Volume upward continuation of potential fields from the minimum-length solution: an optimal tool for continuation through general surfaces.
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Summary
Several methods are used to upward continue potential field data. The leading techniques employ the Fast Fourier transform, accurate and quick for level-to-level upward continuation, and spatially varying scale filters for level-to-draped surfaces. We propose a new approach, based on the minimum-length solution of the inverse potential field problem. The method yields a volume upward continuation, and reveals to be advantageous over the classical techniques when dealing with truncated anomalies and when draped-to-level upward continuation is needed.

Introduction
Upward continuation is used to transform anomalies measured on one surface into those that would have been measured on some higher altitude surface. So, it can be used to merge data at different altitudes, some of them measured on irregular surfaces (i.e. draped airborne surveys), and continue with the whole dataset to a given surface (e.g. Paterson et al., 1990; Pilkington and Roest, 1992; Ridsmill-Smith, 2000; Paoletti et al., 2005). This transformation is also helpful to enhance the effects of deep sources, as it attenuates the highest frequency content of the signal, which is usually associated to shallow sources. Finally, multiscale methods such as the continuous wavelet transform (Martelet et al., 2001; Paoletti et al., 2007; Sailliac et al., 2009; Mauri et al., 2010; Fedi et al., 2010; Fedi and Cascone, 2011), the DEXP transformation (Fedi, 2007; Fedi and Florio, 2006) and the multiridge analysis (Florio et al., 2009; Cella et al., 2009) need potential fields data available on a 3D volume. Obviously, direct measurements at many different altitude levels are not realistic and upward continuation is the tool needed to continue the data to several levels.

Upward continuation originates from Green’s third identity (Blakely, 1996), which states that if \( U \) is an harmonic continuous function, with continuous derivatives throughout a regular region \( R \), then its value at any point \( P \) within a space \( R \), can be evaluated from its behavior on the boundary \( S \):}

\[
U(P) = \frac{1}{4\pi} \iiint_S \left( \frac{1}{r} \frac{\partial U}{\partial n} - U \frac{\partial}{\partial n} \frac{1}{r} \right) dS
\]

where \( n \) is the outward normal direction, \( r \) the distance from \( P \) to the point of integration on \( S \). No information about the source is needed except that it is not located within the region \( R \).

In the following discussion we will refer to different ways of performing upward continuation: a) level-to-level; b) level-to-draped; c) draped-to-level.

The simplest upward continuation operator is defined by an integral formulation when the potential field data are measured on a constant altitude surface \( \zeta_0 \) and continued to some higher altitude plane (level-to-level):

\[
U(x,y,z_0 - \Delta z) = \frac{\Delta z}{2\pi} \iiint_{-\infty}^{\infty} \frac{U(x',y',z_0)}{(x-x')^2 + (y-y')^2 + \Delta z^2} dx'dy'
\]

where \( \Delta z > 0 \) and \( z \) is assumed negative outward. Equation (2) is a convolution integral and can be diagonalized via the Fourier transform and the convolution theorem. The numerical implementation of this formula obviously considers a finite-extent dataset and equi-spaced data, which leads to known types of errors for the continued data (Fedi et al., 2012).

Therefore upward continued data can be calculated by convolution in either the space domain or the Fourier domain. In this last domain the Fourier transform of the data is simply multiplied by the frequency filter:

\[
e^{-j|k|\Delta z}
\]

where \( k \) is the wavenumber vector. As said before, real data are discrete and refer to a finite survey area; so, when using circular convolution to calculate upward continuation in the frequency domain, aliasing errors can affect the low frequency content of upward continued data. These errors can be reduced performing the Fourier transform on a larger dataset, which spreads outside the survey area (Oppenheim and Schafer, 1975; Fedi et al. 2012), built with other surveys data or, through extrapolation algorithms (zero-padding, maximum entropy extension, symmetric extension).

Equations (2) and (3) are strictly valid for level-to-level continuations, however potential fields literature is rich in algorithms performing upward continuation between uneven surfaces. Among them, Cordell (1985), Pilkington and Roest (1992) and Ridsdill-Smith (2000) developed...

In this paper we define a new approach which performs upward continuation using the relation established by Cribb (1976) between the minimum length solution and the upward continuation. This approach has the advantage of generating the field in the upward continuation domain (i.e., at many altitudes) as a unique solution in a 3D volume. Within this volume, all the three types of continuation (level-to-level, level-to-draped and draped-to-level) are naturally defined. In fact, the upward continued data volume can be immediately visualized and used to obtain the field on several surfaces of whatever kind (draped or level).

Due to this feature we will call this method Volume Upward Continuation (VUC). In the VUC the border effects, typical of upward continuation, are controlled in an optimal way: the upward continued data are in fact proportional to a least-square solution of the inverse problem.

Volume Upward continuation

Starting from the continuous inverse geomagnetic problem (e.g., Fedi et al., 2005) discretized over a volume \( \Omega \) of \( N \) cells, each of them with constant magnetization \( m_i \), we can write a linear system of equations:

\[
K m = d
\]  
(4)

Where \( K \) is the kernel coefficients matrix and \( d \) is the measured data vector. This inverse problem is under-determined, as the unknowns are more than data, so it has infinite solutions. The simplest solution is that minimizing the Euclidean norm of the solution \( m \):

\[
m = K^T (K K^T)^{-1} d
\]  
(5).

Cribb (1976) showed, for this solution, this remarkable equation in the frequency domain:

\[
e^{\text{H}_i F [d]} = F[m_i] \frac{V(k)}{4} \quad i = 1, \ldots, L
\]  
(6)

where \( F \) denotes the Fourier transformation, \( L \) is the number of layers, \( m_i \) is the magnetization intensity vector of the \( i \)th layer, \( h_i \) is its depth, \( k \) is the wavevector with components \( k_x, k_y \) and \( f(k) = (tk)n.k/|k|^2 \), with \( t \) and \( n \) unit-vectors along the inducing field and magnetization vectors. Therefore the \( i \)th layer of the magnetization distribution \( m_i \) is directly related to the upward continued field of the data \( d \), to a distance equal to the opposite of the layer depth: \( z = -h_i \). Anti-transforming the second member of equation (6) and assuming, for simplicity, magnetization and inducing field both vertical, we find:

\[
m_i = \frac{d}{4}, \quad i = 1, \ldots, L
\]  
(7)

showing that \( d \) and \( m \) differ only for a numeric constant.

Based on equations (6, 7) we therefore may use the minimum length solution as an effective alternative to common upward continuation techniques.

In Figure 1 we tested the VUC approach in two simple cases: level-to-level and level-to-draped upward continuation of the magnetic anomaly due to a horizontal dipole line located at 30 m depth. We observe that the field obtained from the minimum-length solution well reproduces the computed anomaly at the same altitude.

The result has been achieved following these steps:

a) Inverting the magnetic anomaly to obtain the minimum length solution;

b) Converting the magnetization volume to an upward continued field volume through relation (7);

c) Extracting the \( i \)th layer corresponding to the desired continuation altitude \( z = -h_i \) (level-to-level);

d) Or extracting the field corresponding to the desired draped surface (level-to-draped).

![Figure 1: VUC continuation. a) minimum-length solution section; b) level-to-level VUC profile (green line) converted, in agreement with equation (7), from the minimum-length solution (green line at \( z = -20 \) m in (a)); c) level-to-draped VUC profile (red line) converted from the minimum-length solution (inclined red line in (a) rising from 0 to 20 m). We note a good agreement of VUC field with the true data (blue lines in (b) and (c)).](image)

Level-to-draped upward continuation on synthetic topographic surface: a comparison with other algorithms

We here compare the VUC computed magnetic anomaly on a draped surface with that resulting from other methods. The test magnetic anomaly was originally generated by Ridsdill-Smith, (2000) using a magnetization distribution

\[
\text{m}_{\text{draped}} = \text{m}_{\text{level}} + \text{m}_{\text{topography}}
\]
Volume Upward Continuation (VUC)

with a power-law decay (i.e., containing more energy at low frequencies than at high frequencies), draped over peak topography.

The VUC anomaly field was compared with that produced by the following algorithms: a method based on a spatially varying scale filter defined in the wavelet domain (Ridsdill-Smith, 2000); the chessboard method (Cordell, 1985), which uses a set of constant altitude upward continuations to generate a 3D volume of data around the sampling points of the desired continuation surface and then interpolates around those values; the Taylor series method (Pilkington and Roest, 1992) which approximates the field on the draped surface using a finite number of terms of the Taylor-series of the measured field.

All the above-mentioned approaches produced an accurate upward continued field.

Draped-to-level upward continuation on a real topographic surface

We now show that the most valuable advantage of the VUC method occurs when the field data are on a draped surface (e.g., a topographic surface) and are continued to a constant level. This case is especially important because typical imaging, inversion, differentiation, pole reduction and other algorithms need data at a constant altitude surface (e.g. Paoletti et al., 2009; Paoletti et al., 2013).

To this end, the VUC method was applied to data (Figure 3a) generated by a prismatic source on a real topographic surface located in the surrounding of Naples, Italy (Figure 3b).

Figure 3c shows a vertical section of the minimum length solution, from which it is easy to select the desired level of the draped-to-level upward continuation (red line in Figure 3c). Note that the source-volume has been extended in correspondence with its borders with additional blocks, to circumvent border effects. Note also that the inverted magnetization model is a solution of the least-squares problem and includes also the magnetization of the blocks located at the volume border, at any depth level. On the contrary, typical extrapolation algorithms (zero-padding, maximum entropy extension, symmetric extension) are not physically based and are applied to the first level only, whose effect is propagated at greater altitudes by the continuation process.

After converting the magnetization model into the upward continuation domain (equation 6) we can represent the field at the selected level \( z = -600 \) m (Figure 3e). The result is very close to the true field (Figure 3d), with RMSE=0.0289. Comparing this result with that produced by using the wavelet filter algorithm by Ridsdill-Smith (2000), shown in Figure 3f, we note that the RMSE is 0.0441 and especially that some of the topographic effect still remains in the continued data.

Upward continuation from truncated profiles

As regards extrapolation, which is needed before performing upward continuation and performed by algorithms operating in the Fourier domain, we refer to Fedi et al. (2012), who suggested to extend the data to an additional area sized as the original one. When the anomaly is sufficiently isolated we do not expect substantial differences in using the different extrapolation algorithms.

A special case is that of anomalies severely truncated (Figure 4, dot blue line), because of a limited extent of the survey area. In this case, as Figure 4a shows, we obtain very different results, from the several extrapolation algorithms, where the truncation is important, i.e., on the left side, whereas there is a substantial convergence where the anomaly to be continued is better defined, i.e. from the center to the right side.

In the VUC case, instead, as noted previously, the border effects are circumvented by extending the source-volume with additional blocks at its borders. The border effects are better controlled, because of the constraint, inherent in the inversion process, that the predicted data must fit the measured data. Figure 4b shows that this approach yields, in fact, good results.
In this paper we propose a new approach to upward continuation of potential field data, which yields good results especially in case of draped-to-level upward continuation and when dealing with the continuation of severely truncated anomalies.

One advantage of the novel approach is that the VUC (Volume Upward Continuation) method yields a volume upward continuation, and thus it may be used to extract in a simple way the continuation on any kind of surface and also to multiple levels. We may in fact define all the three kinds of continuation problems (level-to-level, level-to-draped and draped-to-level) by simply picking up the sought surface on the inverted 3D magnetization model and then transforming the corresponding magnetization data into upward continued data, through equations 6 and 7. Inversion obviously implies higher computational costs than standard continuation algorithms and may be prohibitive for large-scale problems. Computational costs are however well compensated when high-quality results are required, such as the case of truncated anomalies and draped-to-level continuation.

Figure 3: (a) Comparison between computed data (blue dot line) at \( z = -10 \text{ m} \) altitude and VUC (green line in (a)). (b) Comparison between computed data (blue dot line) and the following standard upward continuation algorithms in the frequency domain: zero padding (ZPD), smooth extension of order 0 and 1 (S0 and S1), symmetric (SYM) extension. The best results are obtained by the VUC and S1 algorithms.

Figure 4: Magnetic anomaly (a) computed on a complex topographic surface (b); (c) minimum-length solution obtained inverting the magnetic anomaly; (d) magnetic anomaly computed on a constant altitude surface at \( z = -600 \text{ m} \); (e) magnetic anomaly extracted from the VUC model at \( h = 600 \text{ m} \) (red line in (c)); (f) magnetic anomaly upward continued with the wavelet filter method (Ridsdill-Smith, 2000) from the draped surface in (b) to a constant altitude surface at \( z = -600 \text{ m} \).
EDITED REFERENCES
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