Gallai graphs and anti-Gallai graphs

Van Bang Le*
Fachbereich Mathematik, Technische Universität Berlin, Germany

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Abstract

The Gallai graph and the anti-Gallai graph of a graph $G$ have the edges of $G$ as their vertices. Two edges of $G$ are adjacent in the Gallai graph of $G$ if they are incident but do not span a triangle in $G$; they are adjacent in the anti-Gallai graph of $G$ if they span a triangle in $G$. In this paper we show: The Four Color Theorem can be equivalently stated in terms of anti-Gallai graphs; the problems of determining the clique number, and the chromatic number of a Gallai graph are NP-complete. Furthermore, we discuss the relation of Gallai graphs to the theory of perfect graphs. A characterization of Gallai graphs and anti-Gallai graphs is also given.

1. Introduction

By definition, the Gallai graph $\Gamma(G)$ of a graph $G$ is the graph whose vertex-set is the edge-set of $G$; two distinct edges of $G$ are adjacent in $\Gamma(G)$ if they are incident in $G$, but do not span a triangle in $G$. This construction was used by Gallai [8] in his investigation of comparability graphs; the notation was suggested by Sun [20]. Sun used Gallai graphs to describe a nice class of perfect graphs; Gallai graphs are also used in the polynomial time algorithm to recognize $K_{1,3}$-free perfect graphs by Chvátal and Sbihi [4].

Notice that $\Gamma(G)$ is a spanning subgraph of the well-known line graph $L(G)$ of $G$. The anti-Gallai graph $\Delta(G)$ of the graph $G$ is the complement of $\Gamma(G)$ in $L(G)$; that is, the spanning subgraph of $L(G)$ formed by the edges outside $\Gamma(G)$. Thus, $\Delta(G)$ has the edges of $G$ as its vertices; two edges are adjacent in $\Delta(G)$ if they span a triangle in $G$.

In this paper we shall investigate Gallai graphs and anti-Gallai graphs in their own right. In Section 2, a link between anti-Gallai graphs and the Four Color Theorem (4CT) is established; it is shown that the 4CT is equivalent to 'For every planar graph $G$, the chromatic number and the clique number of $\Delta(G)$ are equal'. In Section 3 we show that the computing of the chromatic number and the clique number of a Gallai graph are NP-complete problems. The results in Sections 2 and 3 have been proved

* Current address: Fachbereich Informatik, Universität Rostock, Germany. le@informatik.uni-rostock.de
based on the facts that the chromatic number of $\Delta(G)$, respectively, $\Gamma(G)$, cannot exceed the chromatic number of $G$, respectively, $\overline{G}$. In Section 4 we discuss the role of Gallai graphs in the theory of perfect graphs. In particular, we conjecture that perfect graphs can be characterized in terms of Gallai graphs; our conjecture is weaker than the Perfect Graph Conjecture (PGC) and stronger than the Perfect Graph Theorem (PGT). The last section contains a characterization of Gallai graphs and anti-Gallai graphs similar to that of Krausz for line graphs.

All graphs considered are undirected, have no multiple edges or loops, but are not necessarily finite.

Let $H$ be a subgraph of a graph $G$. The vertex-set and the edge-set of $H$ are denoted by $V(H)$ and $E(H)$, respectively. If $V(H) = V(G)$, then $H$ is called a spanning subgraph of $G$. The edge joining two vertices $x$ and $y$ will simply be denoted by $xy$.

A chord of $H$ is an edge in $E(G) \setminus E(H)$ joining two vertices in $H$; a short chord of $H$ is a chord joining two vertices distance 2 apart in $H$. If $H$ has no chord, then $H$ is an induced subgraph of $G$.

Sometimes, we shall identify an induced subgraph with its vertex-set, and a set of vertices with the subgraph induced by that set.

Graphs not containing an induced subgraph isomorphic to a given graph $H$ are called $H$-free.

Let $G$ and $G'$ be two disjoint graphs. The union $G \cup G'$ is simply the graph with vertex-set $V(G) \cup V(G')$ and edge-set $E(G) \cup E(G')$. The join $G \ast G'$ is the graph with vertex-set $V(G) \cup V(G')$ and edge-set $E(G) \cup E(G') \cup \{vv' | v \in V(G), v' \in V(G')\}$. The complement of $G$ is denoted by $\overline{G}$.

Let $a$ be a cardinal number, and let $n$ be a natural number. Then $K_a$ and $K_{1,a}$ denote the complete graph with $a$ vertices, respectively, the complete bipartite graph having the bipartition in 1-element and $a$-element sets; $K_{1,a}$ is also called an $a$-star (or, briefly, star), $K_3$ is also called a triangle. $C_n$ $(n \geq 3)$ denotes the cycle with $n$ vertices. An odd hole in a graph is an induced cycle $C_n$ of odd length $n \geq 5$.

The chromatic number $\chi(G)$ of the graph $G$ is the smallest cardinality of a family of disjoint independent sets partitioning $V(G)$, and the clique number $\omega(G)$ is the supremum of all natural number $k$ such that $G$ contains a complete subgraph with $k$ vertices. $\omega(G) := \omega(\overline{G})$ is called the independence number of $G$. It is immediate that $\chi(G) \geq \omega(G)$.

The following basic properties of Gallai graphs and anti-Gallai graphs can be obtained easily from the definition; we often use these facts without reference.

**Proposition 1.1.** (i) Let $H$ be a subgraph of the graph $G$. If $H$ has no short chord, then $\Gamma(H)$ is an induced subgraph of $\Gamma(G)$ and $\Delta(H)$ is an induced subgraph of $\Delta(G)$.

(ii) Every $K_a$ in $\Gamma(G)$ stems from an induced star $K_{1,a}$ in $G$; every $K_a$ ($a > 3$) in $\Delta(G)$ stems from a $K_{a+1}$ in $G$.

(iii) For all graphs $G$, $\Gamma(G \ast K_1) \cong \Gamma(G) \cup \overline{G}$ and $G$ is an induced subgraph of $\Delta(G \ast K_1)$. 
2. Anti-Gallai graphs and the 4CT

We begin by giving a formula for the parameter $\omega(A(G))$ in terms of that of $G$. If $Q$ is a complete subgraph in $G$, then the $|Q|-1$ edges in $Q$ having a common endvertex form a complete subgraph in $A(G)$. Thus, $\omega(A(G)) \geq \omega(G)-1$ or $\omega(A(G)) = \omega(G) = 3$. Conversely, if $F \subseteq E(G)$ forms a complete subgraph in $A(G)$, then all edges in $F$ must have a common endvertex (and, therefore, the endvertices of all edges in $F$ induce a complete subgraph with $|F|+1$ vertices in $G$), or else $F$ consists of exactly three edges forming a triangle in $G$. Thus, either $\omega(G) \geq \omega(A(G)) + 1$ or $\omega(G) \geq \omega(A(G)) = 3$. Hence we get

Proposition 2.1. For every graph $G$ with $\omega(G) \geq 1$,

$$\omega(A(G)) = \begin{cases} \omega(G) - 1 & \text{if } \omega(G) \neq 3, \\ 3 & \text{if } \omega(G) = 3. \end{cases}$$

Let $A^0(G)$ denote $G$ and for every integer $t > 1$, $A^t(G)$ denotes $A(A^{t-1}(G))$. Iterated anti-Gallai graphs are investigated in [12, 19].

Corollary 2.2. For every graph $G$ with finite clique number, $A^{\omega(G)-1}(G)$ has no edges or is the union vertex-disjoint triangles.

Proof. By induction on $\omega(G)$. The conclusion is obvious in case $\omega(G) \leq 2$. If $\omega(G) = 3$, then two distinct triangles in $A(G)$ are edge-disjoint and therefore $A^2(G)$ is the union of vertex-disjoint triangles. Let $4 \leq \omega(G) < \infty$. By Proposition 2.1, $\omega(A(G)) = \omega(G) - 1$, hence by induction

$$A^{\omega(G)-1}(G) = A^{\omega(G)-2}(A(G)) = A^{\omega(A(G))-1}(A(G))$$

is the union of vertex-disjoint triangles.

Corollary 2.2 implies that all graphs of bounded degree are $A$-convergent generalizing the result of Jarett [12] (see also [19]).

The next proposition shows that $\chi(A(G))$ cannot exceed $\chi(G)$; the idea is due to Albertson and Collins [1]. Actually, in [1] they investigated the edge-clique graph $K(G)$ of a (finite) graph $G$, which is defined as the graph whose vertices are edges of $G$; two edges of $G$ are adjacent in $K(G)$ if they are contained in a common clique in $G$ (thus, $A(G)$ is a spanning subgraph of $K(G)$.) Albertson and Collins then proved that $\chi(K(G))$ cannot exceed $\chi(G)$ provided $G$ is $K_4$-free. Notice, however, that Proposition 2.1 and Corollary 2.2, as well as the Proposition 2.3 below do not hold true for $K(G)$.

Proposition 2.3. For every graph $G$ with finite chromatic number, $\chi(A(G)) \leq \chi(G)$. Moreover, if $\chi(G)$ is even, then $\chi(A(G)) \leq \chi(G) - 1$. 
Proof. (1) First, it is well known that the chromatic index of every complete graph with \( m \) vertices is \( m \), respectively, \( m - 1 \), if \( m \) is odd, respectively, \( m \) is even [2, Ch. 12, Theorem 1]. That is, every complete graph with \( m \) vertices has an edge-coloring with \( m \) colors such that distinct colors are assigned to incident edges; if \( m \) is even, then \( m - 1 \) colors suffice. For such an edge-coloring of a complete graph on the vertex-set \( \{1, 2, \ldots, m\} \) let \( c(i, j) \) denote the color of the edge \( ij \). Thus, \( c(i, j) \neq c(i, k) \) if and only if \( j \neq k \).

(2) Now, let \( G \) be a graph with \( m = \chi(G) < \infty \). Consider a vertex-coloring \( f : V(G) \rightarrow \{1, 2, \ldots, m\} \) of \( G \), and an edge-coloring \( c \) of the complete graph on vertex-set \( \{1, 2, \ldots, m\} \) as in (1). Then the map \( g : E(G) \rightarrow \{1, 2, \ldots, m\} \), \( g(xy) = c(f(x), f(y)) \) for every edge \( xy \in E(G) \), is a vertex-coloring of \( \Delta(G) \): If two edges \( xy, yz \) span a triangle in \( G \), then \( f(x), f(y) \) and \( f(z) \) are pairwise different, hence \( g(xy) = c(f(x), f(y)) \neq c(f(y), f(z)) = g(yz) \). Thus, adjacent vertices in \( \Delta(G) \) are distinctly colored by \( g \). □

Proposition 2.4. The following statements are equivalent.

(i) For all planar graphs \( G \), \( \chi(G) \leq 4 \) (the Four Color Theorem).

(ii) For all planar graphs \( G \), \( \chi(\Delta(G)) = \omega(\Delta(G)) \).

Proof. Let \( G \) be a 4-colorable graph (not necessarily planar). If \( \omega(G) \leq 2 \), then \( \Delta(G) \) has no edges; hence \( \chi(\Delta(G)) = \omega(\Delta(G)) \). In the other cases either \( 3 = \omega(G) \leq \chi(G) \leq 4 \) or \( \omega(G) = \chi(G) = 4 \), Propositions 2.1 and 2.3 together yield \( \chi(\Delta(G)) = \omega(\Delta(G)) \) again. Thus, the implication (i) \( \Rightarrow \) (ii) is, in particular, proved.

Now, let \( G \) be a finite planar graph satisfying (ii). To prove (i), we may assume that \( G \) is maximal planar [13]. Since \( \omega(G) \) is 3 or 4, \( \omega(\Delta(G)) = 3 \) by Proposition 2.1. By hypothesis (ii), \( \chi(\Delta(G)) = 3 \). That is, the edges of \( G \) can be colored with 3 colors such that the edges in any triangle receive different colors. Tait (see [13]) proved that the existence of such an edge-coloring of a maximal planar graph \( G \) implies \( \chi(G) \leq 4 \). In the infinite case, (i) follows from the finite case and a theorem of De Bruijn and Erdős (see [10, Kapitel 6, Satz 2.4]). □

3. Calculating the parameters \( \chi \) and \( \omega \) of Gallai graphs

It is well known that the problems of determining the independence number, the clique number and the chromatic number of a finite graph are NP-complete. For line graphs, there are polynomial-time algorithms that solve the first two problems; the last remains NP-complete for line graphs [11]. We shall show in this section that the last two problems remain NP-complete for the class of Gallai graphs. For background information on the theory of NP-completeness we refer to Garey and Johnson [9].

We start by bounding the parameters of \( \Gamma(G) \) in terms of the parameters of \( G \). First, since every \( K_n \) in \( \Gamma(G) \) stems from an induced \( K_{1,n} \) in \( G \), we have \( \omega(\Gamma(G)) \leq \omega(\overline{G}) \).
Next, we modify the proof of Proposition 2.3 to obtain \( \chi(\overline{G}) \) as an upper bound for \( \chi(\Gamma(G)) \).

**Proposition 3.1.** For any graph \( G \) whose complement \( \overline{G} \) has finite chromatic number, 
\[ \chi(\Gamma(G)) \leq \chi(\overline{G}). \]

**Proof.** Let \( G \) be a graph such that \( \chi(\overline{G}) \) is finite. Set \( m = \chi(\overline{G}) \), and let \( c \) be an edge-coloring of the complete graph on the vertex-set \( \{1, 2, \ldots, m\} \) as in the proof of Proposition 2.3. Consider a vertex-coloring \( f : V(\overline{G}) \rightarrow \{1, 2, \ldots, m\} \) of the complement \( \overline{G} \). We discuss two cases.

**Case 1:** \( m \) is even. Recall that in this case, \( c \) used \( m - 1 \) colors, say \( 1, 2, \ldots, m - 1 \). For each edge \( xy \) of \( G \), set \( g(xy) = c(f(x), f(y)) \), if \( f(x) \neq f(y) \), and \( g(xy) = m \), otherwise. Then \( g \) is a vertex-coloring of \( \Gamma(G) \) with at most \( m = \chi(\overline{G}) \) colors. Let, namely, \( xy, yz \) be two incident edges such that \( x \) and \( z \) are nonadjacent. Then \( f(x) \neq f(z) \), hence \( g(xy) = c(f(x), f(y)) \) or \( g(yz) = c(f(y), f(z)) \) implying \( g(xy) \neq g(yz) \). This means that two adjacent vertices in \( \Gamma(G) \) are distinctly colored by \( g \).

**Case 2:** \( m \) is odd. In this case, \( c \) used \( m \) colors. For each vertex \( i \) of the complete graph with vertex-set \( \{1, 2, \ldots, m\} \), the \( m - 1 \) edges having the common endvertex \( i \) received \( m - 1 \) colors by \( c \). Hence, there exists an unique color \( c_i \) that is not used for the \( m - 1 \) edges at the vertex \( i \) of that complete graph. Thus, we color the edges of \( G \) with at most \( m = \chi(\overline{G}) \) colors as follows. For each edge \( xy \) of \( G \), set \( g(xy) = c(f(x), f(y)) \), if \( f(x) \neq f(y) \), and \( g(xy) = c_i \), if \( f(x) = f(y) = i \). Then \( g \) is indeed a vertex-coloring of \( \Gamma(G) \): If two vertices \( xy, yz \) of \( \Gamma(G) \) are adjacent, then \( x \) and \( z \) are nonadjacent in \( G \), hence \( f(x) \neq f(z) \). Thus, \( g(xy) = c(f(x), f(y)) \) or \( g(yz) = c(f(y), f(z)) \) implying \( g(xy) \neq g(yz) \) by definition of \( c \); else, say, \( g(xy) = c(f(x), f(y)) \) and \( g(yz) = c_i, i = f(y) = f(z) \). Then \( g(xy) \neq g(yz) \) by definitions of \( c \) and \( c_i \).

In any case, we have produced a vertex-coloring of \( \Gamma(G) \) with at most \( \chi(\overline{G}) \) colors, and Proposition 3.1 is completely proved. \( \square \)

**Corollary 3.2.** The problems of determining the clique number and the chromatic number of a Gallai graph are NP-complete.

**Proof.** Let \( G \) be an arbitrary, finite graph. We know from Proposition 1.1 (iii) that
\[ \Gamma(G * K_1) \approx \Gamma(G) \cup \overline{G}. \] (1)
As we have seen at the beginning of this section, \( \omega(\Gamma(G)) \leq \omega(\overline{G}) \), hence (1) gives
\[ \omega(\Gamma(G * K_1)) = \omega(\overline{G}). \] (2)
Finally, (1) and Proposition 3.1 together yield
\[ \chi(\Gamma(G * K_1)) = \chi(\overline{G}). \] (3)
Note that the Gallai graph of a given graph can be constructed in polynomial time. Thus, Eqs. (2) and (3) give a polynomial reduction of the problems of determining
the clique number and the chromatic number to the special case of the problems for
Gallai graphs. Since we know [9] that the general problems are NP-complete, we have
the results. □

4. Gallai graphs and the PGC

By definition, a graph $G$ is perfect if, for each induced subgraph $G'$ of $G$, the
chromatic number $\chi(G')$ equals the clique number $\omega(G')$. In this section, all graphs
considered are finite.

The notion of perfect graphs has been introduced by Berge; he proposed the follow-
ing two conjectures:

A graph is perfect if and only if its complement is perfect. \hspace{1cm} (4)
A graph is perfect if and only if it has no
odd hole and no complement of an odd hole. \hspace{1cm} (5)

The first of these conjectures has been proved by Lovász and is known as the
perfect graph theorem (PGT); the second, known as the perfect graph conjecture (PGC),
remains open. For more information on the theory of perfect graphs we refer to Berge
and Chvátal [3].

The PGC has been shown to hold for several graph classes; including $K_{1,3}$-free graphs
by Parthasarathy and Ravindra [18] and $K_4$-free graphs by Tucker [22]. However, the
PGC is still open for the class of Gallai graphs.

Let $G$ be a graph and assume that $\Gamma(G)$ has a complement $\overline{C}$ (as an induced
subgraph) of an odd hole $C = x_1x_2 \ldots x_{2m+1}x_1$. Let $e_i$ be the edges in $G$ corresponding
to the $x_i$. If $m \geq 3$, then $x_1x_3x_5$ and $x_1x_4x_6$ are triangles in $\Gamma(G)$. Hence, $e_1,e_3,e_4$ and
e_6 form a star at a vertex $v$ in $G$. Since, in $\overline{C} \subseteq \Gamma(G)$, every vertex $x_i$ is adjacent to
at least two vertices in $\{x_1,x_3,x_4,x_6\}$, all the edges $e_i$ must have the vertex $v$ as an
endvertex. Therefore, the subgraph of $G$ induced by the endvertices of the edges $e_i$ is
isomorphic to $C \ast K_1$; hence $\Gamma(G)$ also has an odd hole isomorphic to $C$. Thus, if the
PGC is true, then Gallai graphs having no odd hole must be perfect.

Question 4.1. Are Gallai graphs without odd holes perfect?

For Gallai graphs of certain graphs, a positive answer to Question 4.1 is known.
Gallai graphs of triangle-free graphs are line graphs and Gallai graphs of $K_{1,4}$-free
graphs are $K_4$-free. Thus, [21,22] give a positive answer to the Question 4.1 in these
cases. Another positive case is the following.

Proposition 4.2. The PGC is true for Gallai graphs of graphs having a 4-colorable
complement.
Proof. Let $G$ be a graph such that $\chi(\overline{G}) \leq 4$ and $\Gamma(G)$ has no odd hole. Consider an induced subgraph $H$ of $\Gamma(G)$. If $H$ is $K_4$-free, then by the result of Tucker [22], $\omega(H) = \chi(H)$. If $\omega(H) \geq 4$, then by Proposition 3.1

$$4 \leq \omega(H) \leq \chi(H) \leq \chi(\Gamma(G)) \leq \chi(\overline{G}) \leq 4,$$

showing $\omega(H) = \chi(H)$. 

Notice that Sun has proved a necessary condition for Gallai graphs having no odd hole: if $\Gamma(G)$ has no odd hole, then $G$ is perfect. On the other hand, a positive answer to Question 4.1 would yield a very simple proof for Sun’s result. Namely, let $G$ be a graph such that $\Gamma(G)$ has no odd hole. Then every cycle of odd length at least five in $G$ must have a short chord. This implies immediately that $\overline{G}$ has no odd hole. (Indeed, if $v_1v_2 \cdots v_{2m+1}v_1$ ($m \geq 2$) is an odd hole in $\overline{G}$, then, in $G$, the cycle $v_1v_{m+2}v_2v_{m+3} \cdots v_{m}v_{2m+1}v_{m+1}v_1$ has no short chord.) Then $\Gamma(G \ast K_1) \approx \Gamma(G) \cup \overline{G}$ also has no odd hole, therefore, assuming a positive answer to Question 4.1, is perfect. Thus, $\overline{G}$ and by the PGT, $G$ is perfect.

In contrast to Gallai graphs, the validity of the PGC is easy to see for anti-Gallai graphs.

**Theorem 4.3.** Anti-Gallai graphs without odd holes are perfect.

Proof. Let $G$ be a graph such that $\Delta(G)$ has no odd hole. Then $\Delta(G)$ is $K_4$-free; otherwise $G$ must contain a $K_5$ (by Proposition 2.1), which yields an odd hole of length 5 in $\Delta(G)$. Therefore, $\Delta(G)$ cannot contain a complement of an odd hole of length at least 9. If $\Delta(G)$ has an induced $C_7$, then it is routine to check that $G$ has an induced subgraph isomorphic to $C_7 \ast K_1$, which yields an odd hole of length 7 in $\Delta(G)$. Thus, $\Delta(G)$ has no odd hole, no complement of an odd hole and no $K_4$. Tucker [22] proved that such graphs are perfect. 

We now are going to give a new conjecture between (4) and (5), which will make clear the role of Gallai graphs in the theory of perfect graphs. Note first, that if $G$ is perfect, then, for every induced subgraph $G'$ of $G$, $\chi(G') = \omega(G')$ and $\chi(\overline{G'}) = \omega(G')$; therefore by Proposition 3.1, $\chi(\Gamma(G')) \leq \omega(G')$ and $\chi(\Gamma(\overline{G'})) \leq \omega(G')$. We believe that these conditions imply, conversely, the perfectness of $G$. The conjecture interposed between (4) and (5) is

A graph $G$ is perfect if and only if $\chi(\Gamma(G')) \leq \omega(G')$ and $\chi(\Gamma(\overline{G'})) \leq \omega(G')$ for each induced subgraph $G'$ of $G$. \hspace{1cm} (4')

Trivially, (4') implies (4). To see that (5) implies (4'), it will suffice to show that if $G$ is an odd hole or a complement of an odd hole, then $\chi(\Gamma(G)) > \omega(G)$ or $\chi(\overline{G}) > \omega(\overline{G})$. Indeed, let $G$ be an odd hole with vertices $v_1, v_2, \ldots, v_{2m+1}$ ($m \geq 2$) and edges $v_1v_2, v_kv_{k+1}$ ($k = 1, \ldots, 2m$). Then in $\overline{G}$, the cycle $v_1v_{m+2}v_2v_{m+3} \cdots v_{m}v_{2m+1}v_{m+1}v_1$ has no short chord; hence $\overline{G}$ has an odd hole showing $\chi(\overline{G}) \geq 3 > \omega(\overline{G})$. 


Similary, if $G$ is a complement of an odd hole, then $G$ has a cycle of odd length at least five without short chord. Then $\chi(\Gamma(G)) \geq 3 > \alpha(G) = 2$. Thus, (5) implies (4').

There is another connection between Gallai graphs and perfect graphs. A minimal imperfect graph is an imperfect graph, in which all proper induced subgraphs are perfect. Clearly, the PGC is true if and only if every minimal imperfect graph is an odd hole or a complement of an odd hole. The latter has an equivalent formulation in terms of Gallai graphs:

**Proposition 4.4.** The following statements are equivalent for every minimal imperfect graph $G$.

(i) $G$ or $G$ is an odd hole (the perfect graph conjecture).

(ii) $\Gamma(G)$ and $\Gamma(\overline{G})$ are triangle-free.

(iii) $\Gamma(G)$ or $\Gamma(\overline{G})$ is triangle-free.

**Proof.** Note that by Proposition 1.1, every complete graph in $\Gamma(G)$ stems from an induced star in $G$. If $G$ or $\overline{G}$ is an odd hole, then $G$ and $\overline{G}$ are $K_{1,3}$-free, hence $\Gamma(G)$ and $\Gamma(\overline{G})$ cannot have a triangle. Thus, the implication (i) $\Rightarrow$ (ii) follows. (ii) $\Rightarrow$ (iii) is trivial. Let $G$ be a minimal imperfect graph satisfying (iii). Then $G$ or $\overline{G}$ is $K_{1,3}$-free. It follows from the result of Parthasarathy and Ravindra [18] that $G$ or $\overline{G}$ is an odd hole; we get (i). □

5. A Krausz-type characterization for Gallai graphs and anti-Gallai graphs

The relation of edges (2-simplices) being incident admits a natural generalization of line graphs, Gallai graphs and anti-Gallai graphs. Accordingly, the $k$-line graph $L_k(G)$, the $k$-Gallai graph $\Gamma_k(G)$ and the anti-$k$-Gallai $\Delta_k(G)$ of a graph $G$ have the $k$-simplices of $G$ as vertices. Two $k$-simplices of $G$ are adjacent in $L_k(G)$ if they have in $G$ $k - 1$ vertices in common; they are adjacent in $\Gamma_k(G)$ if they in addition do not span a $k+1$-simplex in $G$ and they are adjacent in $\Delta_k(G)$ if they in addition span a $k+1$-simplex in $G$. Notice that the 2-line graphs, the 2-Gallai graphs and the anti-2-Gallai graphs are exactly the line graphs, the Gallai graphs and the anti-Gallai graphs, respectively, and $\Delta_k(G)$ is the complement of $\Gamma_k(G)$ in $L_k(G)$. The iteration behavior of $k$-Gallai graphs and anti-$k$-Gallai graphs are investigated in [15,19]. $k$-line graphs are investigated in the literature under the names $k$th interchange graphs [7] and $K_k$-intersection graphs [5,6]. A Krausz-type characterization for $k$-line graphs is communicated in [16]. The papers [14,17] contain results concerning the perfection and iteration behavior of $k$-line graphs.

In this section we shall give a characterization for $k$-Gallai graphs and anti-$k$-Gallai graphs. Notice that for each integer $k \geq 2$, every given graph $G$ is (isomorphic to) a component of $\Gamma_k(\overline{G} \ast K_{k-1})$, and is an induced subgraph of $\Delta_k(G \ast K_{k-1})$. Thus, there is no characterization for $k$-Gallai graphs and anti-$k$-Gallai graphs by forbidden induced subgraphs.
Our characterization, admittedly, leaves much to be desired. Thus, it is perhaps important to note that no better ones are known. However, the complexity of recognizing Gallai graphs (and anti-Gallai graphs) still remains open.

Let $k$ be a natural number. A system of sets has the \textit{$k$-Helly property} if for any $k$ pairwise intersecting sets the total intersection of these sets is nonempty. The following theorem holds for all graphs, finite or not.

**Theorem 5.1.** Let $k > 1$ be an integer. A graph $G$ is a $k$-Gallai graph if and only if there is a family $\{G_i \mid i \in I\}$ of induced subgraphs of $G$ satisfying the following properties.

(i) Every vertex of $G$ lies in exactly $k$ members of the family.

(ii) Every edge of $G$ lies in exactly $k - 1$ members of the family.

(iii) $|\bigcap_{i \in J} G_i| \leq 1$ for any $k$-element subset $J$ of $I$.

(iv) The family has the $k$-Helly property.

(v) For any $k - 1$-element subset $A$ of $I$ and any $i \neq j \in I \setminus A$ and vertices $x \neq y$ of $G$ with

$$\{x\} = G_i \cap \bigcap_{a \in A} G_a \quad \text{and} \quad \{y\} = G_j \cap \bigcap_{a \in A} G_a,$$

the following holds:

$$xy \in E(G) \quad \text{if and only if} \quad G_i \cap G_j = \emptyset. \quad (6)$$

In (iii)-(v) we have identified graphs with their vertex-sets. For $k = 2$, the case of Gallai graphs, the condition (iv) is trivially satisfied, and (v) reads simply as follows: For three members $G_i, G_j, G_a$ of the family with $\{x\} = G_i \cap G_a$ and $\{y\} = G_j \cap G_a$, the following holds: $xy \in E(G) \iff G_i \cap G_j = \emptyset$.

Theorem 5.1 also gives a characterization for anti-$k$-Gallai graphs, when in (v), (6) is replaced by

$$xy \in E(G) \quad \text{if and only if} \quad G_i \cap G_j \neq \emptyset. \quad (6')$$

**Proof of Theorem 5.1.** Necessity: Assume that $G = \Gamma_k(H)$ for some graph $H$. Abbreviate $I = V(H)$. For a vertex $v \in I$, let $S_v$ be the set of all $k$-simplices of $H$ containing the vertex $v$, and let $G_v$ denote the induced subgraph of $G$ with vertex-set $S_v$. We claim that the family $\{G_v \mid v \in I\}$ satisfies the properties listed in the theorem.

If $S$ is any $k$-simplex of $H$, then obviously

$$\{S\} = \bigcap_{v \in S} S_v,$$

that is, the vertex $S$ of $G$ lies in exactly $|S| = k$ members $G_v$, $v \in S$, hence (i). If $S$ and $T$ are two $k$-simplices of $H$ such that $|S \cap T| = k - 1$, then both vertices $S$ and $T$ of $G$ lie in exactly $k - 1$ common members $G_v$, $v \in S \cap T$. Moreover, if $S$ and $T$ are additionally adjacent in $G$, then the edge $ST$ belongs to exactly $k - 1$ induced
subgraphs $G_v$, $v \in S \cap T$; we get (ii). To see (iii), let $J$ be a subset of $I$, and let $S,T$ be two members in $\bigcap_{v \in J} S_v$. Then $J \subseteq S$, $J \subseteq T$. Thus, if $|J| = |S| = |T| = k$, then $S = T$. We now prove (iv). Let $J$ be a subset of $I$ with $|J| = k$. Assume that $S_v \cap S_w \neq \emptyset$ for all vertices $v,w \in J$. Then $v$ and $w$ are adjacent in $H$, so $J$ induces a $k$-simplex, showing $\{J\} = \bigcap_{v \in J} S_v$, that is (iv). Finally, let $A$ be a subset of $I$ with $|A| = k - 1$, let $v \neq w \in I \setminus A$, and let $S, T$ be two $k$-simplices of $H$ such that

$$\{S\} = S_v \cap \bigcap_{u \in A} S_u \quad \text{and} \quad \{T\} = S_w \cap \bigcap_{u \in A} S_u.$$  

Then $S = A \cup \{v\}$, $T = A \cup \{w\}$. Thus, $S$ and $T$ are adjacent in $G$ if and only if $v$ and $w$ are nonadjacent in $H$. The latter is equivalent to $S_v \cap S_w = \emptyset$, and thus (v) is proved.

Sufficiency: Let $\{G_i \mid i \in I\}$ be a family of induced subgraphs of $G$ as in the theorem. Let $H$ be the intersection graph of that family, that is, $V(H) = I$, and for $i \neq j \in I$, $i$ and $j$ are adjacent in $H$ if $G_i \cap G_j \neq \emptyset$. Let $\varphi$ be the map from $V(G)$ to $V(\Gamma_k(H))$ defined as follows: if $x$ is the vertex of $G$ belonging to the $k$ members $G_{i_1}, \ldots, G_{i_k}$, then $\varphi(x)$ is the $k$-simplex $\{i_1, \ldots, i_k\}$ of $H$.

Then, by (i) and (iii), $\varphi$ is well-defined and injective. By definition of $H$, (iv) and (iii), $\varphi$ is surjective. We claim that $\varphi$ is an isomorphism between $G$ and $\Gamma_k(H)$. Indeed, let $x, y$ be adjacent vertices in $G$. By (ii), there exists a $(k - 1)$-element subset $A$ of $I$ such that the edge $xy$ belongs to $G_a$, $a \in A$. By (v), there exist $i \neq j \in I \setminus A$ such that $x$ belongs to $G_i \cap \bigcap_{a \in A} G_a$, and $y$ belongs to $G_j \cap \bigcap_{a \in A} G_a$. By (v), $G_i \cap G_j = \emptyset$ saying $i$ and $j$ are nonadjacent in $H$. Then $\varphi(x) = A \cup \{i\}$ and $\varphi(y) = A \cup \{j\}$ are adjacent in $\Gamma_k(H)$. Conversely, let $x, y$ be two vertices in $G$ such that $\varphi(x)$ and $\varphi(y)$ are adjacent in $\Gamma_k(H)$; say

$$\varphi(x) = A \cup \{i\}, \quad \varphi(y) = A \cup \{j\}$$

for some $(k - 1)$-element subset $A$ of $I$, and some $i \neq j \in I \setminus A$, and such that $i$ and $j$ are nonadjacent in $H$. By definition of $\varphi$, $x$ belongs to $G_i \cap \bigcap_{a \in A} G_a$; $y$ belongs to $G_j \cap \bigcap_{a \in A} G_a$. Since $i, j$ are nonadjacent in $H$, $G_i \cap G_j = \emptyset$. Thus by (v), $x$ and $y$ are adjacent in $G$. \qed

As noted above, Gallai graphs cannot be characterized by forbidden-induced subgraphs: An induced subgraph of a Gallai graph need not be a Gallai graph. The simplest examples are the graphs $K_{1,3}$ and $K_4^-$ ($K_4$ minus an edge); these are no Gallai graphs, but appear as component of $\Gamma(K_{1,3} * K_1)$, respectively, $\Gamma(K_4^- * K_1)$. As an application of Theorem 5.1, we show that a graph and all its induced subgraphs are Gallai graphs if and only if it has no $K_{1,3}$, no $K_4^-$. 

Remark. The following statements are equivalent for every graph $G$.

(i) Every induced subgraph of $G$ is a Gallai graph.
(ii) $G$ has no $K_{1,3}$, no $K_4^-$. 
(iii) $G$ is a Gallai graph of a triangle-free graph (and thus, $G$ is a line graph).
Proof. The implication (i) $\Rightarrow$ (ii) is already pointed out. To see (ii) $\Rightarrow$ (iii), consider the family $\mathcal{F}$ of all maximal complete subgraphs and all vertices belonging to exactly one maximal complete subgraph of $G$. Then (ii) implies directly that $\mathcal{F}$ satisfies the conditions in Theorem 5.1, and moreover, the intersection graph $H$ of $\mathcal{F}$ has no triangle. From the proof of that theorem, we have $G \approx \Gamma(H)$, showing (iii). The implication (iii) $\Rightarrow$ (i) is obvious, because a graph is a Gallai graph of a triangle-free graph if and only if it is a line graph of a triangle-free graph. Note that all induced subgraphs of a line graph of a triangle-free graph are also line graphs of a triangle-free graph. □

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