A SIMPLE SVD ALGORITHM FOR FINDING HIDDEN PARTITIONS

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Abstract. Finding a hidden partition in a random environment is a general and important problem, which contains as subproblems many famous questions, such as finding a hidden clique, finding a hidden coloring, finding a hidden bipartition etc.

In this paper, we provide a simple SVD algorithm for this purpose, answering a question of McSherry. This algorithm is very easy to implement and works for sparse graphs with optimal density.

1. THE PROBLEM AND A NEW ALGORITHM

The hidden partition problem is the following: let $X$ be a set of $n$ vertices with a partition $X = \bigcup_{i=1}^{k} X_i$; for all $1 \leq i \leq j \leq n$ and any $x \in X_i, y \in X_j$, we put a random edge between $x$ and $y$ with probability $p_{ij}$. Given one such random graph, the goal is to recover the sets $X_i$. This problem is of importance in computer science and statistics and contains as special cases several well-studied problems such as the hidden clique, hidden bisection, hidden coloring, clustering etc (see, for instance, [1, 2, 3, 6, 7, 8, 9, 11, 12, 15, 16, 18, 13, 21, 19] and the references therein). In what follows, we refer to $X_i$ as clusters.

In an influential paper [24], McSherry provided a (randomized) polynomial time algorithm that solves the general hidden partition problem for a large range of parameters. As corollary, he derived several earlier results obtained for special cases.

The general idea [24] (and in many earlier works on clustering) is to find a good geometric representation of the vertices. We say that a representation is perfect if there is a number $r > 0$ such that

- Vertices in the same cluster have distance at most $r$ from each other.
- Vertices from different clusters have distance at least $4r$ from each other.

Once a perfect representation is obtained, it is easy to find the clusters. If $r$ is known, then the solution is obvious. If $r$ is not known, then there are several simple algorithms. For instance, one can create a minimal spanning tree (with respect to the distances) on the vertices and then remove the largest $k-1$ edges. In what follows, we put all these simple algorithms under a subroutine called Clustering by Distances and the reader can choose his/her favorite to implement. Our main goal is to present a simple way to obtain a perfect representation.

In the rest of the paper, let $s_u := |X_i|$ if $u \in X_i$ and $s := \min_{u \in X} s_u = \min_i |X_i|$. We assume that $n$ is sufficiently large, whenever needed. Asymptotic notation are used under
the assumption \( n \to \infty \). All explicit constants (such as the 4 above) are adhoc and we make no attempt to optimize them.

A popular way to find a perfect representation is to project the points of \( X \) (seen as vectors in \( \mathbb{R}^n \)) onto a properly chosen low-dimensional subspace \( H \). The main technical part of Mc Sherry’s algorithm is a subroutine called \( CProj \) (Combinatorial Projection), which creates \( H \) in a combinatorial way. The inputs in this subroutine are a matrix \( \hat{A} \), parameters \( k, s \), and a properly chosen threshold \( \tau \).

**Algorithm 1: Combinatorial Projection (CProj)**

1. While there are at least \( s/2 \) unclassified nodes, choose an unclassified node \( v_i \) randomly and define \( T_i := \{ u \mid \| \hat{P}_{\hat{A}}^T (\hat{A}_v^T - \hat{A}_u^T) \| \leq \tau \} \). Mark each \( u \in T_i \) as classified.
2. Assign each remaining node to the \( T_i \) with the closest projected \( v_i \).
3. Let \( \hat{c}_i \) be the characteristic vector of \( T_i \).
4. Return \( \hat{P} \hat{c} \), the projection onto the span of the \( \hat{c}_i \).

**Algorithm 2: Mc Sherry’s algorithm**

1. Randomly partition the set \( \{1, \ldots, n\} \) into two parts \( A \) and \( B \). Let \( \hat{A}, \hat{B} \) be the submatrices of the adjacency matrix formed by columns from \( A \) and \( B \).
2. Let \( P_1 = CProj(\hat{B}), P_2 = CProj(\hat{A}) \) and compute \( \hat{H} = [P_1(\hat{A})|P_2(\hat{B})] \).
3. Run Clustering by Distances on the projected points.

Let \( P \) be the probability matrix \( (p_{ij})_{1 \leq i,j \leq n} \). For a vertex \( u \in X \), \( u \) denotes the corresponding column in \( P \). Define

\[
\Delta := \min \| u - v \|,
\]

where the minimum is taken over all pairs \( u, v \) belonging to different clusters. Mc Sherry proved [24]

**Theorem 1.** Assume that \( \sigma^2 \gg \log^6 n/n \) is an upper bound on the variances of the entries. There is a constant \( C > 0 \) such that if

\[
\Delta \geq C \sigma k^{1/2} (\sqrt{\frac{n}{s}} + \sqrt{\log \frac{n}{\epsilon}}),
\]

the above algorithm (with a proper choice of the threshold \( \tau \)) recovers the partition with probability \( 1 - \epsilon \) with respect to the random graph and \( k^{-1} \) with respect to the auxiliary random bits.

The main open question raised by Mc Sherry in [24] is to find a more natural and simpler algorithm, which does not involve the subroutine \( CProj \) (see [24, Section 4.4]). The goal of this paper is to answer this question.

To this end, \( M_k \) denotes the subspace spanned by the first \( k \) left singular vectors of a matrix \( M \). Let \( \hat{P} \) be our input, namely the adjacency matrix of a random graph generated by \( P \). Arguably, the most natural choice for \( \hat{H} \) would be \( \hat{P}_k \) (SVD), which leads to the algorithm below

While SVD I could well win the contest for being the simplest algorithm, it is not easy to analyze in the general case. In what follows, we analyze a slightly more technical alternative, SVD II, which is a variant of an algorithm proposed in [24, Section 1].
Algorithm 3: SVD I

(1) Project the columns of $\hat{P}$ onto $\hat{P}_k$.
(2) Run Clustering by distances on the projected points.

Algorithm 4: SVD II

(0) Randomly partition $X$ into two subsets $Y$ and $Z$. Let $B$ be the adjacency matrix of the bipartite graph between $Y$ and $Z$. Let $Y_1$ be a random subset of $Y$ by selecting each element with probability $1/2$ independently and let $\hat{A}$ be the submatrix of $B$ formed by the columns indexed by $Y_1$.
(1) Project the columns of $B$ indexed by $Y_2 := Y \setminus Y_1$ on $\hat{A}_k$.
(2) Run Clustering by Distances on the projected points.

Compared to SVD I, the extra steps in SVD II are the random partitions in Step (0) done in order to reduce the correlation. (A careful reading of [24] reveals that one also need an extra partition in Algorithm 2 to make the analysis go through.)

Notice that SVD II gives a partition of $Y_2$, not $X$. There are many ways to extend it to a partition of $X$. For instance, we can run the algorithm $l$ times (for some small $l$) and find partitions of $Y_1^1, \ldots, Y_l^2$, where $Y_p^2$ are random subsets of $X$ with density $1/4$ (the input graph is the same, only the random partitions are different). If a cluster $C$ in $Y_p^2$ and a cluster $C'$ in $Y_p^2$ intersect, then they must belong to the same cluster in $X$ and we can merge them. If we choose $l = 3\log n$, say, then with probability $1 - o(n^{-1})$, all vertices of $X$ must belong to some $Y_p^2$ and we recover the clusters $X_1, \ldots, X_k$ at the end. We can also first find the partitions of $Y_1, Y_2$ and $Z_1, Z_2$ by reversing the role of $Y_1$ and $Y_2$ and $Y$ and $Z$ and find which four clusters must belong to an original cluster by looking at the edge densities; we omit the details.

Beside being simple, SVD II is also very convenient to implement, as its main step, the computation of the projection onto $\hat{A}_k$ (given $\hat{A}$ as input) is a routine operation (SVD) which appears in most standard mathematical packages.

Let us now analyze SVD II. For convenience, we assume that $P$ has rank $k$. The general case when $P$ can have a smaller rank is discussed later. Let $\lambda$ be the least non-trivial singular value of $P$.

Theorem 2. There is a constant $C > 0$ such that the following holds. Assume that $\sigma^2 \geq C\log n/n$ and $s \geq C\log n, k = o((n/\log n)^{1/2})$. Then SVD II clusters $Y_2$ correctly with probability $1 - o(n^{-1})$ if one of the following two conditions is satisfied

- **Condition 1.** $\Delta \geq C(\sigma \sqrt{s} + \sqrt{\log n})$.
- **Condition 2.** $\Delta \geq C(\sigma \sqrt{s} + \sigma \sqrt{k \log n} + \frac{\sigma \sqrt{nk}}{\lambda})$

If we omit the assumption $s \geq C\log n$, the statement still holds but with probability $1 - o(n^{-1}) - c \sum_{i=1}^{k} e^{-|X_i|/c}$ for some constant $c$.

Remark 3. We would like to point out a few remarks

- The lower bound $\sigma^2 \geq C\log n/n$ is optimal, up to the value of $C$. If $\sigma^2 < \log n/n$, then there are many isolated points, which can be assigned to any cluster.
We can reduce the failure probability $O(n^{-1})$ to $O(n^{-K})$ for any constant $K$ at the cost of increasing the constant $C$.

Let us now consider the performance of SVD II on various subproblems. We allow the value of $C$ to be flexible in order to omit smaller order terms for convenience. It is instructive to compare the corollaries below with Corollaries 1, 2, 3 from [24].

**Hidden clique.** In this problem, $k = 2$, $s$ is the size of the clique, and $\Delta = (1 - p)\sqrt{s}$, where $p$ is the density of the random graph. Condition 1 becomes

$$(1 - p)s^{1/2} \geq C(p^{1/2} \sqrt{n} + \sqrt{\log n})$$

which is satisfied if $s \geq C(\sqrt{np} + \sqrt{\log n})$. As $np = \Theta(\sigma^2 n) = \Theta(\log n)$, this simplifies to $s \geq C\sqrt{np}$.

**Corollary 4.** There is a constant $C$ such that for any $p \geq C\log n/n$ and $s \geq C\sqrt{np}$, SVD II finds the hidden clique of size $s$ with probability $1 - o(1)$.

**Hidden Coloring.** Here $k$ is the number of color classes, each has size $n/k$; $\Delta = p\sqrt{2n/k}$; $s = n/k$; $\sigma^2 = p(1 - p)$. The singular values of $P$ are $\frac{k}{n}$, $\frac{1}{n}$, $\ldots$, $\frac{1}{n}$. If $p \geq 1/k$, Condition 1 is

$$p\sqrt{n/k} \geq C(p^{1/2} \sqrt{k} + \sqrt{\log n})$$

which is satisfied for $k = o((n/log n)^{1/3})$.

If $p < 1/k$, then the bound $\lambda \geq \sigma\sqrt{n}$ holds, and the $\Delta$ bound in Condition 2 is

$$p\sqrt{n/k} \geq C(\sqrt{pk\log n} + \frac{k\log n}{\sqrt{n}})$$

which is satisfied if $p \geq C\frac{k^{3/2}\log n}{n^{1/3}}$.

**Corollary 5.** There is a constant $C$ such that the following holds. For any $k = o((n/log n)^{1/3}$ and edge density $.99 > p \geq C\frac{k^{3/2}\log n}{n}$, SVD II finds the hidden $k$-coloring with probability $1 - O(n^{-1})$.

**Hidden Bipartition.** Let the two densities be $.99 \geq p > q > 0$. We have $k = 2$, $\Delta = |p - q|n^{1/2}$, $s = n/2$, $\sigma^2 = \Theta(p)$. The two singular values of $P$ are $(p + q)n$ and $(p - q)n$. Condition 2 requires $\frac{p - q}{p + q} \geq C\frac{\log n}{n}$.

**Corollary 6.** There is a constant $C$ such that the following holds Let $.99 > p > q \geq C\log n/n$ be edge densities such that $\frac{p - q}{p^2} \geq C\frac{\log n}{n}$ then SVD II finds the hidden bipartition with probability $1 - o(n^{-1})$.

One can replace $p^{1/4}$ in the denominator by a better term $p^{1/2}$ by considering an approximate algorithm; see Corollary [11]

The rest of the paper is organized as follows. In the next section, we present a few technical lemmas and prove Theorem 2 and Theorem 10 in Section 3. In Section 4, we discuss variants of SVD II, including an approximate version which works under weaker assumptions.
2. Technical lemmas

**Lemma 7** (Projection of a Random Vector). There are constants $C_1, C_2$ such that the following holds. Let $\xi = (\xi_1, \ldots, \xi_n)$ be a random vector in $\mathbb{R}^n$ whose coordinates $\xi_i$ are independent random variables with mean 0 and variance at most $\sigma^2 \leq 1$. Let $H$ be a subspace of dimension $d$ and $\Pi_H \xi$ be the length of the orthogonal projection of $\xi$ onto $H$. Then

$$\mathbb{P}(\Pi_H X \geq \sigma \sqrt{d} + C_1 \sqrt{\log n}) \leq n^{-3}.$$ 

Furthermore, if $H$ has an orthonormal bases $v_1, \ldots, v_d$ such that $\max_{1 \leq i \leq d} \|v_i\|_{\infty} \leq \alpha$, then

$$\mathbb{P}(\Pi_H X \geq C_2 \sqrt{d}(\sigma + \alpha \log n)) \leq n^{-3}.$$ 

We prove this lemma in the appendix.

**Lemma 8** (Norm of a random matrix). There is a constant $C_0 > 0$ such that the following holds. Let $E$ be a symmetric matrix whose upper diagonal entries $e_{ij}$ are independent random variables where $e_{ij} = 1 - p_{ij}$ or $-p_{ij}$ with probabilities $p_{ij}$ and $1 - p_{ij}$, respectively, where $0 \leq p_{ij} \leq 1$. Let $\sigma^2 := \max_{i,j} p_{ij}(1 - p_{ij})$. If $\sigma^2 \geq C_0 \log n/n$, then

$$\mathbb{P}(\|E\| \geq C_0 \sigma n^{1/2}) \leq n^{-3}.$$ 

If $\sigma^2 \geq \frac{\log^2 n}{n}$, the statement is a corollary of [26, Theorem 1.4]. For smaller $\sigma$, one can prove this lemma using the $\epsilon$-net approach by Kahn and Szemeredi [20]. We omit the details, which is very similar to the proof of Feige and Ofek for [14, Theorem 1.1].

**Lemma 9** (Perturbation bound). Let $M, N$ be matrices where $\delta := \lambda_k(M) - \lambda_{k+1}(M) > 0$. Then

$$\sin \angle(M_k, (M + N)_k) \leq \delta^{-1} \|N\|.$$ 

This lemma is a well known result in numerical linear algebra, known as Davis-Kahan-Wedin theorem; see [5, 10, 28, 17].

3. Proof of Theorems

Let $A$ be the probability matrix $p_{ij}$ corresponding to $\hat{A}$. As $A$ is a large random submatrix of $P$, it is not hard to show that $\lambda_k(A) = \Theta(\lambda_k(P))$ with high probability (we provide a verification of this fact at the end of the proof). In the rest of this proof, we assume

$$\lambda_k(A) \geq c_0 \sigma \sqrt{ns},$$

for some constant $c_0 > 0$.

We view the adjacency matrix $\hat{A}$ (between $Y_1$ and $Z$) as a random perturbation of $A$, $\hat{A} := A + E$, where the entries $e_{ij}$ of $E$ are independent and $e_{ij} = 1 - p_{ij}$ with probability $p_{ij}$ and $-p_{ij}$ with probability $1 - p_{ij}$. We denote by $\mathbf{u}, \mathbf{u}, e_u$ the columns corresponding to
a vertex $u$ in $\hat{A}, A, E$, respectively. All matrices are of size approximately $n/2 \times n/4$ by the definitions of $Y, Z$ and $Y_1, Y_2$.

Our leading idea is that the random perturbation $E$ does not change $A_k$ too much, thus hopefully the projections onto $\hat{A}_k$ and $A_k$ differ by only a small amount. The heart of the matter, of course, is to bound this error term. While inviting, a straightforward application of Lemma 9 is too crude in the general case (it does lead to some simple solution for some subproblems in certain range of parameters). We will still make use of this lemma, but for a quite different purpose.

For simplicity, we assume in the rest of the proof that $s \geq C \log n$. For a sufficiently large $C$, this implies that with probability $1 - o(n^{-1})$, each cluster $X_i$ intersects $Z$ in at least $|X_i|/3$ elements. Thus, the distance between two columns (belonging to different clusters) in $A$ is at least $\Delta/3$. We aim to show that with high probability $\| P_{\hat{A}_k} \hat{u} - u \| < \Delta/12$ for all $u \in Y_2$; this will provide a perfect geometric representation. If there is no lower bound on $s$, then the probability that the random partition has this property is at least $1 - c \sum_{i=1}^{k} e^{-|X_i|/c}$ for some constant $c > 0$.

For a fixed $u$, by the triangle inequality

$$\| P_{\hat{A}_k} \hat{u} - u \| \leq \| P_{\hat{A}_k} (\hat{u} - u) \| + \| (P_{\hat{A}_k} - I)u \| = \| P_{\hat{A}_k} e_u \| + \| (P_{\hat{A}_k} - I)u \|.$$ 

To bound the second term, we follow an argument from [24] and consider

$$(P_{\hat{A}_k} - I)A = (P_{\hat{A}_k} - I)\hat{A} - (P_{\hat{A}_k} - I)E.$$ 

The spectral norm of the first term is $\lambda_{k+1}(A_k) \leq \lambda_{k+1}(A) + \| E \| = \| E \|$, as $A$ has rank at most $k$. The spectral norm of the second term is also at most $\| E \|$. Thus, by Lemma 8 by probability at least $1 - n^{-3}$

$$\| (P_{\hat{A}_k} - I)A \| \leq 2 \| E \| \leq C_0 \sigma n^{1/2},$$

for some constant $C_0$.

Let $\chi_u$ be the unit vector $s_u^{-1/2} \mathbb{I}_u$ where $\mathbb{I}_u$ is the indicator vector for the cluster containing $u$, we have

$$\| (P_{\hat{A}_k} - I)A \| \geq \| (P_{\hat{A}_k} - I)A \chi_u \| = s_u^{1/2} \| (P_{\hat{A}_k} - I)u \|.$$ 

Combining the last two inequalities and using the union bound, we conclude that with probability at least $1 - n^{-2}$

$$\| (P_{\hat{A}_k} - I)u \| \leq C_0 \sigma \sqrt{\frac{n}{s_u}},$$ 

for all $u \in X$.

Now we tend to the first term, whose analysis is more involved. By the first part of Lemma 7
\[ \| P_{A_k} e_u \| \leq \sigma k^{1/2} + C_1 \sqrt{\log n} \]

with probability \(1 - o(n^{-2})\), for a properly chosen constant \(C_1\). As \(sk \leq n\), the term \(\sigma k^{1/2}\) is at most \(\sigma \sqrt{n/s}\) and can be omitted. This yields that

\[ \Delta \geq C_0 \sigma \sqrt{n/s} + C_1 \sqrt{\log n} \]

then the algorithm succeeds with probability at least \(1 - o(n^{-1})\). This proves the first part of the theorem concerning Condition 1.

To prove the second part of the theorem, let us reconsider the distance \(P_{A_k} e_u\). Notice that if \(s \leq 10k \log n\), then Condition 2 implies Condition 1 (with some modification on the value of \(C\)). Thus, in what follows, we can assume \(s \geq 10k \log n\).

Rewrite \(\hat{A} = A + E\) and let \(v\) be a singular vector of \(A\). Recall that \(|X_i \cap Z| \geq \frac{\lambda}{s} |X_i| = s_i/3\) for all \(i\). By symmetry, each coordinate in \(v\) is repeated at least \(s/3\) times, thus \(\|v\|_{\infty} \leq 2s^{-1/2}\). Furthermore, by Lemma 9 and Lemma 8 we have with probability \(1 - o(n^{-2})\) that

\[ \sin(A_k, \hat{A}_k) \leq C_0 \frac{\sigma \sqrt{n}}{\lambda} \]

which implies that for any unit vector \(v \in \hat{A}_k\),

\[ \|v\|_{\infty} \leq 2s^{-1/2} + C_0 \frac{\sigma \sqrt{n}}{\lambda} \leq C_0' s^{-1/2} \]

by the condition on \(\lambda\), with some properly chosen constant \(C_0'\). Using the second part of Lemma 11 we conclude that with probability \(1 - o(n^{-2})\), \(\|P_{A_k} e_u\| \leq C'(\sigma k^{1/2} + s^{-1/2} \log n)\) for all \(u\) and some properly chosen constant \(C\), concluding the proof.

For the sake of completeness, let us show that with high probability, the least (non-trivial) singular value of \(P\) and \(A\) are essentially the same, up to a constant factor. We first compare the singular values of \(P\) with the singular values of \(\hat{P}\), the probability matrix of the bipartite graph spanned by \(Y\) and \(X\). Using Chernoff’s bound, one can easily show that with probability at least \(1 - n^{-2}\)

\[ \| X_i \cap Y | - | X_i | / 2 | \leq 5 \sqrt{|X_i|} \log n \]

for all \(1 \leq i \leq k\).

We use the fact that for any matrix \(M\) of rank \(k\) \(\lambda_k(M) = \inf_{\text{rank}(M')=k-1} \|M - M'\|_F\). For simplicity, let us assume for a moment that \(|X_i \cap Y| = |X_i|/2\). Let \(\tilde{P}'\) be the matrix that define \(\lambda_k(\tilde{P})\). We define \(P'\), a rank \((k-1)\) approximation of \(P\), by extending \(\tilde{P}'\) as follows. For the block indexed by \(X_i \setminus Y\), simply copy the block of \(\tilde{P}'\) corresponding to \(X_i \cap Y\). It is trivial that \(P'\) has rank \(k - 1\) and

\[ \| P - P' \|_F^2 = 2 \| \tilde{P} - \tilde{P}' \|_F^2 \]

which implies \(\lambda_k \leq \sqrt{2} \lambda_k(\tilde{P})\). With the same argument, we can compare \(\lambda_k(\tilde{P})\) with \(\lambda_k(B)\) and the later with \(\lambda_k(A)\), each time losing a factor of \(\sqrt{2}\). At the end it would give \(\lambda_k(P) \leq 2^{3/2} \lambda_k(A)\).
To make the argument precise, we need to remove the assumption \( |X_i \cap Y| = |X_i|/2 \). Using [3], we can create a matrix \( P' \) such that

\[
\|P - P'\|_F^2 \leq 2\|\hat{P} - \hat{P}'\|_F^2 + 5 \sum_{i=1}^{k} \sqrt{|X_i|} \log n \sigma^4.
\]

On the other hand, the extra term \( 5 \sum_{i=1}^{k} \sqrt{|X_i|} \log n \sigma^4 \) is less than \( \frac{1}{4} \lambda_k(P)^2 \) by the assumption of the theorem. Thus, we can use the above estimate to get a slightly weaker bound \( \lambda_k(P) \leq 2\lambda_k(\hat{P}) \), completing the proof.

4. Variants

4.1. Dimension and Density. In the case rank \( A = l < k \), it makes more sense to project onto \( A_l \) rather than onto \( A_k \); the rest of the algorithm and analysis remains the same.

If we do not know either \( k \) or \( l \) in advance, we can modify SVD II slightly as follows. The idea is to define the essential rank of \( \hat{A} \) for a properly chosen \( \sigma \). If this information is not known, we can still solve the problem by considering a sequence \( r \) at most \( l \).

Given Theorem 10.

4.2. An approximate Solution. In practice, one is often satisfied with an approximate solution. We say that a partition \( X = \bigcup_{i=1}^{k} X'_i \) is \( \epsilon \)-correct if \( |X_i \setminus X'_i| \leq \epsilon |X_i| \). Similarly, we say that a geometric representation of \( X \) is \( \epsilon \)-perfect if there are points \( x_1, \ldots, x_k \) with distance at least \( 4r \) from each other so that at least \( (1 - \epsilon) |X_i| \) points from \( X_i \) has distance at most \( r \) to \( x_i \). One can use an \( \epsilon \)-perfect representation to find an \( \epsilon \)-correct partition.

**Theorem 10.** Given \( \epsilon > 0 \), there is a constant \( C > 0 \) such that the following holds. If \( \sigma^2 \geq C \frac{\log n}{\epsilon^2} \) and

\[
\Delta \geq C \sigma \sqrt{\frac{n}{s}},
\]

then with probability \( 1 - o(n^{-1}) \) the projection in SVD II produce an \( (1 - \epsilon) \)-perfect representation of the point \( \sin Y_2 \).

It is worth mentioning that in various situations, an \( \epsilon \)-correct partition can be upgraded to a fully correct one by a simple “correction” procedure, as shown in the following example:

**Hidden bipartition.** Assume \( p > q \). Let \( X = X'_1 \cup X'_2 \) be an \( \epsilon \)-correct partition, for some small \( \epsilon \) (say \( \epsilon = .1 \)). Then both \( X'_i \) have size at most \( \frac{1}{2} (1 - \epsilon)n \). Assume \( |X_i \setminus X'_i| \leq \epsilon n/2 \); it follows that \( |X'_i \setminus X_i| \leq \epsilon n/2 \). With probability \( 1 - n^{-2} \), the following holds. For any \( u \in X \), let \( d_u \) be the number of its neighbors in \( X'_1 \). If \( u \in X'_1 \), then...
\[ d_u \geq |X_1 \cap X'_1|p + |X'_1 \setminus X_1|q - 10\sqrt{np \log n} = D_1. \]

On the other hand, if \( u \in X_2 \), then
\[ d_u \leq |X_1 \cap X'_1|q + |X'_1 \setminus X_1|p + 10\sqrt{np \log n} = D_2. \]

It is clear that if \( (p - q) \geq 30\sqrt{p \log n/n} \), then \( D_1 > D_2 \). Thus, one can correct the partition by defining \( X_1 \) be the set of \( n/2 \) vertices \( u \) with largest \( d_u \).

Corollary 11. There is a constant \( C \) such that the following holds: Let \( .99 > p > q \geq C \log n/n \) be edge densities such that \( \frac{p-q}{p^{1/2}} \geq C \sqrt{\log n/n} \), then the approximation algorithm with correction finds the hidden bipartition with probability \( 1 - o(n^{-1}) \).

Proof of Theorem 10. We first bound \( \| P_{A_k} e_u \| \). Recall that
\[ E[\| P_{A_k} e_u \|^2] \leq \sigma^2 k. \]

By Markov’s inequality, it follows that \( P(\| P_{A_k} e_u \| \geq K\sigma^{1/2}) \leq K^{-2} \). We call a vertex \( u \) good if \( \| P_{A_k} e_u \| \leq K\sigma^{1/2} \). For a sufficiently large \( C \) (depending on \( K \)), all good vertices will be clustered correctly. Moreover, choosing \( K \geq 2\epsilon^{-1/2} \), the probability for \( u \) being good is at least \( 1 - \epsilon/4 \), thus the expectation of the number of good elements in \( X_i \) is at least \( |X_i|(1 - \epsilon/4) \). As the good events are independent, Chernoff’s bound implies that with probability \( 1 - n^{-2} \), at least \( |X_i|(1 - \epsilon) \) points from \( X_i \) are good. This completes the proof.

Appendix A. Proof of Lemma 7

Notice that the function \( \Pi_H(X) \) is 1-Lipschitz and convex, thus by Talagrand’s inequality \[ 25 \] for any \( t > 0 \)
\[ P(\Pi_H X \geq \mu + t) \leq 2 \exp(-t^2/4) \]
where \( \mu \) is the mean of \( \Pi_H(X) \). We do not know \( \mu \); however, we can bound from above. Slightly abusing the notation, let \( \Pi := (\pi_{ij}) \) denote the projection matrix onto \( H \), then
\[ E[\| \Pi H X \|^2] = EX^T \Pi X = \sum_{i=1}^n \pi_{ii} E\xi_i^2 \leq \sigma^2 \sum_{i=1}^n \pi_{ii} = d\sigma^2. \]

Combining this with the concentration inequality, it is not hard to show that \( \mu \leq \sigma d^{1/2} + O(1) \), concluding the proof of the first part of the lemma.

The second part follows immediately from

Claim 12. Let \( (a_1, \ldots, a_n) \) be real numbers such that \( \sum_i a_i^2 = 1 \) and \( |a_i| \leq \alpha \) for all \( i \). Let \( \xi_i \) be independent random variables with mean 0 and \( E[\xi_i]^k \leq \sigma^2 \) for all \( k \geq 2 \). Let \( S := \sum_{i=1}^n a_i \xi_i \). Then
\[ P(|S| \geq 4(\sigma \sqrt{\log n} + \alpha \log n) \leq 2n^{-3}. \]

To prove Claim 12, notice that for any \( 0 < t \leq \alpha^{-1} \) we have

\[
E \exp(tS) = \prod_i E \exp(ta_i \xi_i) = \prod_i \left( 1 + \frac{a_i^2 t^2}{2!} + \frac{a_i^3 t^3 \xi_i^3}{6!} + \ldots \right)
\]

Since \( E\xi_i^k \leq \sigma^2 \) for all \( k \geq 2 \) and \( t|a_i| \leq 1 \), the right most formula is

\[
\leq \prod_i \left( 1 + \sigma^2 t^2 a_i^2 \right) \leq \exp(\sigma^2 t^2). \]

Markov’s inequality yields

\[
P(S \geq T) \leq \exp(-tT + t^2 \sigma^2). \]

To optimize the RHS, let us consider two cases

**Case 1.** \( \sigma \geq \alpha \sqrt{\log n} \). Take \( T = 4\sigma \sqrt{\log n} \) and \( t = \frac{\sqrt{\log n}}{\sigma} \leq \alpha^{-1} \). With this setting, \(-tT + t^2 \sigma^2 = -3 \log n\). 

**Case 2.** \( \sigma < \alpha \sqrt{\log n} \). Take \( T = 4\alpha \log n \) and \( t = \alpha^{-1} \). In this setting, \(-tT + t^2 \sigma^2 \leq -4 \log n + \log n = -3 \log n\).

One can bound \( P(-S \leq T) \) the same way.

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**References**


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