(t), w_t(t) \rangle.
\end{align*}
$$

We have, similar to (2.12),

$$
|b(\|u^\alpha u - v^\alpha v\|, w_t(t))| \leq \frac{1}{2} \|w_t(t)\|^2 + 9b^2(\alpha + 1)^2 d_0^{2\alpha + 2} D^{2\alpha} \|\nabla w(t)\|^2.
$$

By using (3.4) and the Cauchy–Schwarz inequality we get

$$
|\hat{\gamma} \langle \Delta v(t), w_t(t) \rangle| = \frac{\hat{\gamma}|}{\gamma_2} \|v_t + av + b|v^\alpha v, w_t(t)|
$$
\[ \leq \frac{1}{2} \| w(t) \|^2 + \frac{\hat{\gamma}^2}{2\gamma_2} \| v(t) + av(t) + b|v(t)|^\alpha v(t) \|^2 \]
\[ \leq \frac{1}{2} \| w(t) \|^2 + \frac{3\hat{\gamma}^2}{2\gamma_2} \left[ \| v(t) \|^2 + a^2 \| v(t) \|^2 + b^2 \| v(t) \|^{2\alpha + 2} \right]. \quad (3.14) \]

Since \(2\alpha + 2 \leq 6\), using (S) and (2.1) in (3.14) we have
\[ |\langle \Delta v(t), w(t) \rangle| \leq \frac{1}{2} \| w(t) \|^2 + \frac{3\hat{\gamma}^2}{2\gamma_2} \| v(t) \|^2 + \frac{3\hat{\gamma}^2}{2\gamma_2} \left[ (d_0 a D)^2 + b^2 (d_0 D)^2 \right]. \quad (3.15) \]

Employing (3.13) and (3.15) in (3.12) we obtain
\[ \frac{d}{dt} \left[ \gamma_1 \| \nabla w(t) \|^2 + a \| w(t) \|^2 \right] \leq K_4 \hat{\gamma}^2 + \frac{3\hat{\gamma}^2}{\gamma_2} \| v(t) \|^2 + L_0 \| \nabla w(t) \|^2, \quad (3.16) \]

where \(K_4 = 3\gamma_2^{-2} [(d_0 a D)^2 + b^2 (d_0 D)^2 \] and \(L_0 = 18b^2(\alpha+1)^2 d_0^2 \). Let us multiply (3.11) by \(\eta = 2L_0\gamma_1^{-1}\) and add to (3.16):
\[ \frac{d}{dt} \left[ \gamma_1 \| \nabla w(t) \|^2 + (\eta + a) \| w(t) \|^2 \right] + L_0 \| \nabla w(t) \|^2 + 2a\eta \| w(t) \|^2 \]
\[ \leq \hat{\gamma}^2 \left[ \eta\gamma_1^{-1} \| \nabla v(t) \|^2 + \frac{3}{\gamma_2} \| v(t) \|^2 + K_4 \right]. \quad (3.17) \]

It is clear that
\[ L_0 \| \nabla w(t) \|^2 + 2a\eta \| w(t) \|^2 \geq k_0 \left[ \gamma_1 \| \nabla w(t) \|^2 + (\eta + a) \| w(t) \|^2 \right], \]

where \(k_0 = \min(L_0\gamma_1^{-1}, 2\eta\alpha(\eta + a)^{-1})\). Thus it follows from (3.17) that the function
\[ Z(t) = \gamma_1 \| \nabla w(t) \|^2 + (\eta + a) \| w(t) \|^2 \]
satisfies the inequality
\[ Z'(t) + k_0 Z(t) \leq \hat{\gamma}^2 \left[ \eta\gamma_1^{-1} \| \nabla v(t) \|^2 + \frac{3}{\gamma_2} \| v(t) \|^2 + K_4 \right]. \quad (3.18) \]

Integrating (3.18) and taking into account (2.1) and (I2) we obtain
\[ Z(t) \leq \hat{\gamma}^2 \int_0^t \left[ \eta\gamma_1^{-1} \| \nabla v(\tau) \|^2 + \frac{3}{\gamma_2} \| v(\tau) \|^2 \right] d\tau + K_4 k_0^{-1} \hat{\gamma}^2 \]
\[ \leq \hat{\gamma}^2 \left( \eta\gamma_1^{-1} D + \frac{3D}{\gamma_2} + K_4 k_0^{-1} \right). \]

The last estimate implies the desired inequality (E2). \( \square \)

**Acknowledgement**

The authors would like to thank the referee for valuable comments and suggestions.

**References**


