A new method for ranking DMUs by interval DEA

Vahideh Rezaie, Member of Young Researchers Club, Islamic Azad University, Yasooj Branch

Email: vahidehrezaie@yahoo.com
Samad Nejatian, Islamic Azad University, Yasooj Branch
Yaghoub Zeraatkish, Islamic Azad University, Yasooj Branch

Abstract: The Interval DEA model has been formulated to obtain an efficiency interval consisting of evaluations from both the optimistic and pessimistic viewpoints. DMUs which are not rated as efficient in the conventional sense are improved so that their lower bounds become as large as possible under the condition that their upper bounds attain the maximum value one. The points obtained by this method are called ideal points. In order to improve the lower bound of the efficiency interval, different ideal points are defined for different DMUs. The purpose of this paper is to rank DMUs by these ideal points for each DMU.

Keywords: Data envelopment analysis (DEA), Decision making unit (DMU), Interval DEA, Ideal points

1. Introduction

DEA is a non-parametric technique for measuring the efficiency of DMUs with common input and output terms [1,2]. The radial models include the CCR and the BCC models, and the non-radial models include the additive model, the multiplication model, the range-adjusted measure (RAM), and the slack-based measure (SBM) [5]. Russell [7] discussed four conditions that are desirable in measuring "technical efficiency". Fare and Lovell [6] proposed a model for evaluation from the pessimistic viewpoint. Many methods have been proposed in order to rank best performers; Andersen and Petersen (AP) [8] and Mehrabian et al. [9] (MAJ) presented two most popular of these methods. The method we propose in this paper, ranks such DMUs, and does not have the above-mentioned problems. The next section addresses obtaining ideal points by interval DEA. In section 3 we give a ranking method by ideal points. And the paper concludes in section 4.

2. Obtaining ideal points by interval DEA

In this section we want to obtain ideal points by interval DEA. Entani and Tanaka [11] have proposed the interval DEA model to obtain the efficiency interval. The problem that obtains the upper bound of the efficiency interval is formulated as follows:

\[
\theta^* = \max \frac{U^r_j}{\sum U^r_j}, \quad \text{s.t.} \quad U \geq 0, V \geq 0.
\]

Where Xu and Yu are the input and output vectors of DMUj, respectively, whose elements are all positive, and the decision variables are the weight vectors U and V.

Also, the lower bound of the Efficiency interval for DMUo can be determined as follows:

\[
\theta^* = \frac{U^r_j}{\sum U^r_j}, \quad \text{s.t.} \quad U \geq 0, V \geq 0.
\]

According to [11], Model (2) can be changed to the following problem:

\[
\theta^o \geq \max \frac{y_r}{\sum y_r}, \quad \text{s.t.} \quad U \geq 0, V \geq 0.
\]

Where the ith element of the input weight vector V and the rth element of the output vector U are One and the other elements are all zero.

Theorem 1. The optimal value of (2) and (3) are equal.

Proof: The proof of the theorem is provided in [11]. DMUs are improved so that their lower bounds become so large as to attain the maximum value One. The points obtained by this method are called ideal points. The ith input element and the rth output element of the ideal point for DMUo are denoted as follows:

\[
X^o_i = \min \left( \frac{y_r}{\sum y_r} \right), \quad i = 1, \ldots, m.
\]

\[
Y^o_r = \max \left( \frac{y_r}{\sum y_r} \right), \quad r = 1, \ldots, s.
\]

3. A new proposed method for ranking by ideal points

We are dealing with n DMUs with the input and output matrices \( X \equiv (x_{ij}) \in \mathbb{R}^{m \times n} \) and \( Y \equiv (y_{ij}) \in \mathbb{R}^{n \times m} \) respectively. The data set is positive, i.e. \( X > 0 \) and \( Y > 0 \). It is often necessary in real performance assessment practice to rank n DMUs in terms of their efficiencies. So, we first obtain the efficiency of DMUs by adjusting the Russell measure model. We know Inefficient DMUs
that have higher $\theta^*_{\text{Russell}}$ will have better ranks. But very often more than one DMU is evaluated as DEA efficient, which makes DEA efficient units unable to be compared or ranked.

Then we calculate the ideal points of efficient DMUs, i.e., $\theta^*_{\text{Russell}}=1$ by models (4), (5). According to the definition of ideal points, we have the following theorem:

**Theorem 2.** A DMU does not dominate its own ideal point $(\text{DMU}_o, o \in \{1, \ldots, n\})$

**Proof:** The proof of the theorem is provided in [3].

For ranking efficiency DMUs, we obtain the distance them of the their ideal points by following distance:

$$d(o, X^*_o, Y^*_o) = \frac{1}{m} \left( \sum_{i=1}^{m} \frac{\theta_i}{\sum_{s=1}^{s} \phi_s} \right)$$

s.t.

$$\begin{align*}
x_i &= \theta X_i, \quad i = 1, \ldots, m \\
y_r &= \phi Y_r, \quad r = 1, \ldots, s \\
\theta_i &\geq 1, \quad i = 1, \ldots, m \\
0 &\leq \phi_r \leq 1, \quad r = 1, \ldots, s
\end{align*}$$

where $d(o, X^*_o, Y^*_o)$ is distance of $\text{DMU}_o = (X^*_o, Y^*_o)$. The Performance of a DMU will be better if its ideal point has a smaller distance to itself.

Because, if $\text{DMU}_o$ has higher output then $\text{DMU}_o$ will have greater input and if $\text{DMU}_o$ has less input then $\text{DMU}_o$ will have smaller output i.e if $\text{DMU}_o$ have better performance then its ideal point is closer to itself.

To elaborate, we apply our proposed model in the following small example, then we compare it with AP and MAJ ranking methods.

**Example1:** We consider 7 DMUs with two inputs and two outputs. The data and the adjusting Russell measure ($\theta^*_{\text{Russell}}$) of the DMUs are shown in Table 1.

### Table 1: inputs, outputs and $\theta^*_{\text{Russell}}$ of DMUs in example 1

<table>
<thead>
<tr>
<th>DMU</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$x_2$</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$y_1$</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>$y_2$</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>$\theta^*_{\text{Russell}}$</td>
<td>0.7576</td>
<td>0.6481</td>
<td>0.8100</td>
<td>0.5303</td>
<td>1</td>
<td>0.5072</td>
<td></td>
</tr>
<tr>
<td>Rank</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>1</td>
<td>7</td>
</tr>
</tbody>
</table>

According to the results of the adjusting Russell measure model, DMUD and DMUF are evaluated as efficient. The ideal points, and $d(o, X^*_o, Y^*_o)$ are shown in Table 2.

### Table 2: Ideal points and $d(o, X^*_o, Y^*_o)$

<table>
<thead>
<tr>
<th>Ideal DMU</th>
<th>$\text{Input}_1$</th>
<th>$\text{Input}_2$</th>
<th>$\text{Output}_1$</th>
<th>$\text{Output}_2$</th>
<th>$d(o, X^<em>_o, Y^</em>_o)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{DMU}_o$</td>
<td>1.33</td>
<td>1.33</td>
<td>6</td>
<td>9</td>
<td>2.9421</td>
</tr>
<tr>
<td>$\text{DMU}_f$</td>
<td>1.60</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>1.2500</td>
</tr>
</tbody>
</table>

The important property of this method is its ability to rank extreme and non-extreme DMUs. We show this property with the following example.