Space of linear differential operators on the real line as a module over the Lie algebra of vector fields

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Abstract

Let $\mathcal{D}^k$ be the space of k-th order linear differential operators on $\mathbf{R}$: $A = a_k(x)\frac{d^k}{dx^k} + \cdots + a_0(x)$. We study a natural 1-parameter family of Diff($\mathbf{R}$)- (and Vect($\mathbf{R}$))-modules on $\mathcal{D}^k$. (To define this family, one considers arguments of differential operators as tensor-densities of degree λ.) In this paper we solve the problem of isomorphism between Diff($\mathbf{R}$)-module structures on $\mathcal{D}^k$ corresponding to different values of λ. The result is as follows: for $k = 3$ Diff($\mathbf{R}$)-module structures on $\mathcal{D}^3$ are isomorphic to each other for every values of $\lambda \neq 0,\ 1,\ \frac{1}{2},\ \frac{1}{2} \pm \frac{\sqrt{21}}{6}$, in this case there exists a unique (up to a constant) intertwining operator $T : \mathcal{D}^3 \to \mathcal{D}^3$. In the higher order case ($k \geq 4$) Diff($\mathbf{R}$)-module structures on $\mathcal{D}^k$ corresponding to two different values of the degree: λ and λ', are isomorphic if and only if $\lambda + \lambda' = 1$.

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1 Introduction

Space of linear differential operators on a manifold M has various algebraic structures: the structure of associative algebra and of Lie algebra, in the 1-dimensional case it can be considered as an infinite-dimensional Poisson space (with respect to so-called Adler-Gelfand-Dickey bracket).

1.1 Diff(M)**-module structures**. One of the basic structures on the space of linear differential operators is a natural family of module structures over the group of diffeomorphisms Diff(M) (and of the Lie algebra of vector fields Vect(M)). These Diff(M)- (and Vect(M))-module structures are defined if one considers the arguments of differential operators as *tensor-densities* of degree λ on M.

In this paper we consider the space of differential operators on $\mathbf{R}$.[1] Denote $\mathcal{D}^k$ the space of k-th order linear differential operators:

$$A(\phi) = a_k(x)\frac{d^k\phi}{dx^k} + \cdots + a_0(x)\phi \tag{1}$$

where $a_i(x), \phi(x) \in C^\infty(\mathbf{R})$.

Define a 1-parameter family of Diff($\mathbf{R}$)-module structures on $C^\infty(\mathbf{R})$ by:

$$g_\lambda^*\phi := \phi \circ g^{-1} \cdot \left(\frac{dg^{-1}}{dx}\right)^{-\lambda}$$

where $\lambda \in \mathbf{R}$ (or $\lambda \in \mathbf{C}$) is a parameter. Geometrically speaking, ϕ is a *tensor-density* of degree $-\lambda$:

$$\phi = \phi(x)(dx)^{-\lambda}.$$

A 1-parameter family of actions of Diff($\mathbf{R}$) on the space of differential operators (1) is defined by:

$$g(A) = g_\lambda^* A (g_\lambda^*)^{-1}.$$

[1]Particular cases of actions of Diff($\mathbf{R}$) and Vect($\mathbf{R}$) on this space were considered by classics (see [1], [14]). The well-known example is the Sturm-Liouville operator $\frac{d^2}{dx^2} + a(x)$ acting on $-\frac{1}{2}$-densities (see e.g. [1], [14], [13]). Already this simplest case leads to interesting geometric structures and is related to so-called Bott-Virasoro group (cf. [9], [12]).

Denote $\mathcal{D}^k_\lambda$ the space of operators (1) endowed with the defined Diff($\mathbf{R}$)-module structure.

Infinitesimal version of this action defines a 1-parameter family of Vect($\mathbf{R}$)-module structures on $\mathcal{D}^k$ (see Sec. 3 for details).

1.2 The problem of isomorphism. Let M be a manifold, $\dim M \geq 2$. The problem of isomorphism of Diff(M)- (and Vect(M))-module structures for different values of λ was stated in [4] and saved in the case of second order differential operators. In this case, different Diff(M)-module structures are isomorphic to each other for every λ except 3 critical values: $\lambda = 0, -\frac{1}{2}, -1$ (corresponding to differential operators on: functions, $\frac{1}{2}$-densities and volume forms respectively).

Geometric quantization gives an example of such a special Diff(M)-module: differential operators are considered as acting on $\frac{1}{2}$-densities (see [7]).

Recently P.B.A. Lecomte, P. Mathonet, and E. Tousset [8] showed that in the case of differential operators of order ≥ 3, Diff(M)-modules corresponding to λ and λ'-densities are isomorphic if and only if $\lambda + \lambda' = 1$. The unique isomorphism in this case is given by conjugation of differential operators.

These results solve the problem of isomorphism in the multi-dimensional case.

It was shown in [4], [8], that the case $\dim M = 1$ ($M = \mathbf{R}$ or S^1) is particular. It is reacher in algebraic structures and therefore is of a special interest.

In this paper we solve the problem of isomorphism of Diff($\mathbf{R}$)-modules $\mathcal{D}^k_\lambda$ for any k. The result is as follows.

1) The modules $\mathcal{D}^3_\lambda$ of *third order* differential operators (1) are isomorphic to each other for all values of λ except 5 critical values:

$$\{0, \quad -1, \quad -\frac{1}{2}, \quad -\frac{1}{2} + \frac{\sqrt{2}1}{6}, \quad -\frac{1}{2} - \frac{\sqrt{2}1}{6}\}.$$

(this result was announced in [4]).

2) The Diff($\mathbf{R}$)-modules $\mathcal{D}^k_\lambda$ and $\mathcal{D}^k_{\lambda'}$ on the space of differential operators (1) of order $k \geq 4$ are isomorphic if and only if $\lambda + \lambda' = -1$.

1.3 Intertwining operator. The most important result of the paper is a construction of the unique (up to a constant) equivariant linear operator on the space of third order differential operators:

$$T : \mathcal{D}^3_\lambda \to \mathcal{D}^3_\mu \tag{2}$$

for $\lambda, \mu \neq 0, -1, -\frac{1}{2}, \ -\frac{1}{2} \pm \frac{\sqrt{21}}{6}$, see the explicit formulæ (3), (7) and (8) below. It has nice geometric and algebraic properties and seems to be an interesting object to study.

Operator T is an analogue of the *second order Lie derivative* from [4] intertwining different $\mathrm{Diff}(M)$-actions on the space of second order differential operators on a multi-dimensional manifold M.

1.3 Normal symbols. The main tool of this paper is the notion of a *normal symbol*, which we define in the case of 4-th order differential operators. We define a sl_2-equivariant way to associate a polynomial function of degree 4 on $T^*\mathbf{R}$ to a differential operator $A \in \mathcal{D}^4_\lambda$. In the case of second order operators the notion of normal symbol was defined in [4]. This construction is related with the results of [3]. We discuss the geometric properties of the normal symbol and its relations to the intertwining operator (2).

2 Main results

We formulate here the main results of this paper, all the proofs will be given in Sec. 3-7.

2.1 Classification of $\mathrm{Diff}(\mathbf{R})$-modules. First, remark that for each value of k, there exists an isomorphism of $\mathrm{Diff}(\mathbf{R})$-modules:

$$\mathcal{D}^k_\lambda \cong \mathcal{D}^k_{-1-\lambda}.$$

It is given by *conjugation* $A \mapsto A^*$:

$$A^* = \sum_{i=1}^{k} (-1)^i \frac{d^i}{dx^i} \circ a_i(x)$$

The following two theorems give a solution of the problem of isomorphism of $\mathrm{Diff}(\mathbf{R})$-modules $\mathcal{D}^k_\lambda$ on space $\mathcal{D}^k$.

The first result was announced on [4]:

Theorem 1. *(i) All the* Diff(**R**)*-modules* $\mathcal{D}^3_\lambda$ *with* $\lambda \neq 0, -1, -\frac{1}{2}, -\frac{1}{2} + \frac{\sqrt{21}}{6}, -\frac{1}{2} - \frac{\sqrt{21}}{6}$ *are isomorphic to each other.*
(ii) The modules $\mathcal{D}^3_0$, $\mathcal{D}^3_{-\frac{1}{2}}$, $\mathcal{D}^3_{-\frac{1}{2}+\frac{\sqrt{21}}{6}}$ *are not isomorphic to* $\mathcal{D}^3_\lambda$ *for general* λ.

It follows from the general isomorphism $* : \mathcal{D}^k_\lambda \cong \mathcal{D}^k_{-1-\lambda}$, that

$$\mathcal{D}^3_0 \cong \mathcal{D}^3_{-1} \quad \text{and} \quad \mathcal{D}^3_{-\frac{1}{2}+\frac{\sqrt{21}}{6}} \cong \mathcal{D}^3_{-\frac{1}{2}-\frac{\sqrt{21}}{6}}.$$

Therefore, there exist 4 non-isomorphic Diff(**R**)-module structures on the space $\mathcal{D}^3$.

Theorem 2. *For* $k \geq 4$*, the* Diff(**R**)*-modules* $\mathcal{D}^k_\lambda$ *and* $\mathcal{D}^k_{\lambda'}$ *are isomorphic if and only if* $\lambda + \lambda' = -1$.

This result shows that operators of order 3 play a special role in the 1-dimensional case (as operators of order 2 in the case of a manifold of dimension ≥ 2, cf. [4], [8]).

2.2 Intertwining operator T.

Theorem 3. *For* $\lambda, \mu \neq 0, -1, -\frac{1}{2}, -\frac{1}{2} \pm \frac{\sqrt{21}}{6}$ *there exists a unique (up to a constant) isomorphism of* Diff(**R**)*-modules* $\mathcal{D}^3_\lambda$ *and* $\mathcal{D}^3_\mu$.

Let us give an explicit formula for the operator (2).

Every differential operator of order 3 can be written (not in a canonical way) as a linear combination of four operators:
1) a zero order operator of multiplication by a function: $\phi(x) \mapsto \phi(x)f(x)$,
2) a first order operator of Lie derivative:

$$L^\lambda_X = X(x)\frac{d}{dx} - \lambda X'(x),$$

where $X' = \frac{dX}{dx}$,
3) symmetric "anti-commutator" of Lie derivatives:

$$[L^\lambda_X, L^\lambda_Y]_+ := L^\lambda_X \circ L^\lambda_Y + L^\lambda_Y \circ L^\lambda_X$$

4) symmetric third order expression:

$$[L^\lambda_X, L^\lambda_Y, L^\lambda_Z]_+ := Sym_{X,Y,Z}(L^\lambda_X \circ L^\lambda_Y \circ L^\lambda_Z)$$

for some vector fields $X(x)\frac{d}{dx}, Y(x)\frac{d}{dx}, Z(x)\frac{d}{dx}$.

Theorem 4. *The following formula:*

$$
\begin{aligned}
T(f) &= \frac{\mu(\mu+1)(2\mu+1)}{\lambda(\lambda+1)(2\lambda+1)} f \\
T(L_X^\lambda) &= \frac{3\mu^2+3\mu-1}{3\lambda^2+3\lambda-1} L_X^\mu \\
T([L_X^\lambda, L_Y^\lambda]_+) &= \frac{2\mu+1}{2\lambda+1}[L_X^\mu, L_Y^\mu]_+ \\
T([L_X^\lambda, L_Y^\lambda, L_Z^\lambda]_+) &= [L_X^\mu, L_Y^\mu, L_Z^\mu]_+
\end{aligned}
\tag{3}
$$

defines an intertwining operator (2).

A remarkable fact is that the formula (3) does not depend on the choice of X, Y, Z and f representing the third order operator. (Indeed, the formulæ (7) and (8) below give the expression of T directly in terms of coefficients of differential operators.) Moreover, this property fixes the coefficients in (3) in a unique way (up to a constant).

Remarks. 1) In the case of multi-dimensional manifold M, almost all Diff(M)-module structures on the space of second order differential operators are isomorphic to each other and the corresponding isomorphism is unique (up to a constant) [4]; there is no isomorphism between different Diff(M)-module structures on the space of third order operators, except the conjugation [8].

2) The formula (3) gives an idea that it would be interesting to study the commutative algebra structure (defined by the anti-commutator) on the Lie algebra of all differential operators.

3 Action of $\mathrm{Vect}(\mathbf{R})$ on space $\mathcal{D}^4$

To prove Theorems 1-4, it is sufficient to consider only the Vect$(\mathbf{R})$-action on $\mathcal{D}^k$. Indeed, since the Diff$(\mathbf{R})$-action on the space of differential operators is local, therefore, the two properties: of Vect$(\mathbf{R})$- and of Diff$(\mathbf{R})$-equivariance are equivalent.

3.1 Definition of the family of Vect(**R**)**-actions**. Let Vect(**R**) be the Lie algebra of smooth vector fields on **R**:

$$X = X(x)\frac{d}{dx}$$

with the commutator

$$[X(x)\frac{d}{dx}, Y(x)\frac{d}{dx}] = (X(x)Y'(x) - X'(x)Y(x))\frac{d}{dx},$$

where $X' = dX/dx$.

The action of Vect(**R**) on space $\mathcal{D}^k$ is defined by:

$$\mathrm{ad}L_X^\lambda(A) := L_X^\lambda \circ A - A \circ L_X^\lambda$$

where

$$L_X^\lambda \phi = X(x)\phi'(x) - \lambda X'(x)\phi(x)$$

The last formula defines a 1-parameter family of Vect(**R**)-actions on $C^\infty(\mathbf{R})$.

One obtains a 1-parameter family of Vect(**R**)-modules on $\mathcal{D}^k$.

Notation. 1. The operator L_X^λ is called the *operator of Lie derivative* of tensor-densities of degree $-\lambda$. Denote $\mathcal{F}_\lambda$ the corresponding Vect(**R**)-module structure on $C^\infty(\mathbf{R})$.

2. As in the case of Diff(**R**)-module structures, we denote $\mathcal{D}_\lambda^k$ space $\mathcal{D}^k$ as a Vect(**R**)-module.

3.2 Explicit formula. Let us calculate explicitly the action of Lie algebra Vect(**R**) on space $\mathcal{D}^4$. Given a differential operator $A \in \mathcal{D}^4$, let us use the following notation for the Vect(**R**)-action $\mathrm{ad}L_X$:

$$\mathrm{ad}L_X(A) = a_4^X(x)\frac{d^4}{dx^4} + a_3^X(x)\frac{d^3}{dx^3} + a_2^X(x)\frac{d^2}{dx^2} + a_1^X(x)\frac{d}{dx} + a_0^X(x).$$

Lemma 3.1. *The action* $\mathrm{ad}L_X^\lambda$ *of* Vect(**R**) *on space* $\mathcal{D}^4$ *is given by :*

$$\begin{aligned}
a_4^X &= L_X^4(a_4)\\
a_3^X &= L_X^3(a_3) + 2(2\lambda - 3)a_4 X''\\
a_2^X &= L_X^2(a_2) + 3(\lambda - 1)a_3 X'' + 2(3\lambda - 2)a_4 X'''\\
a_1^X &= L_X^1(a_1) + (2\lambda - 1)a_2 X'' + (3\lambda - 1)a_3 X''' + (4\lambda - 1)a_4 X^{IV}\\
a_0^X &= L_X^0(a_0) + \lambda(a_1 X'' + a_2 X''' + a_1 X^{IV} + a_0 X^{V})
\end{aligned} \tag{4}$$

Proof. One gets easily the formula (4) from the definition:

$$\begin{aligned}\mathrm{ad}L_X^\lambda(A) = [L_X^\lambda, A] = \quad & (X\frac{d}{dx} - \lambda X')(a_4\frac{d^4}{dx^4} + \cdots + a_0) \\ & -(a_4\frac{d^4}{dx^4} + \cdots + a_0)(X\frac{d}{dx} - \lambda X')\end{aligned}$$

3.3 Remarks. It is convenient to interpret the action (4) as a *deformated* standard action of $\mathrm{Vect}(\mathbf{R})$ on the direct sum:

$$\mathcal{F}_4 \oplus \mathcal{F}_3 \oplus \mathcal{F}_2 \oplus \mathcal{F}_1 \oplus \mathcal{F}_0.$$

(given by the first term of the right hand side of each equality in the formula (4)). This interpretation is the motivation of the main construction of Sec. 4, it will be discussed in Sec. 7.2.

The main idea of proof of Theorems 1 and 2 is to find some *normal form* (cf. [4]) of the coefficients $a_4(x), \ldots, a_0(x)$ for 4-order differential operators on $\mathbf{R}$ which reduce the action (4) to a canonical form.

4 Normal form of a symbol

It is convenient to represent differential operators as polynomials on the cotangent bundle. The standard way to define a (total) symbol of an an operator (1) is to associate to A the polynomial

$$P_A(x, \xi) = \sum_{i=0}^{k} \xi^i a_i(x),$$

on $T^*\mathbf{R} \cong \mathbf{R}^2$ (where ξ is a coordinate on the fiber). However, this formula depends on coordinates, only the higher term $\xi^k a_k(x)$ of P_A (the principal symbol) has a geometric sense.

4.1 The main idea. Lie algebra $\mathrm{Vect}(\mathbf{R})$ naturally acts on $C^\infty(T^*\mathbf{R})$ (it acts on the cotangent bundle).

Consider a linear differential operator $A \in \mathcal{D}^4$. Let us look for a natural definition of a symbol of A in the following form:

$$\overline{P}_A(x, \xi) = \xi^4 \bar{a}_4(x) + \xi^3 \bar{a}_3(x) + \xi^2 \bar{a}_2(x) + \xi \bar{a}_1(x) + \bar{a}_0(x),$$

where the functions $\bar{a}_i(x)$ are linear expressions of the coefficients $a_i(x)$ and their derivatives.

Any symbol $\overline{P}(x,\xi)$ can be considered as a linear mapping

$$\mathcal{D}^4 \to \mathcal{F}_4 \oplus \mathcal{F}_3 \oplus \mathcal{F}_2 \oplus \mathcal{F}_1 \oplus \mathcal{F}_0.$$

Indeed, the Lie algebra $\mathrm{Vect}(\mathbf{R})$ acts on each coefficient $\bar{a}_i(x)$ of the polynomial $\overline{P}_A(x,\xi)$ as on a tensor-density of degree $-i$:

$$L_X(\overline{P}_A) = \sum_{i=0}^{4} \xi^i L_X^i(\bar{a}_i).$$

However, there is no such a mapping which is $\mathrm{Vect}(\mathbf{R})$-equivariant.

4.2 Definition. *The normal symbol of $A \in \mathcal{D}^4_\lambda$ as a polynomial $\overline{P}_A(x,\xi)$ such that the linear mapping $A \mapsto \overline{P}_A$ is equivariant with respect to the subalgebra $sl_2 \subset \mathrm{Vect}(\mathbf{R})$ generated by the vector fields*

$$\{\frac{d}{dx},\ \ x\frac{d}{dx},\ \ x^2\frac{d}{dx}\}.$$

Proposition. *4.1. (i) The following formula defines a normal symbol of a differential operator $A \in \mathcal{D}^4_\lambda$:*

$$\begin{aligned}
\bar{a}_4 = \ & a_4\\
\bar{a}_3 = \ & a_3 + \tfrac{1}{2}(2\lambda-3)a_4'\\
\bar{a}_2 = \ & a_2 + (\lambda-1)a_3' + \tfrac{2}{7}(\lambda-1)(2\lambda-3)a_4''\\
\bar{a}_1 = \ & a_1 + \tfrac{1}{2}(2\lambda-1)a_2' + \tfrac{3}{10}(\lambda-1)(2\lambda-1)a_3''\\
& +\tfrac{1}{15}(\lambda-1)(2\lambda-1)(2\lambda-3)a_4'''\\
\bar{a}_0 = \ & a_0 + \lambda a_1' + \tfrac{1}{3}\lambda(2\lambda-1)a_2'' + \tfrac{1}{6}\lambda(\lambda-1)(2\lambda-1)a_3'''\\
& +\tfrac{1}{30}\lambda(\lambda-1)(2\lambda-1)(2\lambda-3)a_4^{(IV)}
\end{aligned} \tag{5}$$

(ii) The normal symbol is defined uniquely (up to multiplication of each function $\bar{a}_i(x)$ by a constant).

Proof. Direct calculation shows that the $\mathrm{Vect}(\mathbf{R})$-action adL^λ on $\mathcal{D}^4$

given by the formula (4) reads in terms of $\bar{a}_i$ as:

$$\begin{array}{rcl} \bar{a}_4^X &=& L_X^4(\bar{a}_4) \\ \bar{a}_3^X &=& L_X^3(\bar{a}_3) \\ \bar{a}_2^X &=& L_X^2(\bar{a}_2)+\frac{2}{7}(6\lambda^2+6\lambda-5)J_3(X,\bar{a}_4) \\ \bar{a}_1^X &=& L_X^1(\bar{a}_1)+\frac{2}{5}(3\lambda^2+3\lambda-1)J_3(X,\bar{a}_3) \\ && \qquad\qquad +\frac{1}{6}\lambda(\lambda+1)(2\lambda+1)J_4(X,\bar{a}_4) \\ \bar{a}_0^X &=& L_X^0(\bar{a}_0)+\frac{2}{3}\lambda(\lambda+1)J_3(X,\bar{a}_2) \\ && \qquad\qquad +\frac{1}{6}\lambda(\lambda+1)(2\lambda+1)J_4(X,\bar{a}_3) \\ && \qquad\qquad +\frac{1}{420}\lambda(\lambda+1)(12\lambda^2+12\lambda+11)J_5(X,\bar{a}_4) \end{array} \tag{6}$$

where $\bar{a}_i^X$ are coefficients of the normal symbol of the operator $adL_X^\lambda(A)$ and the expressions J_m are:

$$\begin{array}{l} J_3(X,\bar{a}_s)=X'''\bar{a}_s \\ J_4(X,\bar{a}_s)=sX^{(IV)}\bar{a}_s+2X'''\bar{a}_s' \\ J_5(X,\bar{a}_s)=s(2s-1)X^{(V)}\bar{a}_s+5(2s-1)X^{(IV)}\bar{a}_s'+10X'''\bar{a}_s'' \end{array}$$

It follows that the mapping $\mathcal{D}^4\to\mathcal{F}_4\oplus\cdots\oplus\mathcal{F}_0$ defined by (5) is sl_2-equivariant. Indeed, for a vector field $X\in sl_2$ (which is a polynomial in x of degree ≤ 2) all the terms $J_m(X,\bar{a}_s)$ in (6) vanish.

Proposition 4.1 (i) is proven.

Let us prove the uniqueness. By definition, the functions $\bar{a}_i(x)$ are linear expressions in $a_s(x)$ and their derivatives:

$$\bar{a}_i(x)=\sum_{s,t}\alpha_t^s(x)a_s^{(t)}(x),$$

where $a_s^{(t)}=d^ta_s/dx^t$, $\alpha_k^j(x)$ are some functions. The fact that the normal symbol $\overline{P}_A$ is sl_2-equivariant means that for a vector field $X\in sl_2$, $\bar{a}_i^X=L_X^i(a_i)$.
a) Substitute $X=d/dx$ to obtain that the coefficients α_k^j does not depend on x;
b) substitute $X=xd/dx$ to obtain the condition $j-k=i$:

$$\bar{a}_i(x)=\sum_{j=4}^{i}\alpha^ja_j^{(j-i)}(x),$$

and $\alpha^i \neq 0$;
c) put $\alpha^i = 1$ and, finally, substitute $X = x^2 d/dx$ to obtain the coefficients from (5).

Proposition 4.1 (ii) is proven.

The notion of normal symbol of a 4-th order differential operator plays a central role in this paper.

4.3 Remark: the transvectants. Operations $J_3(X, a_s)$, $J_4(X, a_s)$, $J_5(X, a_s)$ are particular cases of the following remarkable bilinear operations on tensor-densities. Consider the expressions:

$$j_n(\phi, \psi) = \sum_{i+j=n} (-1)^i \binom{n}{i} \frac{(2\lambda - i)!(2\mu - j)!}{(2\lambda - n)!(2\mu - n)!} \phi^{(i)} \psi^{(j)}$$

where $\phi = \phi(x), \psi = \psi(x)$ are smooth functions.

This operations defines unique (up to a constant) sl_2-equivariant mapping:

$$\mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu-n}$$

Operations $j_n(\phi, \psi)$ were discovered by Gordan [6], they are also known as Rankin-Cohen brackets (see [11], [2]).

Note, that the operations J_m from the formula (6) are proportional to j_n for $X \in \mathcal{F}_1, a_s \in \mathcal{F}_{-s}$.

5 Diagonalization of operator T

We will obtain here an important property of the intertwining operator (2): in terms of normal symbol it has a diagonal form. We will also prove the part (i) of Theorem 1 and Theorem 4.

5.1 Proof of Theorem 1, part (i). Let us define an isomorphism of modules $\mathcal{D}^3_\lambda$ and $\mathcal{D}^3_\mu$ for $\lambda \neq 0, -1, -\frac{1}{2}, -\frac{1}{2} \pm \frac{\sqrt{21}}{6}$. Associate to $A \in \mathcal{D}^3_\lambda$ the operator $T(A) \in \mathcal{D}^3_\mu$:

$$T: \; a_3 \frac{d^3}{dx^3} + a_2 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_0 \;\longmapsto\; a^T_3 \frac{d^3}{dx^3} + a^T_2 \frac{d^2}{dx^2} + a^T_1 \frac{d}{dx} + a^T_0$$

such that its standard symbol

$$\overline{P}_{T(A)} = \xi^3 \overline{a^T}_3(x) + \xi^2 \overline{a^T}_2(x) + \xi \overline{a^T}_1(x) + \overline{a^T}_0(x)$$

is given by:

$$
\begin{aligned}
\overline{a^T}_3(x) &= \bar{a}_3(x)\\
\overline{a^T}_2(x) &= \frac{2\mu+1}{2\lambda+1}\bar{a}_2(x)\\
\overline{a^T}_1(x) &= \frac{3\mu^2+3\mu-1}{3\lambda^2+3\lambda-1}\bar{a}_1(x)\\
\overline{a^T}_0(x) &= \frac{\mu(\mu+1)(2\mu+1)}{\lambda(\lambda+1)(2\lambda+1)}\bar{a}_0(x)
\end{aligned}
\tag{7}
$$

It follows immediately from the formula (6), that this formula defines an isomorphism of Vect($\mathbf{R}$)-modules: $T : \mathcal{D}^3_\lambda \cong \mathcal{D}^3_\mu$.

Theorem 1 (i) is proven.

5.2 Proof of Theorem 4. Let us show that the operator (7) in terms of symmetric expressions of Lie derivatives is given by (3).

The first equality in (3) coincides with the last equality in (7).

1) Consider a first order operator of a Lie derivative

$$L^\lambda_X = X(x)\frac{d}{dx} - \lambda X'(x).$$

Its normal symbol defined by (5) is

$$\overline{P}_{L^\lambda_X} = \xi X(x).$$

One obtains the second equality of the formula (3).

2) The anti-commutator

$$[L^\lambda_X, L^\lambda_Y]_+ = 2XY\frac{d^2}{dx^2} + (1-2\lambda)(XY)'\frac{d}{dx} - \lambda(XY'' + X''Y) + 2\lambda^2 X'Y'$$

has the following normal symbol:

$$\overline{P}_{[L^\lambda_X, L^\lambda_Y]_+} = 2\xi^2 XY - \frac{2}{3}\lambda(\lambda+1)(XY'' + X''Y - X'Y')$$

which also following from (5). The third equality of (3) follows now from the second and the fourth ones of (7).

3) The normal symbol of a third order expression $[L_X^\lambda, L_Y^\lambda, L_Z^\lambda]_+ := \mathrm{Sym}_{X,Y,Z}(L_X^\lambda L_Y^\lambda L_Z^\lambda)$ can be also easily calculated from (5). The result is:

$$\begin{aligned}\overline{P}_{[L_X^\lambda, L_Y^\lambda, L_Z^\lambda]_+} &= 6\xi^3 XYZ \\ &-(3\lambda^2+3\lambda-1)\xi(XYZ''+XY''Z+X''YZ-\tfrac{1}{5}(XYZ)'') \\ &-\lambda(\lambda+1)(2\lambda+1)(XYZ'''+XY'''Z+X'''YZ)\end{aligned}$$

This formula implies the last equality of (3).

6.3 Remarks. a) The normal symbols of $[L_X^\lambda, L_Y^\lambda]_+$ and $[L_X^\lambda, L_Y^\lambda, L_Z^\lambda]_+$ are given by very simple and harmonic expressions (which implies the diagonal form (3) of operator T). It would be interesting to understand better the geometric reason of this fact.

b) Comparing the formulæ (3) and (7), one finds a coincidence between coefficients. This fact shows that, in some sense, the symmetric expressions of Lie derivatives and the normal symbol represent the same thing in terms of differential operators and in terms of polynomial functions on $T^*\mathbf{R}$, respectively. We do not see any reason *a-priori* for this remarkable coincidence.

6 Uniqueness of operator T

In this section we prove that the isomorphism T defined by the formula (7) is unique (up to a constant). We also show that in the higher order case $k \geq 4$ there is no analogues of this operator.

6.1 Proof of Theorem 3. The normal symbol of an operator $A \in \mathcal{D}_\lambda^3$ *is* at the same time a normal symbol of $T(A) \in \mathcal{D}_\mu^3$, since operator T is equivariant. The normal symbol is unique up to normalization (cf. Proposition 4.1, part (ii)), therefore the polynomial $\overline{P}_{T(A)}(x,\xi)$ defined by the formula (5), is of the form:

$$\overline{P}_{T(A)}(x,\xi) = \alpha_3\xi^3\bar{a}_3(x) + \alpha_2\xi^2\bar{a}_2(x) + \alpha_1\xi\bar{a}_1(x) + \alpha_0\bar{a}_0(x),$$

where $\alpha_i \in \mathbf{R}$ are some constants depending on λ and μ. Choose $\alpha_3 = 1$. It follows immediately from the formula (6) (after substitution $a_4 \equiv 0$) that the formula (7) gives the unique choice of the constants $\alpha_2, \alpha_1, \alpha_0$ such that operator T is equivariant.

Theorem 3 is proven.

6.2 Proof of Theorem 2. Suppose now that $\Phi : \mathcal{D}^4_\lambda \to \mathcal{D}^4_\mu$ is an isomorphism. In the same way, it follows that in terms of normal symbols, operator Φ is diagonal. More precisely, if $A \in \mathcal{D}^4_\lambda$, then the normal symbol of the operator $\Phi(A) \in \mathcal{D}^4_\mu$ is:

$$\overline{P}_{\Phi(A)}(x,\xi) = \alpha_4\xi^4\bar{a}_4(x) + \alpha_3\xi^3\bar{a}_3(x) + \alpha_2\xi^2\bar{a}_2(x) + \alpha_1\xi\bar{a}_1(x) + \alpha_0\bar{a}_0(x),$$

where $\bar{a}_i$ are the components of the normal symbol of A, $\alpha_i \in \mathbf{R}$. The condition of equivariance implies :

$$\begin{array}{ll}
\dfrac{\alpha_2}{\alpha_0} = \dfrac{\lambda(\lambda+1)}{\mu(\mu+1)}, & \dfrac{\alpha_3}{\alpha_0} = \dfrac{\lambda(\lambda+1)(2\lambda+1)}{\mu(\mu+1)(2\mu+1)} \\
\dfrac{\alpha_4}{\alpha_0} = \dfrac{\lambda(\lambda+1)(12\lambda^2+12\lambda+11)}{\mu(\mu+1)(12\mu^2+12\mu+11)}, & \dfrac{\alpha_4}{\alpha_2} = \dfrac{6\lambda^2+6\lambda-5}{6\mu^2+6\mu-5} \\
\dfrac{\alpha_3}{\alpha_1} = \dfrac{3\lambda^2+3\lambda-1}{3\mu^2+3\mu-1}, & \dfrac{\alpha_4}{\alpha_1} = \dfrac{\lambda(\lambda+1)(2\lambda+1)}{\mu(\mu+1)(2\mu+1)}
\end{array}$$

This system of equation has solutions if and only if $\lambda = \mu$ or $\lambda + \mu = -1$. For $\lambda = \mu$ one has: $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ and for $\lambda + \mu = -1$ one has: $\alpha_0 = -\alpha_1 = \alpha_2 = -\alpha_3 = \alpha_4$.

Theorem 2 is proven for $k = 4$.

Theorem 2 follows now from one of the results of [8]: given an isomorphism $\Phi : \mathcal{D}^k_\lambda \to \mathcal{D}^k_\mu$, then the restriction of Φ to $\mathcal{D}^4_\lambda$ is an isomorphism of Vect($\mathbf{R}$)-modules: $\mathcal{D}^4_\lambda \to \mathcal{D}^4_\mu$. (To prove this, it is sufficient to suppose equivariance of Φ with respect to the affine algebra with generators $\frac{d}{dx}, x\frac{d}{dx}$, see [8]).

This implies that $\lambda = \mu$ or $\lambda + \mu = -1$.

Theorem 2 is proven.

7 Relation with the cohomology group $H^1(\mathrm{Vect}(\mathbf{R}); \mathbf{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu))$

The problem of isomorphism of Vect($\mathbf{R}$)-modules $\mathcal{D}^k_\lambda$ for different values of λ is related to the first cohomology group $H^1(\mathrm{Vect}(\mathbf{R}); \mathrm{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu))$. This cohomology group has already been calculated by B.L. Feigin and D.B. Fuchs (in the case of formal series) [5].

7.1 Nontrivial cocycles.The relation of $\mathrm{Vect}(\mathbf{R})$-action on the space of differential operators and the cohomology groups $H^1(\mathrm{Vect}(\mathbf{R}); \mathrm{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu))$ is given by the following construction.

Let us associate to the bilinear mappings J_m defined by the formula (6), a linear mapping $c_m : \mathrm{Vect}(\mathbf{R}) \to \mathrm{Hom}(\mathcal{F}_\mathbf{s}, \mathcal{F}_{\mathbf{s+1-m}})$:

$$c_m(X)(a) := J_m(X, a),$$

where $a \in \mathcal{F}_s$.

A remarkable property of transvectants J_3 and J_4 is:

Lemma 7.1. *For each value of s, the mappings c_3 and c_4 are non-trivial cocycles on* $\mathrm{Vect}(\mathbf{R})$*:*
(i) $c_3 \in Z^1(\mathrm{Vect}(\mathbf{R}); \mathrm{Hom}(\mathcal{F}_\mathbf{s}, \mathcal{F}_{\mathbf{s-2}}))$,
(ii) $c_4 \in Z^1(\mathrm{Vect}(\mathbf{R}); \mathrm{Hom}(\mathcal{F}_\mathbf{s}, \mathcal{F}_{\mathbf{s-3}}))$.

Proof. From the fact that the formula (6) defines a $\mathrm{Vect}(\mathbf{R})$ action one checks that for any $X, Y \in \mathrm{Vect}(\mathbf{R})$:

$$[L_X, c_m(Y)] - [L_Y, c_m(X)] = c_m([X, Y])$$

with $m = 3, 4$. This means, that c_3 and c_4 are cocycles.

The cohomology classes $[c_3], [c_4] \neq 0$. Indeed, verify that c_3 and c_4 are cohomological to the non-trivial cocycles:

$$\begin{aligned} \widetilde{c}_3(X)(a) &= X'''a + 2X''a', \\ \widetilde{c}_4(X)(a) &= X'''a' + X''a'' \end{aligned}$$

from [5].

Lemma 7.1 is proven.

7.2 Proof of Theorem 1, part (ii). First, remark that for every $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$, the formula

$$\begin{aligned} \rho_X(a_3) &= L_X^3(a_3) \\ \rho_X(a_2) &= L_X^2(a_2) \\ \rho_X(a_1) &= L_X^1(a_1) + \alpha_1 J_3(X, a_3) \\ \rho_X(a_0) &= L_X^0(a_0) + \alpha_2 J_3(X, a_2) + \alpha_3 J_4(X, \bar{a}_3) \end{aligned}$$

defines a Vect($\mathbf{R}$)-action. Indeed, this formula coincides with (6) in the case $a_4 \equiv 0$ and for the special values of $\alpha_1, \alpha_2, \alpha_3$, however, the constants $\alpha_1, \alpha_2, \alpha_3$ are independent.

The Vect($\mathbf{R}$)-action ρ is a *non-trivial 3-parameter deformation* of the standard action on the direct sum $\mathcal{F}_3 \oplus \mathcal{F}_2 \oplus \mathcal{F}_1 \oplus \mathcal{F}_0$.

The fact, that the cocycles c_3 and c_4 are non-trivial, is *equivalent* to the fact that the defined Vect($\mathbf{R}$)-modules with:

1) $\alpha_1, \alpha_2, \alpha_3 \neq 0$,

2) $\alpha_1 = 0, \alpha_2 \neq 0, \alpha_3 \neq 0$,

3) $\alpha_1 \neq 0, \alpha_2 = 0, \alpha_3 \neq 0$,

4) $\alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_3 = 0$,

are not isomorphic to each other.

The Vect($\mathbf{R}$)-modules $\mathcal{D}^3_\lambda$ (given by the formula (6) with $a_4 \equiv 0$) corresponds to the case 1) for general values of λ, to the case 2) for $\lambda = -\frac{1}{2} \pm \frac{\sqrt{21}}{6}$, to the case 3) for $\lambda = -\frac{1}{2}$ and to the case 4) for $\lambda = 0, -1$. Therefore, one obtains 5 critical values of the degree for which Vect($\mathbf{R}$)-module structure on the space of third order operators is special.

Theorem 1 (ii) is proven.

Remark. For each value of λ at least one of constants $\alpha_1, \alpha_2, \alpha_3 \neq 0$. This implies that the module $\mathcal{D}^3_\lambda$ is not isomorphic to the direct sum $\mathcal{F}_3 \oplus \mathcal{F}_2 \oplus \mathcal{F}_1 \oplus \mathcal{F}_0$.

8 Explicit formula for the intertwining operator

We give here the explicit formula for the operator (2) intertwining Vect($\mathbf{R}$)-actions on $\mathcal{D}^3$ which follows from the expression for the operator T in terms of the normal symbol (7).

For every $A \in \mathcal{D}^3_\lambda$, the operator $T(A) \in \mathcal{D}^3_\mu$

$$T(A) = a_3^T \frac{d^3}{dx^3} + a_2^T \frac{d^2}{dx^2} + a_1^T \frac{d}{dx} + a_0^T$$

is given by the following formula:

$$
\begin{aligned}
a_3^T &= a_3 \\
a_2^T &= \frac{2\mu+1}{2\lambda+1}a_2 + \frac{3(\mu-\lambda)}{2\lambda+1}a_3' \\
a_1^T &= \frac{3\mu^2+3\mu-1}{3\lambda^2+3\lambda-1}a_1 \\
&\quad + \frac{(\lambda-\mu)(\mu(12\lambda-1)-\lambda+3)}{2(2\lambda+1)(3\lambda^2+3\lambda-1)}a_2' \\
&\quad + \frac{3}{2}\frac{\mu^2(5\lambda-1)-\mu(6\lambda^2+\lambda-1)+\lambda^3+2\lambda^2-\lambda}{(2\lambda+1)(3\lambda^2+3\lambda-1)}a_3'' \\
a_0^T &= \frac{\mu(\mu+1)(2\mu+1)}{\lambda(\lambda+1)(2\lambda+1)}a_0 \\
&\quad - \frac{\mu^3(3\lambda+5)-\mu^2(3\lambda^2-6)-\mu(5\lambda^2+6\lambda)}{(\lambda+1)(2\lambda+1)(3\lambda^2+3\lambda-1)}a_1' \\
&\quad + \frac{\mu^3(3-\lambda)-\mu^2(6\lambda^2+7\lambda-5)+\mu(7\lambda^3+4\lambda^2-5\lambda)}{2(\lambda+1)(2\lambda+1)(3\lambda^2+3\lambda-1)}a_2'' \\
&\quad - \frac{\mu^3(3\lambda^2+1)-3\mu^2(\lambda^2+2\lambda-1)-\mu(3\lambda^4-3\lambda^3-5\lambda^2+3\lambda)}{2(\lambda+1)(2\lambda+1)(3\lambda^2+3\lambda-1)}a_3'''
\end{aligned}
\tag{8}
$$

We de not prove this formula since we do not use it in this paper.

Remarks. 1) If $\lambda = \mu$, then the operator T defined by this formula is identity, if $\lambda + \mu = -1$, then T is the operator of conjugation.

2) The fact that operator T is equivariant implies that *the formula (8) does not depend on the choice of the coordinate x.*

9 Discussion and final remarks

Let us give here few examples and applications of the normal symbols (5).

9.1 Examples. The notion of normal symbol was introduced in [4] in the case of second order differential operators. In this case, for $\lambda = \frac{1}{2}$ (operators on $-\frac{1}{2}$-densities), the normal symbol (5) is just the standard total symbol:

$\bar{a}_2 = a_2, \bar{a}_1 = a_1, \bar{a}_0 = a_0$. This corresponds to the classical example of second order operators on $-\frac{1}{2}$-densities (cf. the footnote at the introduction).

In the same way, the semi-integer values $\lambda = 1, \frac{3}{2}, 2, \ldots$ corresponds to particularly simple expressions of the normal symbol for operators of order $k = 3, 4, 5, \ldots$.

9.2 Modules of second order differential operators on R. a) The module of second order operators $\mathcal{D}^2_\lambda$ with $\lambda = 0, -1$, is decomposed to a direct sum:

$$\mathcal{D}^2_0 \cong \mathcal{D}^2_{-1} \cong \mathcal{F}_2 \oplus \mathcal{F}_1 \oplus \mathcal{F}_0.$$

Indeed, the coefficients $\bar{a}_2, \bar{a}_1, \bar{a}_0$ transform as tensor-densities (cf formula (6)). This module is special: $\mathcal{D}^2_\lambda$ is not isomorphic to $\mathcal{D}^2_0$ for $\lambda \neq 0, -1$ (see [4]).

b) For every $\lambda, \mu \neq 0, -1$, $\mathcal{D}^2_\lambda \cong \mathcal{D}^2_\mu$.

9.3 Operators on $\frac{1}{2}$-densities. For every $k \geq 3$, the module $\mathcal{D}^k_{-\frac{1}{2}}$ (corresponding to $\frac{1}{2}$-densities) is special. It is decomposed into a sum of submodules: of symmetric operators and of skew-symmetric operators.

9.4 Normal symbol and Weil symbol. The Weil quantization defines a 1-parameter family of mappings from the space of polynomials $C[\xi, x]$ to the space of differential operators on $\mathbf{R}$ with polynomial coefficients. One associate to a polynomial the symmetric expression in $\hbar\frac{d}{dx}$ and x: $F(\xi, x) \mapsto \mathrm{Sym}F(\hbar\frac{d}{dx}, x)$. This (1,1)-correspondence between differential operators and polynomials is sl_2-equivariant. However, in the Weil quantization the action of the Lie algebra sl_2 on differential operators is generated by $x^2, x\frac{d}{dx} + \frac{d}{dx}x, \frac{d^2}{dx^2}$ and therefore, is completely different from the normal symbol.

9.5 Automorphic (pseudo)differential operators. The notion of canonical symbol is related (and in some sense inverse) to the construction of the recent work P. Cohen, Yu. Manin and D. Zagier [3] of a PSL_2-equivariant (pseudo)differential operator associated to a holomorphic tensor-density on the upper half-plane.

9.6 Exotic $\star$-product. Another way to understand this sl_2-equivariant correspondence between linear differential operators and polynomials in ξ, x leads to a $\star$-product on the algebra of Laurent polynomials on $T^*\mathbf{R}$ which is not equivalent to the standard Moyal-Weil quantization (see [10]).

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