Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions∗

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Abstract

Estimates on the initial coefficients are obtained for normalized analytic functions \( f \) in the open unit disk with \( f \) and its inverse \( g = f^{-1} \) satisfying the conditions that \( zf'(z)/f(z) \) and \( zg'(z)/g(z) \) are both subordinate to a starlike univalent function whose range is symmetric with respect to the real axis. Several related classes of functions are also considered, and connections to earlier known results are made.

Keywords: Univalent functions, bi-univalent functions, bi-starlike functions, bi-convex functions, subordination

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1. Introduction

Let \( \mathcal{A} \) be the class of all analytic functions \( f \) in the open unit disk \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \). The Koebe one-quarter theorem ensures that the image of \( \mathbb{D} \) under every univalent function \( f \in \mathcal{A} \) contains a disk of radius 1/4. Thus every univalent function \( f \) has an inverse \( f^{-1} \) satisfying \( f^{-1}(f(z)) = z \), \( z \in \mathbb{D} \) and

\[
f(f^{-1}(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq 1/4).
\]

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{D} \) if both \( f \) and \( f^{-1} \) are univalent in \( \mathbb{D} \). Let \( \sigma \) denote the class of bi-univalent functions defined in the unit disk \( \mathbb{D} \). A domain \( D \subset \mathbb{C} \) is convex if the line segment joining any two points in \( D \) lies entirely in \( D \), while a domain is starlike with respect to a point \( w_0 \in D \) if the line segment joining any point of \( D \) to \( w_0 \) lies inside \( D \). A function \( f \in \mathcal{A} \) is starlike if \( f(\mathbb{D}) \) is a starlike domain with respect to the origin, and convex if \( f(\mathbb{D}) \) is convex. Analytically, \( f \in \mathcal{A} \) is starlike if and only if \( \Re zf'(z)/f(z) > 0 \), whereas \( g \in \mathcal{A} \) is convex if and only if \( 1 + \Re zf''(z)/f'(z) > 0 \). The classes consisting of starlike and convex functions are denoted by \( \mathcal{ST} \) and \( \mathcal{CV} \) respectively. The classes \( \mathcal{ST}(\alpha) \) and \( \mathcal{CV}(\alpha) \) of starlike and convex functions of order \( \alpha, 0 \leq \alpha < 1 \), are respectively characterized by \( \Re zf'(z)/f(z) > \alpha \) and \( 1 + \Re zf''(z)/f'(z) > \alpha \). Various subclasses of starlike and convex functions are often investigated. These functions are typically characterized by the quantity \( zf'(z)/f(z) \) or \( 1 + zf''(z)/f'(z) \) lying in a certain domain starlike with respect to 1 in the right-half plane. Subordination is useful to unify these subclasses.

An analytic function \( f \) is subordinate to an analytic function \( g \), written \( f(z) \prec g(z) \), provided there is an analytic function \( w \) defined on \( \mathbb{D} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) satisfying \( f(z) = g(w(z)) \). Ma and Minda unified various subclasses of starlike and convex functions for which either of the quantity \( zf'(z)/f(z) \) or \( 1 + zf''(z)/f'(z) \) is subordinate to a more general superordinate function. For this purpose, they considered an analytic function \( \varphi \) with positive real part in the unit disk \( \mathbb{D} \), \( \varphi(0) = 1, \varphi'(0) > 0 \), and \( \varphi \) maps \( \mathbb{D} \) onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions \( f \in \mathcal{A} \) satisfying the subordination \( zf'(z)/f(z) \prec \varphi(z) \). Similarly, the class of Ma-Minda convex functions consists of functions \( f \in \mathcal{A} \) satisfying the subordination

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Let \( f \in H \) with \( |f(z)| < 1 \) for \( z \in D \), and let \( f(0) = 0 \) and \( f'(0) = 1 \). Then there are analytic functions \( u, v : D \to D \), with \( u(0) = v(0) = 0 \), satisfying

\[
 f'(z) = \varphi(u(z)) \quad \text{and} \quad g'(w) = \varphi(v(w)).
\]

Define the functions \( p_1 \) and \( p_2 \) by

\[
p_1(z) := \frac{1 + u(z)}{1 - u(z)} = 1 + c_1z + c_2z^2 + \cdots \quad \text{and} \quad p_2(z) := \frac{1 + v(z)}{1 - v(z)} = 1 + b_1z + b_2z^2 + \cdots,
\]

or, equivalently,

\[
u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2} \left( c_1z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots \right),
\]

and

\[
v(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2} \left( b_1z + \left( b_2 - \frac{b_1^2}{2} \right) z^2 + \cdots \right).
\]
Then $p_1$ and $p_2$ are analytic in $\mathbb{D}$ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \mathbb{D} \to \mathbb{D}$, the functions $p_1$ and $p_2$ have positive real part in $\mathbb{D}$, and $|b_i| \leq 2$ and $|c_i| \leq 2$. In view of (2.4), (2.5) and (2.6), clearly

$$f'(z) = \varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) \quad \text{and} \quad g'(w) = \varphi \left( \frac{p_2(w) - 1}{p_2(w) + 1} \right).$$

(2.7)

Using (2.5) and (2.6) together with (2.1), it is evident that

$$\varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = 1 + \frac{1}{2} B_1 c_1 z + \left( \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right) z^2 + \cdots$$

(2.8)

and

$$\varphi \left( \frac{p_2(w) - 1}{p_2(w) + 1} \right) = 1 + \frac{1}{2} B_1 b_1 w + \left( \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2 \right) w^2 + \cdots.$$ 

(2.9)

Since $f \in \sigma$ has the Maclaurin series given by (2.2), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 + \cdots.$$ 

Since

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + \cdots \quad \text{and} \quad g'(w) = 1 - 2a_2 w + 3(2a_2^2 - a_3) w^2 + \cdots.$$ 

it follows from (2.7), (2.8) and (2.9) that

$$2a_2 = \frac{1}{2} B_1 c_1,$$ 

(2.10)

$$3a_3 = \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2,$$ 

(2.11)

$$-2a_2 = \frac{1}{2} B_1 b_1$$

(2.12)

and

$$3(2a_2^2 - a_3) = \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2.$$ 

(2.13)

From (2.10) and (2.12), it follows that

$$c_1 = -b_1.$$ 

(2.14)

Now (2.11), (2.13), (2.14) and (2.14) yield

$$a_2^2 = \frac{B_1^2 (b_2 + c_2)}{4(3B_1^2 - 4B_2 + 4B_1)},$$

which, in view of the well-known inequalities $|b_2| \leq 2$ and $|c_2| \leq 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.3).

By subtracting (2.13) from (2.11), further computations using (2.10) and (2.14) lead to

$$a_3 = \frac{1}{12} B_1 (c_2 - b_2) + \frac{1}{16} B_2^2 c_1^2,$$

and this yields the estimate given in (2.3).
Remark 2.1. For the class of strongly starlike functions, the function $\varphi$ is given by

$$\varphi(z) = \left(\frac{1 + z}{1 - z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots \quad (0 < \gamma \leq 1),$$

which gives $B_1 = 2\gamma$ and $B_2 = 2\gamma^2$. Hence the inequalities in (2.3) reduce to the result in [12, Theorem 1, inequality (2.4), p.3]. In the case

$$\varphi(z) = \frac{1 + (1 - 2\gamma)z}{1 - z} = 1 + 2(1 - \gamma)z + 2(1 - \gamma)z^2 + \cdots,$$

then $B_1 = B_2 = 2(1 - \gamma)$, and thus the inequalities in (2.3) reduce to the result in [12, Theorem 2, inequality (3.3), p.4].

A function $f \in \sigma$ is said to be in the class $ST_\sigma(\alpha, \varphi)$, $\alpha \geq 0$, if the following subordinations hold:

$$\frac{zf'(z)}{f(z)} + \alpha z^2 f''(z) = \varphi(z) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} = \varphi(w), \quad g(w) := f^{-1}(w).$$

Note that $ST_\sigma(\varphi) \equiv ST_\sigma(0, \varphi)$. For functions in the class $ST_\sigma(\alpha, \varphi)$, the following coefficient estimates are obtained.

Theorem 2.2. Let $f$ given by (2.2) be in the class $ST_\sigma(\alpha, \varphi)$. Then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{|B_1|^2 (1 + 4\alpha) + (B_1 - B_2)(1 + 2\alpha)^2},$$

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{(1 + 4\alpha)}. \quad (2.16)$$

Proof. Let $f \in ST_\sigma(\alpha, \varphi)$. Then there are analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\frac{zf'(z)}{f(z)} + \alpha z^2 f''(z) = \varphi(u(z)) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} = \varphi(v(w)), \quad (g = f^{-1}). \quad (2.17)$$

Since

$$\frac{zf'(z)}{f(z)} + \alpha z^2 f''(z) = 1 + a_2(1 + 2\alpha)z + (2(1 + 3\alpha)a_3 - (1 + 2\alpha)a_2^2)z^2 + \cdots$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} = 1 - (1 + 2\alpha)a_2w + ((3 + 10\alpha)a_2^2 - 2(1 + 3\alpha)a_3)w^2 + \cdots,$$

then (2.8), (2.9) and (2.17) yield

$$a_2(1 + 2\alpha) = \frac{1}{2} B_1 c_1, \quad (2.18)$$

$$2(1 + 3\alpha)a_3 - (1 + 2\alpha)a_2^2 = \frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4} B_2 c_1^2, \quad (2.19)$$

$$(1 + 2\alpha)a_1 = \frac{1}{2} B_1 b_1, \quad (2.20)$$

and

$$(3 + 10\alpha)a_2^2 - 2(1 + 3\alpha)a_3 = \frac{1}{2} B_1 \left(\frac{b_1}{2} - \frac{b_1^2}{2}\right) + \frac{1}{4} B_2 b_1^2. \quad (2.21)$$
It follows from (2.18) and (2.20) that
\[ c_1 = -b_1. \]

Equations (2.19), (2.20), (2.21) and (2.22) lead to
\[ a_2^2 = \frac{B_1^3(b_2 + c_2)}{4(B_1^2(1 + 4\alpha) + (B_1 - B_2)(1 + 2\alpha)^2)}, \]
which, in view of the inequalities \(|b_2| \leq 2\) and \(|c_2| \leq 2\) for functions with positive real part, yield
\[ |a_2|^2 \leq \frac{B_1^3}{|B_1^2(1 + 4\alpha) + (B_1 - B_2)(1 + 2\alpha)^2|}. \]
Since \(B_1 > 0\), the last inequality, upon taking square roots, gives the desired estimate on \(|a_2|\) given in (2.15).

Now, further computations from (2.19), (2.20), (2.21) and (2.22) lead to
\[ a_3 = \frac{(B_1/2)((3 + 10\alpha)c_2 + (1 + 2\alpha)b_2) + b_1^2(1 + 3\alpha)(B_2 - B_1)}{4(1 + 3\alpha)(1 + 4\alpha)}, \]
which, using the inequalities \(|b_1| \leq 2\), \(|b_2| \leq 2\) and \(|c_2| \leq 2\) for functions with positive real part, yields
\[ |a_3| \leq \frac{(B_1/2)(2(3 + 10\alpha) + 2(1 + 2\alpha)) + 4(1 + 3\alpha)(B_2 - B_1)}{4(1 + 3\alpha)(1 + 4\alpha)} = \frac{B_1 + |B_2 - B_1|}{(1 + 4\alpha)}. \]
This completes the proof of the estimate in (2.16).

For \(\alpha = 0\), Theorem 2.2 readily yields the following coefficient estimates for Ma-Minda bi-starlike functions.

**Corollary 2.1.** Let \(f\) given by (2.2) be in the class \(ST_{\sigma}(\varphi)\). Then
\[ |a_2| \leq \frac{B_1\sqrt{B_1}}{|B_1^2 + B_1 - B_2|} \quad \text{and} \quad |a_3| \leq B_1 + |B_2 - B_1|. \]

**Remark 2.2.** For the class of strongly starlike functions, the function \(\varphi\) is given by
\[ \varphi(z) = \left(1 + \frac{z}{1 - z}\right)^\gamma = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots \quad (0 < \gamma \leq 1), \]
and so \(B_1 = 2\gamma\) and \(B_2 = 2\gamma^2\). Hence, when \(\alpha = 0\) (bi-starlike function), the inequality in (2.15) reduces to the estimates in [3, Theorem 2.1]. On the other hand, when \(\alpha = 0\) and
\[ \varphi(z) = \frac{1 + (1 - 2\gamma)z}{1 - z} = 1 + 2(1 - \gamma)z + 2(1 - \gamma)z^2 + \cdots, \]
then \(B_1 = B_2 = 2(1 - \gamma)\) and thus the inequalities in (2.15) and (2.16) reduce to the estimates in [3, Theorem 3.1].

Next, a function \(f \in \sigma\) belongs to the class \(M_\sigma(\alpha, \varphi)\), \(\alpha \geq 0\), if the following subordinations hold:
\[ (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \]
and
\[ (1 - \alpha)\frac{wg'(w)}{g(w)} + \alpha \left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \varphi(w), \]
g\(w) := f^{-1}(w)\). A function in the class \(M_\sigma(\alpha, \varphi)\) is called bi-Mocanu-convex function of Ma-Minda type. This class unifies the classes \(ST_{\sigma}(\varphi)\) and \(CV_{\sigma}(\varphi)\).

For functions in the class \(M_\sigma(\alpha, \varphi)\), the following coefficient estimates hold.
**Theorem 2.3.** Let $f$ given by (2.1) be in the class $\mathcal{M}_\sigma(\alpha, \varphi)$. Then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{(1 + \alpha)|B_1^2 + (1 + \alpha)(B_1 - B_2)|}}$$  \hspace{1cm} (2.23)

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{1 + \alpha}.$$  \hspace{1cm} (2.24)

**Proof.** If $f \in \mathcal{M}_\sigma(\alpha, \varphi)$, then there are analytic functions $u, v : \mathbb{D} \rightarrow \mathbb{D}$, with $u(0) = v(0) = 0$, such that

$$(1 - \alpha)zf'(z) + \alpha \left(1 + zf''(z)\right) = \varphi(u(z))$$  \hspace{1cm} (2.25)

and

$$(1 - \alpha)wg'(w) + \alpha \left(1 + wg''(w)\right) = \varphi(v(w)).$$  \hspace{1cm} (2.26)

Since

$$(1 - \alpha)zf'(z) + \alpha \left(1 + zf''(z)\right) = 1 + (1 + \alpha)a_2 z + \cdots$$

and

$$(1 - \alpha)wg'(w) + \alpha \left(1 + wg''(w)\right) = 1 - (1 + \alpha)a_2 w + \cdots,$$

from (2.25), (2.26), (2.27) and (2.28), it follows that

$$(1 + \alpha)a_2 = \frac{1}{2} B_1 c_1,$$  \hspace{1cm} (2.27)

$$2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2 = \frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4} B_2 c_1^2,$$  \hspace{1cm} (2.28)

$$-(1 + \alpha)a_2 = \frac{1}{2} B_1 b_1,$$  \hspace{1cm} (2.29)

and

$$(3 + 5\alpha)a_2^2 - 2(1 + 2\alpha)a_3 = \frac{1}{2} B_1 \left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4} B_2 b_1^2.$$  \hspace{1cm} (2.30)

The equations (2.27) and (2.29) yield

$$c_1 = -b_1.$$  \hspace{1cm} (2.31)

From (2.28), (2.30) and (2.31), it follows that

$$a_2^2 = \frac{B_1^2(b_2 + c_2)}{4(1 + \alpha)(B_1^2 + (1 + \alpha)(B_1 - B_2))},$$

which yields the desired estimate on $|a_2|$ as described in (2.23).

As in the earlier proofs, use of (2.25), (2.26), (2.27) and (2.30) shows that

$$a_3 = \frac{(B_1/2)((1 + 3\alpha)b_2 + (3 + 5\alpha)c_2) + b_1^2(1 + 2\alpha)(B_2 - B_1)}{4(1 + \alpha)(1 + 2\alpha)},$$

which yields the estimate (2.24).
For $\alpha = 0$, Theorem 2.3 gives the coefficient estimates for Ma-Minda bi-starlike functions, while for $\alpha = 1$, it gives the following estimates for Ma-Minda bi-convex functions.

**Corollary 2.2.** Let $f$ given by (2.1) be in the class $CV, (\varphi)$. Then

$$|a_2| \leq \frac{B_1}{\sqrt{2|B_1^2 + 2B_1 - 2B_2|}} \quad \text{and} \quad |a_3| \leq \frac{1}{2}(B_1 + |B_2 - B_1|).$$

**Remark 2.3.** For $\varphi$ given by

$$\varphi(z) = \frac{1 + (1 - 2\gamma)z}{1 - z} = 1 + 2(1 - \gamma)z + 2(1 - \gamma)z^2 + \cdots,$$
evidently $B_1 = B_2 = 2(1 - \gamma)$, and thus when $\alpha = 1$ (bi-convex functions), the inequalities in (2.23) and (2.24) reduce to a result in [5, Theorem 4.1].

Next, function $f \in \sigma$ is said to be in the class $L_{\sigma}(\alpha, \varphi), \alpha \geq 0$, if the following subordinations hold:

$$\left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} \prec \varphi(z)$$

and

$$\left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\alpha} \prec \varphi(w),$$

g(w) := f^{-1}(w). This class also reduces to the classes of Ma-Minda bi-starlike and bi-convex functions. For functions in this class, the following coefficient estimates are obtained.

**Theorem 2.4.** Let $f$ given by (2.1) be in the class $L_{\sigma}(\alpha, \varphi)$. Then

$$|a_2| \leq \frac{2B_1\sqrt{B_1}}{\sqrt{2|B_1^2 + 4(\alpha - 2)^2(B_1 - B_2)|}}$$

and

$$|a_3| \leq \frac{2(3 - 2\alpha)(B_1 + |B_1 - B_2|)}{|(3 - 2\alpha)(\alpha^2 - 3\alpha + 4)|}.$$ (2.33)

**Proof.** Let $f \in L_{\sigma}(\alpha, \varphi)$. Then there are analytic functions $u, v : \mathbb{D} \rightarrow \mathbb{D}$, with $u(0) = v(0) = 0$, such that

$$\left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} = \varphi(u(z))$$

and

$$\left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\alpha} = \varphi(v(w)).$$ (2.35)

Since

$$\left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} = 1 + (2 - \alpha)a_2z + \left( 2(3 - 2\alpha) + \frac{(\alpha - 2)^2 - 3(1 - \alpha^2)}{2} a_2^2 \right) z^2 + \cdots$$

and

$$\left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\alpha} = 1 - (2 - \alpha)a_2w + \left( 2(3 - 2\alpha) + \frac{1}{2}(\alpha^2 - 3\alpha - 5) a_2^2 - 2(3 - 2\alpha)a_3 \right) w^2 + \cdots,$$

from (2.28), (2.29), (2.34) and (2.35), it follows that

$$(2 - \alpha)a_2 = \frac{1}{2} B_1 c_1, \quad (2.36)$$
\[2(3 - 2\alpha)a_3 + \left((\alpha - 2)^2 - 3(4 - 3\alpha)\right)\frac{a_2^2}{2} = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \quad (2.37)\]

\[-(2 - \alpha)a_2 = \frac{1}{2}B_1 b_1 \quad (2.38)\]

and

\[\left(8(1 - \alpha) + \frac{1}{2}\alpha(\alpha + 5)\right)a_2^2 - 2(3 - 2\alpha)a_3 = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2. \quad (2.39)\]

Now (2.36) and (2.38) clearly yield

\[c_1 = -b_1. \quad (2.40)\]

Equations (2.37), (2.39) and (2.40) lead to

\[a_2^2 = \frac{B_1^2(b_2 + c_2)}{2(\alpha^2 - 3\alpha + 4)B_1^2 + 4(\alpha - 2)^2(B_1 - B_2)}, \]

which yields the desired estimate on \(|a_2|\) as asserted in (2.32).

Proceeding similarly as in the earlier proof, using (2.37), (2.38), (2.39) and (2.40), it follows that

\[a_3 = \frac{(B_1/2)(16(1 - \alpha) + \alpha(\alpha + 5)c_2 + 3(4 - 3\alpha) - (\alpha - 2)^2b_2) + 2B_1^2(3 - 2\alpha)(B_1 - B_2)}{4(3 - 2\alpha)(\alpha^2 - 3\alpha + 4)}, \]

which yields the estimate (2.33). \(\square\)

**Remark 2.4.** The determination of the sharp estimates for the coefficients \(|a_2|, |a_3|\) and for other coefficients of functions belonging to the classes investigated in this paper are open problems. In fact, some estimate (not necessarily sharp) for \(|a_n|, (n \geq 4)\) would be interesting.

**References**


