NEW PASSIVITY CRITERIA FOR FUZZY BAM NEURAL NETWORKS WITH MARKOVIAN JUMPING PARAMETERS AND TIME-VARYING DELAYS

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This paper addresses the problem of passivity analysis issue for a class of fuzzy bidirectional associative memory (BAM) neural networks with Markovian jumping parameters and time varying delays. A set of sufficient conditions for the passiveness of the considered fuzzy BAM neural network model is derived in terms of linear matrix inequalities by using the delay fractioning technique together with the Lyapunov function approach. In addition, the uncertainties are inevitable in neural networks because of the existence of modeling errors and external disturbance. Further, this result is extended to study the robust passivity criteria for uncertain fuzzy BAM neural networks with time varying delays and uncertainties. These criteria are expressed in the form of linear matrix inequalities (LMIs), which can be efficiently solved via standard numerical software. Two numerical examples are provided to demonstrate the effectiveness of the obtained results.

Keywords: BAM neural networks, linear matrix inequality, delay fractioning technique, Markovian jump, passivity analysis.

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1. Introduction

Neural networks have found successful applications in various fields of science and engineering. In particular, bidirectional associative memory (BAM) neural networks

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is a type of recurrent neural network which was introduced by Kosko in 1988 [11, 12]. Recently, BAM neural networks have attracted the attention of many researchers due to their applications in many fields such as image processing, pattern recognition, automatic control, associative memory, signal processing and combinatorial optimization [20]. Particularly, when neural networks are employed as associative memories, the equilibrium points represent the stored patterns and the stability of each equilibrium point means that each stored pattern can be retrieved even in the presence of noise [21]. Some of these applications require that the designed network model has a unique stable equilibrium point [13, 14, 23, 26]. Thus, the stability analysis of delayed BAM neural network models has been investigated by many researchers and various sufficient conditions have been reported in the literature, see [5, 24, 25] and references therein.

On the other hand, the passivity theory which originated from circuit theory plays an important role in electrical networks and many other dynamical systems [17]. The passivity approach can also be used to neural networks [7, 28]. The passivity theory is a main tool for analyzing stability of uncertain dynamical systems. The main idea of passivity theory is that the passive properties of a system can keep the system internally stable [9]. Further, the time delay can be frequently encountered in the implementations of artificial neural networks due to the finite switching speed of amplifiers. Therefore, time delays whether constant or variable are ubiquitous in neural networks and will cause instability, divergence and oscillations in the models. Extensive research has been conducted on the stability analysis of neural networks with time varying delay [19, 22, 27]. Moreover, uncertainties are inevitable in neural networks because of the existence of modeling errors and external disturbance [31]. Therefore, it is of both theoretical and practical importance to study the problem of passivity analysis of neural networks with time delays and uncertainties. Recently, passivity analysis for delayed neural networks has been discussed by many researchers [8, 15, 18]. Based on the Lyapunov–Krasovskii theory, passivity conditions for neural networks with time varying delays and uncertainties have been presented via LMIs in [36, 37].

However, in mathematical modeling of real world problems, we will encounter some other inconveniences, for example, the complexity and the uncertainty or vagueness. Fuzzy theory is considered as a more suitable setting for the sake of taking vagueness into consideration. Fuzzy systems in the form of the Takagi–Sugeno (TS) model [30] have attracted rapidly growing interest in recent years. Some nonlinear dynamic systems can be approximated by the overall fuzzy linear TS models for the purpose of dynamical analysis [32]. Recently, the concept of incorporating fuzzy logic into a neural network via TS fuzzy model is proposed as an effective tool for the neural networks. So far, a great number of results have been reported for the stability of TS fuzzy neural networks [2, 6]. Syed Ali and Balasubramaniam [29] studied the robust stability problem of TS fuzzy Cohen–Grossberg BAM neural networks with discrete and distributed time-varying delays by using the Lyapunov functional theory and LMI technique.
Further, the neural networks often display a feature of network modes jumping and such type of jumping are commonly considered to be determined by an ideal homogeneous Markov chain [1]. The Markovian jump systems have a strong practical background, since many physical systems are subjected to abrupt changes in their structures, due to random failures, repairs of components, sudden environmental disturbances, changing subsystem interconnections, and abrupt variations at the operating point of a nonlinear plant, etc. Wang et al. [33] studied the exponential stability of delayed recurrent neural networks with Markovian jumping parameters. The stability problem for a class of continuous time neural networks with time delays and Markovian jumping have been established in [16]. Very recently, Zhang et al. [36] derived a set of sufficient conditions for the passivity analysis of stochastic Markovian switching genetic regularity neural networks with time varying delays. Zhu and Shen [37] discussed the passivity of stochastic delayed neural networks with Markovian switching by applying Lyapunov functional and free-weighting matrix and the passivity criteria presented in terms of linear matrix inequalities. Moreover, an important index for checking the conservatism of passivity criteria is to find the maximum delay bounds in which system can be guaranteed to be passive. Therefore, the choice of a Lyapunov functional and derivation of a passivity condition from the time-derivative of such a functional play an important role in the reduction of conservatism. In this regard, Wang et al. [34] proposed a delay fractioning technique to reduce the conservatism of the stability criteria. Recently, Li et al. [15] discussed the passivity analysis problem for a class of neural networks with discrete and distributed time-varying delays by employing the delay fractioning idea and the obtained results are formulated in LMIs. Hu et al. [10] studied the exponential stability of BAM neural networks by employing the delay fractioning idea together with the free-weighting matrix approach to get the reduced less conservative results via Lyapunov stability theory. More recently, the robust stability of stochastic genetic regulatory networks with time delays by using a delay fractioning approach has been discussed in [35]. As a result, it is shown that the size of delay fractioning gets thinner which leads much to less conservative results.

To the best knowledge of the authors, so far, no result on the passiveness of fuzzy BAM neural networks with Markovian jumping parameters in the presence of time varying delays and uncertainties is available in the existing literature. This motivates our research. By constructing a new Lyapunov–Krasovskii functional and utilizing some advanced techniques, a set of sufficient conditions is derived to ensure the passivity performance for fuzzy BAM neural networks with Markovian jumping parameters along with time-varying delays. Further, this paper considers robust passivity analysis for uncertain fuzzy BAM neural networks with time varying delays and uncertainties. The parametric uncertainties are assumed to be norm bounded. The passivity conditions are formulated and obtained in terms of LMIs, which can be easily solved numerically by using the MATLAB LMI control toolbox. Finally, we provide two numerical examples to demonstrate the effectiveness of the proposed results.
Notation: The superscripts “T” and “(-1)” stand for matrix transposition and matrix inverse; \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space; \( P > 0 \) means that \( P \) is real symmetric and positive definite; \( I_n \) and \( 0_n \) represent \( n \)-dimensional identity matrix and zero matrix. In symmetric block matrices or long matrix expressions, we use an asterisk (\( * \)) to represent a term that is induced by symmetry, \( \text{sym}(A) \) is defined as \( A + A^T \); \( \text{diag}\{.\} \) stands for a block-diagonal matrix and \( (\Omega_1, \mathcal{F}, \mathcal{P}) \) be a complete probability space, in which \( \Omega \) is the sample space, \( \mathcal{F} \) is the \( \sigma \)-algebra of subsets of the sample space, and \( \mathcal{P} \) is the probability measure on \( \mathcal{F} \).

2. Formulation of the problem and preliminaries

Let \( \eta(t) = \eta_t \) (\( t \geq 0 \)) be a right-continuous homogeneous Markov chain on the probability space \( (\Omega, \mathcal{F}, \mathcal{P}) \) which takes values in a finite state space \( \mathbb{S} = \{1, 2, \ldots, s\} \) with generator \( \Pi = [\pi_{kk'}] \), \( k, k' \in \mathbb{S} \) denotes the transition rate matrix with transition probability

\[
\text{Pr}(\eta_{t+\Delta t} = k' / \eta_t = k) = \begin{cases} 
\pi_{kk'} \Delta t + o(\Delta t), & k \neq k', \\
1 + \pi_{kk} \Delta t + o(\Delta t), & k = k',
\end{cases}
\]

where \( \Delta t > 0 \), \( \lim_{\delta t \to 0} (o(\delta t) / \delta t) = 0; \), \( \pi_{kk'} \geq 0 \), is the known transition rate from the mode \( k \) to \( k' \) for \( k \neq k' \) with \( \pi_{kk} = -\sum_{k'=1, k' \neq k}^{s} \pi_{kk'}, k, k' \in \mathbb{S} \).

Consider the following fuzzy BAM neural networks with Markovian jumping parameters and time-varying delay in the form

\[
\begin{align*}
\dot{u}_i(t) &= -a_i(\eta(t)) u_i(t) + \sum_{j=1}^{n} b_{ji}(\eta(t)) F_j(v_j(t)) \\
&\quad + \sum_{j=1}^{n} c_{ji}(\eta(t)) F_j(v_j(t - \rho(t))) + I_i(t), \\
\dot{v}_j(t) &= -d_j(\eta(t)) v_j(t) + \sum_{i=1}^{m} e_{ij}(\eta(t)) G_i(u_i(t)) \\
&\quad + \sum_{i=1}^{m} f_{ij}(\eta(t)) G_i(u_i(t - \tau(t))) + J_j(t)
\end{align*}
\]

for \( i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n, \ t > 0 \), where \( u_i(t) \) and \( v_j(t) \) denote activations of the \( i \)-th and \( j \)-th neurons. \( F_j(\cdot) \) and \( G_i(\cdot) \) stand for the signal functions of the \( j \)-th and \( i \)-th neurons; \( a_i(\eta(t)) \) and \( d_j(\eta(t)) \) are positive constants stand for the rate with which the cell \( i \) and \( j \) reset their potentials to the resting state when isolated from the other cells and inputs; \( b_{ji}(\eta(t)), c_{ji}(\eta(t)), e_{ij}(\eta(t)) \) and \( f_{ij}(\eta(t)) \) denote the synaptic connection weights; \( I_i(t) \) and \( J_j(t) \) denote the external inputs at time \( t \). The bounded functions \( \tau(t) \) and \( \rho(t) \) represent the time varying delay satisfying

\[
0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \quad \dot{\tau}(t) \leq \mu_1, \quad 0 \leq \rho_1 \leq \rho(t) \leq \rho_2, \quad \dot{\rho}(t) \leq \mu_2.
\]

Moreover, the time varying delay is represented into two parts: Constant part and time-varying part, that is, \( \tau(t) = \tau_1 + \tau^*(t) \), and for \( \rho(t) = \rho_1 + \rho^*(t) \), where \( \tau^*(t) \),
\[ \rho^*(t) \text{ satisfy } 0 \leq \tau^*(t) \leq \tau_2 - \tau_1, \quad 0 \leq \rho^*(t) \leq \rho_2 - \rho_1 \text{ and } \tau_2 > \tau_1 > 0, \quad \rho_2 > \rho_1 > 0, \quad \mu_1, \mu_2 > 0 \text{ are constants.} \]

The initial values for system (1) are given by
\[ u_i(t) = \phi_{ui}(t), \quad t \in [-\tau_2, 0], \quad i = 1, 2, \ldots, m \]
\[ v_j(t) = \phi_{vj}(t), \quad t \in [-\rho_2, 0], \quad j = 1, 2, \ldots, n, \]
where \( \phi_{ui}(t) \) and \( \phi_{vj}(t) \) are continuous functions defined on \([-\tau_2, 0]\) and \([-\rho_2, 0]\), respectively.

The system (1) is equivalent to the vector form as follows,
\[
\begin{align*}
\dot{u}(t) &= -A(\eta_t)u(t) + B(\eta_t)F(v(t)) + C(\eta_t)F(v(t - \rho(t))) + I(t), \\
\dot{v}(t) &= -D(\eta_t)v(t) + E(\eta_t)G(u(t)) + F(\eta_t)G(u(t - \tau(t))) + J(t),
\end{align*}
\]
where \( u = [u_1, u_2, \ldots, u_m]^T, \quad v = [v_1, v_2, \ldots, v_n]^T, \quad A(\eta_t) = \text{diag}\{a_1(\eta_t), a_2(\eta_t), \ldots, a_m(\eta_t)\}, \quad D(\eta_t) = \text{diag}\{d_1(\eta_t), d_2(\eta_t), \ldots, d_n(\eta_t)\} \]
represent the feedback connection weights,
\[ B(\eta_t) = [(b_{ji}(\eta_t))]_{n \times m}^T, \quad C(\eta_t) = [(c_{ji}(\eta_t))]_{n \times m}^T, \]
\[ E(\eta_t) = [(e_{ij}(\eta_t))]_{m \times n}^T, \quad F(\eta_t) = [(f_{ij}(\eta_t))]_{m \times n}^T \]
are the connection weights and delayed connection weights and \( I = [I_1, I_2, \ldots, I_m]^T, \quad J = [J_1, J_2, \ldots, J_n]^T \) denote the external input vectors.

As mentioned above, it is reasonable to assume that the network model (3) has only one equilibrium point \( u^* = [u_1^*, u_2^*, \ldots, u_m^*]^T, \quad v^* = [v_1^*, v_2^*, \ldots, v_n^*]^T \). Then we will shift the equilibrium point \( u^* \) and \( v^* \) to the origin. The transformation \( x(\cdot) = u(\cdot) - u^*, \quad y(\cdot) = v(\cdot) - v^* \), \( U_1(\cdot) = I(\cdot) - I^*(\cdot) \) and \( U_2(\cdot) = J(\cdot) - J^*(\cdot) \), \( i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \), change the system (3) to the following form
\[
\begin{align*}
\dot{x}(t) &= -A(\eta_t) x(t) + B(\eta_t) f(y(t)) + C(\eta_t) f(y(t - \rho(t))) + U_1(t), \\
\dot{y}(t) &= -D(\eta_t) y(t) + E(\eta_t) g(x(t)) + F(\eta_t) g(u(t - \tau(t))) + U_2(t),
\end{align*}
\]
where \( g(x(t)) = G(x(t) + u^*) - G(u^*) \) and \( f(y(t)) = F(y(t) + v^*) - F(v^*) \). Further the activation functions satisfy the following assumption:

(A1) For \( i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \), there exist constants \( G_i^-, G_i^+, F_j^- \) and \( F_j^+ \), such that
\[
G_i^- \leq \frac{(g_i(x_1) - g_i(x_2))}{x_1 - x_2} \leq G_i^+, \quad \forall x_1 \neq x_2,
\]
\[
F_j^- \leq \frac{(f_j(y_1) - f_j(y_2))}{y_1 - y_2} \leq F_j^+, \quad \forall y_1 \neq y_2.
\]
Based on the above discussions, in this paper, we consider time delayed fuzzy BAM neural networks with Markovian jumping parameters described by the following nonlinear differential equations. The $p$-th rule of this TS fuzzy model is given by,

**Plant Rule i:** If $u_1(t)$ and $v_1(t)$ are $\eta_1^j$ and ...and $u_p(t)$ and $v_p(t)$ are $\eta_p^j$, then

\[
\begin{align*}
\dot{x}(t) &= -(A_i(\eta_t) + \Delta A_i(t, \eta_t))x(t) + (B_i(\eta_t) + \Delta B_i(t, \eta_t))f(y(t)) \\
&\quad + (C_i(\eta_t) + \Delta C_i(t, \eta_t))f(y(t - \rho(t))) + U_1(t), \\
\dot{y}(t) &= -(D_i(\eta_t) + \Delta D_i(t, \eta_t))y(t) + (E_i(\eta_t) + \Delta E_i(t, \eta_t))g(x(t)) \\
&\quad + (F_i(\eta_t) + \Delta F_i(t, \eta_t))g(x(t - \tau(t))) + U_2(t),
\end{align*}
\]

where $\eta_j^i$, $j = 1, 2, \ldots, p$, are fuzzy sets, $(u_1(t), u_2(t), \ldots, u_p(t), v_1(t), v_2(t), \ldots, v_p(t))^T$ is the premise variable vector, $x(t)$ and $y(t)$ are the state variables, $r$ is the number of IF-THEN rules.

For each possible values of $\eta_t = k$, where $k \in S$, in the succeeding discussion, we will denote the matrices associated with the $k$-th mode by

\[
\begin{align*}
A_{i,k} &= A_i(\eta_t), & \Delta A_{i,k} &= \Delta A_i(t, \eta_t), & B_{i,k} &= B_i(\eta_t), & \Delta B_{i,k} &= \Delta B_i(t, \eta_t), \\
C_{i,k} &= C_i(\eta_t), & \Delta C_{i,k} &= \Delta C_i(t, \eta_t), & D_{i,k} &= D_i(\eta_t), & \Delta D_{i,k} &= \Delta D_i(t, \eta_t), \\
E_{i,k} &= E_i(\eta_t), & \Delta E_{i,k} &= \Delta E_i(t, \eta_t), & F_{i,k} &= F_i(\eta_t), & \Delta F_{i,k} &= \Delta F_i(t, \eta_t),
\end{align*}
\]

where $A_{i,k}, B_{i,k}, C_{i,k}, D_{i,k}, E_{i,k}$ and $F_{i,k}$ are known constant matrices of appropriate dimensions that describe the nominal system. $\Delta A_{i,k}, \Delta B_{i,k}, \Delta C_{i,k}, \Delta D_{i,k}, \Delta E_{i,k}$ and $\Delta F_{i,k}$ for $k \in S$, are unknown matrices that represent the time-varying parameter uncertainties and are assumed to be of the form

\[
\begin{align*}
\begin{bmatrix}
\Delta A_{i,k}, \Delta B_{i,k}, \Delta C_{i,k}, \Delta D_{i,k}, \Delta E_{i,k}, \Delta F_{i,k}
\end{bmatrix}
= M_k Z_k(t) \begin{bmatrix}
E_{1i,k}, E_{2i,k}, E_{3i,k}, E_{4i,k}, E_{5i,k}, E_{6i,k}
\end{bmatrix},
\end{align*}
\]

where $M_k, E_{1i,k}, E_{2i,k}, E_{3i,k}, E_{4i,k}, E_{5i,k}$ and $E_{6i,k}$ are known real constant matrices, and $Z_k(t)$ denotes unknown time-varying matrix function satisfying

\[
Z_k^T(t)Z_k(t) \leq I, \quad \forall k \in S.
\]

It is assumed that all elements $Z_k(t)$ are Lebesgue measurable, and $\Delta A_{i,k}, \Delta B_{i,k}, \Delta C_{i,k}, \Delta D_{i,k}, \Delta E_{i,k}$ and $\Delta F_{i,k}$ are said to be admissible if both (6) and (7) hold.

The defuzzified output of the system (5) is inferred as follows

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \left[ -(A_{i,k} + \Delta A_{i,k})x(t) + (B_{i,k} + \Delta B_{i,k})f(y(t)) \\
&\quad + (C_{i,k} + \Delta C_{i,k})f(y(t - \rho(t))) + U_1(t) \right], \\
\dot{y}(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \left[ -(D_{i,k} + \Delta D_{i,k})y(t) + (E_{i,k} + \Delta E_{i,k})g(x(t)) \\
&\quad + (F_{i,k} + \Delta F_{i,k})g(x(t - \tau(t))) + U_2(t) \right],
\end{align*}
\]
where \( h_i(\theta(t)) = \frac{M_i(\theta(t))}{\sum_{i=1}^{r} M_i(\theta(t))} \); \( M_i(\theta(t)) = \prod_{j=1}^{p} \eta_{ij}(\theta_j(t)) \), \( \eta_{ij}(\theta_j(t)) \) is the grade of membership of \( \theta_j(t) \) in \( \eta_{ij} \). It is assumed that \( M_i(\theta(t)) \geq 0 \), \( i = 1, 2, \ldots, r \); \( \sum_{i=1}^{r} M_i(\theta(t)) > 0 \) for all \( t \). Therefore, \( h_i(\theta(t)) \geq 0 \) (for \( i = 1, 2, \ldots, r \)) and \( \sum_{i=1}^{r} h_i(\theta(t)) = 1 \).

Before ending this section, the following definition is recalled which will be used to prove our main results in the next section.

**Definition 2.1** ([36]). The system (4) is called passive if there exist a scalar \( \lambda \) such that

\[
2 \int_{0}^{t_p} \begin{bmatrix} f^T(y(s)) & g^T(x(s)) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} ds \geq -\lambda \int_{0}^{t_p} \begin{bmatrix} U_1^T(s) & U_2^T(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} ds
\]

for all \( t_p \geq 0 \) and for all solution of fuzzy BAM neural networks (4) with \( x_0 = 0 \) and \( y_0 = 0 \).

### 3. Passivity results

In this section, we consider the passivity criteria for fuzzy delayed BAM neural networks with Markovian jumping parameters and generalized activation functions. One of the main issue in passivity criteria is how to further reduce the possible conservatism induced by the introduction of the Lyapunov functional when dealing with time delays. By employing the idea of delay fractioning technique as in [34, 35], we introduce a new Lyapunov–Krasovskii functional candidate for fuzzy BAM neural networks. Based on the Lyapunov–Krasovskii functional candidate, we first analyze passivity criteria for fuzzy BAM neural networks (8) with time varying delay and without uncertainties, that is, \( \Delta A_{i,k} = \Delta B_{i,k} = \Delta C_{i,k} = \Delta D_{i,k} = \Delta E_{i,k} = \Delta F_{i,k} = 0 \). Further, this result is extended to obtain a new robust passivity condition for BAM fuzzy neural networks with norm-bounded uncertainties. For convenience, in the following, we denote

\[
F_1 = \text{diag} \left\{ F_1^{-}, F_2^{-}, \ldots, F_n^{-} \right\},
\]

\[
F_2 = \text{diag} \left\{ \frac{F_1^{+} + F_1^{-}}{2}, \frac{F_2^{+} + F_2^{-}}{2}, \ldots, \frac{F_n^{+} + F_n^{-}}{2} \right\},
\]

\[
G_1 = \text{diag} \left\{ G_1^{+}, G_2^{+}, \ldots, G_m^{+} \right\},
\]

\[
G_2 = \text{diag} \left\{ \frac{G_1^{-} + G_1^{+}}{2}, \frac{G_2^{-} + G_2^{+}}{2}, \ldots, \frac{G_m^{-} + G_m^{+}}{2} \right\}.
\]

The following lemmas will be essential in establishing the desired LMI based passivity criteria.

**Lemma 3.1** ([4]). Given constant matrices \( \Omega_1, \Omega_2 \) and \( \Omega_3 \) with appropriate dimensions, where \( \Omega_1^T = \Omega_1 \) and \( \Omega_2^T = \Omega_2 > 0 \), then \( \Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0 \) if and
only if
\[
\begin{bmatrix}
\Omega_1 & \Omega_3^T \\
* & -\Omega_2
\end{bmatrix} < 0 \quad \text{or} \quad
\begin{bmatrix}
-\Omega_2 & \Omega_3 \\
* & \Omega_1
\end{bmatrix} < 0.
\]

**Lemma 3.2** ([3]). For any constant matrix \( M > 0 \), any scalars \( a \) and \( b \) with \( a < b \), and a vector function \( x(t) : [a, b] \rightarrow \mathbb{R}^n \) such that the integrals concerned are well defined, then the following hold
\[
\int_a^b x(s)ds^T M \int_a^b x(s)ds \leq (b - a) \int_a^b x^T(s) M x(s) ds.
\]

**Lemma 3.3** ([3]). Assume that \( \Omega, M_k \) and \( E \) are real matrices with appropriate dimensions and \( Z_k(t) \) is a matrix function satisfying \( Z_k^T(t) Z_k(t) \leq I \). Then, \( \Omega + M_k Z_k(t) E + [M_k Z_k(t) E]^T < 0 \) holds if and only if there exists a scalar \( \epsilon > 0 \) satisfying \( \Omega + \epsilon^{-1} M_k M_k^T + \epsilon E^T E < 0 \).

**Theorem 3.4.** Given scalars \( l, h \geq 1 \), the fuzzy BAM neural networks (8) with Markovian jumping parameters and without uncertainty is passive if there exist matrices \( P_{lk} > 0, P_{lk} > 0, Q_{lj} > 0, (j = 1, 2, \ldots, 6) \), \( R_l > 0, (j = 1, 2) \) and \( Z_j > 0, (j = 1, 2, 3, 4) \), and positive diagonal matrices \( X_j, (j = 1, 2, 3, 4) \) and a scalar \( \lambda > 0 \), and any appropriately dimensioned matrices \( L_1, L_2 \) such that the following LMI hold for \( k = 1, 2, \ldots, s \) and \( i = 1, 2, \ldots, r \)
\[
\Omega < 0,
\]
where
\[
\Omega = W_{P_1}^T P_1 W_{P_1} + W_{P_2}^T P_2 W_{P_2} + W_{Q_1}^T Q_1 W_{Q_1} + W_{Q_2}^T Q_2 W_{Q_2} + W_{R_1}^T R_1 W_{R_1} + W_{R_2}^T R_2 W_{R_2} + W_{T_1}^T Z_1 W_{T_1} + W_{T_2}^T Z_2 W_{T_2} + W_{T_3}^T Z_3 W_{T_3} + W_{T_4}^T Z_4 W_{T_4} + \text{sym}\left(W_{c_1}^T L W_{L_1} + W_{c_2}^T L W_{L_2} - W_{Z_5}^T W_\lambda\right) + W_{G_{11}}^T G_1 W_{G_{12}} + W_{G_{21}}^T G_2 W_{G_{22}} + W_{G_{31}} G_3 W_{G_{32}} + W_{G_{41}} G_4 W_{G_{42}} - \lambda W_\lambda^T W_\lambda
\]
with
\[
W_{P_1} = 
\begin{bmatrix}
I_n & 0_{n,(l+6)n+(h+7)m} \\
0_{n,(l+3)n} & I_n & 0_{n,3n+(h+7)m} \\
0_{m,(l+7)n} & I_m & 0_{m,(h+6)m} \\
0_{m,(l+7)n+(h+3)m} & I_m & 0_{m,3m}
\end{bmatrix}, \quad
P_1 = 
\begin{bmatrix}
0 & P_1 & 0 & 0 \\
P_1 & 0 & 0 & 0 \\
0 & 0 & 0 & P_2 \\
0 & 0 & P_2 & 0
\end{bmatrix},
\]
and
\[
W_{P_2} = 
\begin{bmatrix}
I_n & 0_{n,(l+6)n+(h+7)m} \\
0_{n,(l+7)n} & I_m & 0_{m,(h+6)m}
\end{bmatrix}, \quad
P_2 = 
\begin{bmatrix}
\sum_{k'=1}^s \pi_{kk'} P_{1k'} & 0 \\
0 & \sum_{k'=1}^s \pi_{kk'} P_{2k'}
\end{bmatrix}.
\]
\[ W_{Q_1} = \begin{bmatrix} I_n & 0_{n,7n+(h+7)m} \\ 0_{n,n} & I_n & 0_{n,6n+(h+7)m} \\ I_n & 0_{n,(l+6)n+(h+7)m} \\ 0_{n,(l+1)n} & \sqrt{1-\mu_1}I_n & 0_{n,5n+(h+7)m} \\ I_n & 0_{n,(l+6)n+(h+7)m} \\ 0_{n,(l+2)n} & I_n & 0_{n,4n+(h+7)m} \end{bmatrix}, \]

\[ W_{Q_2} = \begin{bmatrix} 0_{hm,(l+7)n} & I_{hm} & 0_{hm,7m} \\ 0_{hm,(l+7)n+m} & I_{hm} & 0_{hm,6m} \\ 0_{m,(l+7)n} & I_m & 0_{m,(h+6)m} \\ 0_{m,(l+7)n+(h+1)m} & \sqrt{1-\mu_2}I_m & 0_{m,5m} \\ 0_{m,(l+7)n} & I_m & 0_{m,(h+6)m} \\ 0_{m,(l+7)n+(h+2)m} & I_m & 0_{m,4m} \end{bmatrix}, \]

\[ \overline{Q}_1 = \text{diag}\{Q_1, -Q_1, Q_2, -Q_2, Q_3, -Q_3\}, \]

\[ \overline{Q}_2 = \text{diag}\{Q_4, -Q_4, Q_5, -Q_5, Q_6, -Q_6\}, \]

\[ W_{R_1} = \begin{bmatrix} 0_{n,(l+4)n} & I_n & 0_{n,2n+(h+7)m} \\ 0_{n,(l+5)n} & \sqrt{1-\mu_1}I_n & 0_{n,n+(h+7)m} \end{bmatrix}, \]

\[ W_{R_2} = \begin{bmatrix} 0_{m,(l+7)n+(h+4)m} & I_m & 0_{m,2m} \\ 0_{m,(l+7)n+(h+5)m} & \sqrt{1-\mu_2}I_m & 0_{m,m} \end{bmatrix}, \]

\[ \overline{R}_1 = \text{diag}\{R_1, -R_1\}, \quad \overline{R}_2 = \text{diag}\{R_2, -R_2\}, \]

\[ W_{Z_1} = \begin{bmatrix} 0_{n,(l+3)n} & \sqrt{\frac{\tau_1}{l}}I_n & 0_{n,3n+(h+7)m} \\ 0_{n,(l+3)n} & \sqrt{\frac{\tau_2-\tau_1}{l}}I_n & 0_{n,3n+(h+7)m} \end{bmatrix}, \]

\[ W_{Z_2} = \begin{bmatrix} \sqrt{\frac{l}{\tau_1}}I_n & -\sqrt{\frac{l}{\tau_1}}I_n & 0_{n,(l+5)n+(h+7)m} \\ 0_n & \frac{1}{\sqrt{\tau_2-\tau_1}}I_n & \frac{1}{\sqrt{\tau_2-\tau_1}}I_n & 0_{n,5n+(h+7)m} \\ 0_n & \frac{1}{\sqrt{\tau_2-\tau_1}}I_n & \frac{1}{\sqrt{\tau_2-\tau_1}}I_n & 0_{n,4n+(h+7)m} \end{bmatrix}. \]
\[
W_{Z_3} = \begin{bmatrix}
0_{m,(l+7)n+(h+3)m} & \sqrt{\frac{\rho_1}{h}} I_m & 0_{m,3m} \\
0_{m,(l+7)n+(h+3)m} & \sqrt{\rho_2 - \rho_1} I_m & 0_{m,3m}
\end{bmatrix},
\]

\[
W_{Z_4} = \begin{bmatrix}
0_{m,(l+7)n} & \sqrt{\frac{h}{\rho_1}} I_m & -\sqrt{\frac{h}{\rho_1}} I_m & 0_{m,(h+5)m} \\
0_{m,(l+7)n+h_m} & \sqrt{\rho_2 - \rho_1} I_m & -\sqrt{\rho_2 - \rho_1} I_m & 0_{m,5m} \\
0_{m,(l+7)n+(h+1)m} & \sqrt{\rho_2 - \rho_1} I_m & -\sqrt{\rho_2 - \rho_1} I_m & 0_{m,4m}
\end{bmatrix},
\]

\[
\bar{Z}_1 = \text{diag}\{Z_1, Z_2\}, \quad \bar{Z}_2 = \text{diag}\{Z_1, Z_2, Z_2\}, \quad \bar{Z}_3 = \text{diag}\{Z_3, Z_4\}, \quad \bar{Z}_4 = \text{diag}\{Z_3, Z_4, Z_4\},
\]

\[
W_{L_1} = \begin{bmatrix}
I_{n,7n+(h+7)m} \\
0_{n,(l+1)n} I_n \\
0_{n,(l+3)n} I_n
\end{bmatrix}, \quad W_{L_2} = \begin{bmatrix}
0_{m,(l+4)n} E_{i,k} F_{i,k} 0_m -D_{i,k} 0_{m,(h+2)m} -I_m 0_{m,2m} I_m
\end{bmatrix},
\]

\[
W_{G_{11}} = \begin{bmatrix}
0_{(l+7)n,m} & 0_{(l+7)n,m} \\
F_{m,m} & -F_2 \\
0_{(h+3)m,m} & 0_{(h+3)m,m} \\
-F_2 & I_m
\end{bmatrix}, \quad W_{G_{21}} = \begin{bmatrix}
F_1 & -F_2 \\
0_{3m,m} & 0_{3m,m} \\
-F_2 & I_m \\
0_{m,m} & 0_{m,m}
\end{bmatrix},
\]

\[
W_{G_{12}} = \begin{bmatrix}
0_{m,(l+7)n} I_m & 0_{m,(h+6)m} \\
0_{m,(l+7)n+(h+4)m} I_m & 0_{m,2m}
\end{bmatrix}, \quad W_{G_{22}} = \begin{bmatrix}
0_{m,(l+7)n+(h+1)m} I_m & 0_{m,5m} \\
0_{m,(l+7)n+(h+5)m} I_m & 0_{m,m}
\end{bmatrix},
\]

\[
W_{G_{31}} = \begin{bmatrix}
G_1 & -G_2 \\
0_{(l+3)n,n} & 0_{(l+3)n,n} \\
-G_2 & I_n \\
0_{2n+(h+7)m,n} & 0_{2n+(h+7)m,n}
\end{bmatrix}, \quad W_{G_{41}} = \begin{bmatrix}
0_{(l+1)n,n} & 0_{(l+1)n,n} \\
G_1 & -G_2 \\
0_{3n,n} & 0_{3n,n} \\
-G_2 & I_n \\
0_{n+(h+7)m,n} & 0_{n+(h+7)m,n}
\end{bmatrix},
\]

\[
\]
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\[ W_{G_{32}} = \begin{bmatrix} I_n & 0_{n,(l+6)n+(h+7)m} \\ 0_{n,(l+4)n} & I_n & 0_{n,2n+(h+7)m} \end{bmatrix}, \quad W_{G_{42}} = \begin{bmatrix} 0_{n,(l+1)n} & I_n & 0_{n,5n+(h+7)m} \\ 0_{n,(l+5)n} & I_n & 0_{n,n+(h+7)m} \end{bmatrix}, \]

\[ \overline{G}_1 = \text{diag}\{-X_3, -X_3\}, \quad \overline{G}_2 = \text{diag}\{-X_4, -X_4\}, \]

\[ \overline{G}_3 = \text{diag}\{-X_1, -X_1\}, \quad \overline{G}_4 = \text{diag}\{-X_2, -X_2\}, \]

\[ W_\lambda = \begin{bmatrix} 0_{n,(l+6)n} & I_n & 0_{n,(h+7)m} \\ 0_{m,(l+7)n+(h+6)m} & I_m & 0_{m,2m} \end{bmatrix}, \quad W_{Z_5} = \begin{bmatrix} 0_{n,(l+4)n} & I_n & 0_{n,2n+(h+7)m} \\ 0_{m,(l+7)n+(h+4)} & I_m & 0_{m,2m} \end{bmatrix}. \]

**Proof:** In order to prove the passivity result, we consider the following Lyapunov-Krasovskii functional for the model (8) based on the delay fractioning idea

\[ V(t, x(t), y(t), k) = \sum_{q=1}^{5} V_q(t, x(t), y(t)), \quad (10) \]

where

\[ V_1(t, x(t), y(t), k) = x^T(t) P_{1k} x(t) + y^T(t) P_{2k} y(t), \]

\[ V_2(t, x(t), y(t), k) = \int_{t-\tau_1}^{t} \Gamma_1^T(s) Q_1 \Gamma_1(s) ds + \int_{t-\tau(t)}^{t} x^T(s) Q_2 x(s) ds \]

\[ + \int_{t-\tau_2}^{t} x^T(s) Q_3 x(s) ds, \]

\[ V_3(t, x(t), y(t), k) = \int_{t-\rho_1}^{t} \Gamma_2^T(s) Q_4 \Gamma_2(s) ds + \int_{t-\rho(t)}^{t} y^T(s) Q_5 y(s) ds \]

\[ + \int_{t-\rho_2}^{t} y^T(s) Q_6 y(s) ds, \]

\[ V_4(t, x(t), y(t), k) = \int_{t-\tau(t)}^{t} g^T(x(s)) R_1 g(x(s)) ds + \int_{t-\rho(t)}^{t} f^T(y(s)) R_2 f(y(s)) ds, \]

\[ V_5(t, x(t), y(t), k) = \int_{t-\rho_1}^{t} \int_{t+\theta}^{t} \dot{x}^T(s) Z_1 \dot{x}(s) ds d\theta + \int_{t-\tau_1}^{t} \int_{t+\theta}^{t} \dot{x}^T(s) Z_2 \dot{x}(s) ds d\theta \]

\[ + \int_{t-\rho_1}^{t} \int_{t+\theta}^{t} \dot{y}^T(s) Z_3 \dot{y}(s) ds d\theta + \int_{t-\rho_2}^{t} \int_{t+\theta}^{t} \dot{y}^T(s) Z_4 \dot{y}(s) ds d\theta, \]

here

\[ \Gamma_1(t) = \left[ x^T(t), x^T\left(t - \frac{\tau_1}{l}\right), \ldots, x^T\left(t - \frac{(l-1)\tau_1}{l}\right) \right]^T, \]
By using Lemma 3.2, the integrals in 

\[ \Gamma_2(t) = \left[ y^T(t), y^T \left( t - \frac{\rho_1}{h} \right), \ldots, y^T \left( t - \frac{(h-1)\rho_1}{h} \right) \right]^T, \]

where \( l, h > 1 \) are integers.

Let \( \mathcal{L} \) be the weak infinitesimal generator of random process \( \{x(t), y(t), \eta(t), t \geq 0\} \). We can obtain

\[ \mathcal{L}V_1(t, x(t), y(t), k) = \dot{x}^T(t)P_{1k}x(t) + \dot{x}^T(t)P_{1k}\dot{x}(t) + \dot{x}^T(t) \sum_{k'=1}^{s} \Pi_{1k'}P_{1k'}x(t) \]

\[ + \dot{y}^T(t)P_{2k}y(t) + y^T(t)P_{2k}\dot{y}(t) + y^T(t) \sum_{k'=1}^{s} \Pi_{2k'}P_{2k'}y(t), \quad (11) \]

\[ \mathcal{L}V_2(t, x(t), y(t), k) \leq \Gamma_1^T(t)Q_1\Gamma_1(t) - \Gamma_1^T(t - \frac{\tau_1}{l})Q_1\Gamma_1(t - \frac{\tau_1}{l}) + x^T(t)Q_2x(t) \]

\[ - (1 - \mu_1)x^T(t - \tau(t))Q_2x(t - \tau(t)) + x^T(t)Q_3x(t) \]

\[ - x^T(t - \tau_2)Q_3x(t - \tau_2), \quad (12) \]

\[ \mathcal{L}V_3(t, x(t), y(t), k) \leq \Gamma_2^T(t)Q_4\Gamma_2(t) - \Gamma_2^T(t - \frac{\rho_1}{h})Q_4\Gamma_2(t - \frac{\rho_1}{h}) + y^T(t)Q_5y(t) \]

\[ - (1 - \mu_2)y^T(t - \rho(t))Q_5y(t - \rho(t)) + y^T(t)Q_6y(t) \]

\[ - y^T(t - \rho_2)Q_6y(t - \rho_2), \quad (13) \]

\[ \mathcal{L}V_4(t, x(t), y(t), k) \leq g^T(x(t))R_1g(x(t)) - (1 - \mu_1)g^T(x(t - \tau(t)))R_1g(x(t - \tau(t))) \]

\[ + f^T(y(t))R_2f(y(t)) \]

\[ -(1 - \mu_2)f^T(y(t - \rho(t)))R_2f(y(t - \rho(t))), \quad (14) \]

\[ \mathcal{L}V_5(t, x(t), y(t), k) \leq \dot{x}^T(t) \left( \frac{\tau_1}{l}Z_1 + (\tau_2 - \tau_1)Z_2 \right) \dot{x}(t) - \int_{t - \tau_1}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds \]

\[ - \int_{t - \tau_2}^{t - \tau_1} \dot{x}^T(s)Z_2\dot{x}(s)ds + \dot{y}^T(t) \left( \frac{\rho_1}{h}Z_3 + (\rho_2 - \rho_1)Z_4 \right) \dot{y}(t) \]

\[ - \int_{t - \rho_2}^{t - \rho_1} \dot{y}^T(s)Z_3\dot{y}(s)ds - \int_{t - \tau_2}^{t - \tau_1} \dot{y}^T(s)Z_4\dot{y}(s)ds. \quad (15) \]

By using Lemma 3.2, the integrals in \( \mathcal{L}V_5 \) can be written as

\[ - \int_{t - \tau_1}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds \leq - \frac{1}{\tau_1} \left( \int_{t - \tau_1}^{t} \dot{x}(s)ds \right)^T Z_1 \left( \int_{t - \tau_1}^{t} \dot{x}(s)ds \right), \quad (16) \]

\[ - \int_{t - \tau_2}^{t - \tau_1} \dot{x}^T(s)Z_2\dot{x}(s)ds = - \int_{t - \tau_2}^{t - \tau_1} \dot{x}^T(s)Z_2\dot{x}(s)ds - \int_{t - \tau(t)}^{t - \tau_1} \dot{x}^T(s)Z_2\dot{x}(s)ds, \]

\[ - \int_{t - \tau_2}^{t - \tau_1} \dot{x}^T(s)Z_2\dot{x}(s)ds \leq - \frac{1}{(\tau_2 - \tau_1)} \left( \int_{t - \tau_2}^{t - \tau_1} \dot{x}(s)ds \right)^T Z_2 \left( \int_{t - \tau_2}^{t - \tau_1} \dot{x}(s)ds \right), \quad (17) \]
Substituting (16)–(21) into (15) we obtain
\[
- \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s) Z_2 \dot{x}(s) ds \leq - \frac{1}{(\tau_2 - \tau_1)} \left( \int_{t-\tau(t)}^{t-\tau_1} \dot{x}(s) ds \right)^T Z_2 \left( \int_{t-\tau(t)}^{t-\tau_1} \dot{x}(s) ds \right),
\]
(18)
\[
- \int_{t-\rho(t)}^{t-\rho_1} \dot{y}^T(s) Z_3 \dot{y}(s) ds \leq - \frac{h}{\rho_1} \left( \int_{t-\rho(t)}^{t-\rho_1} \dot{y}(s) ds \right)^T Z_3 \left( \int_{t-\rho(t)}^{t-\rho_1} \dot{y}(s) ds \right),
\]
(19)
\[
- \int_{t-\rho(t)}^{t-\rho_1} \dot{y}^T(s) Z_4 \dot{y}(s) ds = - \int_{t-\rho_2}^{t-\rho(t)} \dot{y}^T(s) Z_4 \dot{y}(s) ds - \int_{t-\rho(t)}^{t-\rho_1} \dot{y}^T(s) Z_4 \dot{y}(s) ds,
\]
(20)
\[
- \int_{t-\rho(t)}^{t-\rho_1} \dot{y}^T(s) Z_4 \dot{y}(s) ds \leq - \frac{1}{(\rho_2 - \rho_1)} \left( \int_{t-\rho(t)}^{t-\rho_1} \dot{y}(s) ds \right)^T Z_4 \left( \int_{t-\rho(t)}^{t-\rho_1} \dot{y}(s) ds \right),
\]
(21)

Substituting (16)–(21) into (15) we obtain
\[
\mathcal{L} V(t, x(t), y(t), k) \leq \dot{x}^T(t) \left( \frac{\tau_1}{t} Z_1 + (\tau_2 - \tau_1) Z_2 \right) \dot{x}(t)
\]
\[
- \frac{l}{\tau_1} \left[ x(t) - x \left( t - \frac{\tau_1}{l} \right) \right]^T Z_1 \left[ x(t) - x \left( t - \frac{\tau_1}{l} \right) \right]
\]
\[
- \frac{1}{(\tau_2 - \tau_1)} \left\{ [x(t - \tau(t)) - x(t - \tau_2)]^T Z_2 [x(t - \tau(t)) - x(t - \tau_2)]
\]
\[
+ [x(t - \tau_1) - x(t - \tau(t))]^T Z_2 [x(t - \tau_1) - x(t - \tau(t))]
\}
\]
\[
+ \dot{y}^T(t) \left( \frac{\rho_1}{h} Z_3 + (\rho_2 - \rho_1) Z_4 \right) \dot{y}(t)
\]
\[
- \frac{h}{\rho_1} \left[ y(t) - y \left( t - \frac{\rho_1}{h} \right) \right]^T Z_3 \left[ y(t) - y \left( t - \frac{\rho_1}{h} \right) \right]
\]
\[
- \frac{1}{(\rho_2 - \rho_1)} \left\{ [y(t - \rho(t)) - y(t - \rho_2)]^T Z_4 [y(t - \rho(t)) - y(t - \rho_2)]
\]
\[
+ [y(t - \rho_1) - y(t - \rho(t))]^T Z_4 [y(t - \rho_1) - y(t - \rho(t))]
\}
\]
(22)

It follows, by using (11)–(14) and (22) in (10) that
\[
\mathcal{L} V \leq \zeta^T(t) \left\{ W_{p_1}^T \overline{P}_1 W_{p_1} + W_{p_2}^T \overline{P}_2 W_{p_2} + W_{q_1}^T \overline{Q}_1 W_{q_1} + W_{q_2}^T \overline{Q}_2 W_{q_2}
\]
\[
+ W_{r_1}^T \overline{R}_1 W_{r_1} + W_{r_2}^T \overline{R}_2 W_{r_2}
\]
\[
+ W_{z_1}^T \overline{Z}_1 W_{z_1} + W_{z_2}^T \overline{Z}_2 W_{z_2} + W_{z_3}^T \overline{Z}_3 W_{z_3} + W_{z_4}^T \overline{Z}_4 W_{z_4} \right\} \zeta(t),
\]
(23)
where
\[
\zeta(t) = \begin{bmatrix}
\Gamma_1^T(t), T(t) - \tau_1, x^T(t), \dot{x}^T(t), \dot{y}^T(t), \dot{y}(t), g^T(x(t)), \\
g^T(x(t - \tau(t))), \Gamma_1^T(t), \Gamma_2^T(t), y^T(t - \rho_1), y^T(t - \rho(t)), y^T(t - \rho_2), \\
\dot{y}^T(t), f^T(y(t)), f^T(y(t - \rho(t))), \Gamma_1^T(t), \Gamma_2^T(t)
\end{bmatrix}^T.
\]

For any matrices \( L_1 \) and \( L_2 \) with appropriate dimensions, the following equalities hold
\[
2\alpha_1^T(t)L_1\begin{bmatrix}
-A_{i,k}x(t) + B_{i,k}f(y(t)) + C_{i,k}f(y(t - \rho(t))) + U_1(t) - \dot{x}(t)
\end{bmatrix} = 0, \quad (24)
\]
\[
2\alpha_2^T(t)L_2\begin{bmatrix}
-D_{i,k}y(t) + E_{i,k}g(x(t)) + F_{i,k}g(x(t - \tau(t))) + U_2(t) - \dot{y}(t)
\end{bmatrix} = 0, \quad (25)
\]
where
\[
\alpha_1^T(t) = \begin{bmatrix}
\Gamma_1^T(t), T(t) - \tau(t), \dot{x}^T(t)
\end{bmatrix}
\]
and
\[
\alpha_2^T(t) = \begin{bmatrix}
\Gamma_2^T(t), y^T(t - \rho(t)), \dot{y}^T(t)
\end{bmatrix}.
\]

From the assumption (A1), we have
\[
[f_j(y_j(t)) - F^-_j y_j(t)][f_j(y_j(t)) - F^+_j(t)] \leq 0,
\]
which is equivalent to
\[
\begin{bmatrix}
y(t) \\
f(y(t))
\end{bmatrix}^T
\begin{bmatrix}
F^-_j e_j e^T_j - \frac{F^-_j + F^+_j}{2} e_j e^T_j \\
-\frac{F^-_j + F^+_j}{2} e_j e^T_j
\end{bmatrix}
\begin{bmatrix}
y(t) \\
f(y(t))
\end{bmatrix} \leq 0,
\]
where \( e_j \) denotes the unit column vector having 1 on its \( j \)-th row and zero elsewhere.

Let \( X_1 = \text{diag}\{\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_m\} \), \( X_2 = \text{diag}\{\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_m\} \), \( X_3 = \text{diag}\{r_1, r_2, \ldots, r_n\} \) and \( X_4 = \text{diag}\{h_1, h_2, \ldots, h_n\} \) then
\[
\sum_{j=1}^n \bar{r}_j \begin{bmatrix}
y(t) \\
f(y(t))
\end{bmatrix}^T
\begin{bmatrix}
F^-_j e_j e^T_j - \frac{F^-_j + F^+_j}{2} e_j e^T_j \\
-\frac{F^-_j + F^+_j}{2} e_j e^T_j
\end{bmatrix}
\begin{bmatrix}
y(t) \\
f(y(t))
\end{bmatrix} \leq 0,
\]
\[
\begin{bmatrix}
y(t) \\
f(y(t))
\end{bmatrix}^T
\begin{bmatrix}
F_1 X_3 & -F_2 X_3 \\
-F_2 X_3 & X_3
\end{bmatrix}
\begin{bmatrix}
y(t) \\
f(y(t))
\end{bmatrix} \leq 0. \quad (26)
\]

Similarly, we can obtain
\[
\begin{bmatrix}
x(t) \\
g(x(t))
\end{bmatrix}^T
\begin{bmatrix}
G_1 X_1 & -G_2 X_1 \\
-G_2 X_1 & X_1
\end{bmatrix}
\begin{bmatrix}
x(t) \\
g(x(t))
\end{bmatrix} \leq 0, \quad (27)
\]
Combining (23)–(29), we get

\[
\mathcal{L}V(t, x(t), y(t), k) - 2 \begin{bmatrix} f^T(y(t)) & g^T(x(t)) \end{bmatrix} \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} - \lambda \begin{bmatrix} U_1^T(t) & U_2^T(t) \end{bmatrix} \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} \\
\leq \sum_{i=1}^{r} h_i(\theta(t)) \zeta^T(t) \Omega \zeta(t),
\]

where \( \Omega \) is given in (9). In view of the LMI (9), we obtain,

\[
\mathcal{L}V(t, x(t), y(t), k) - 2 \begin{bmatrix} f^T(y(t)) & g^T(x(t)) \end{bmatrix} \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} \\
- \lambda \begin{bmatrix} U_1^T(t) & U_2^T(t) \end{bmatrix} \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} \leq 0.
\]

By integrating Eq. (31) with respect to \( t \) over the time period \([0, t_p]\), we get

\[
2 \int_0^{t_p} \begin{bmatrix} f^T(y(s)) & g^T(x(s)) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} ds \geq V(t_p) - V(0) - \lambda \int_0^{t_p} \begin{bmatrix} U_1^T(s) & U_2^T(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} ds
\]

\[
\geq -\lambda \int_0^{t_p} \begin{bmatrix} U_1^T(s) & U_2^T(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} ds.
\]

When \( t = 0 \), we have \( V(0) = 0 \) and hence from (33) the fuzzy BAM neural network (8) is passive in the sense of Definition 2.1. This completes the proof. \( \square \)

In practical implementation, uncertainties are inevitable in neural networks because of the existence of modeling errors and external disturbance. To consider such a reality, the parameter uncertainties are unavoidable while modeling the network and should be taken into account. Therefore, it is important to study the passivity result in the presence of parameter uncertainties. In the following theorem, we obtain sufficient conditions to ensure the passivity of the fuzzy BAM neural networks with parameter
uncertainties described in (6). Moreover, the main results in this paper can be easily extended to obtain the robust passivity conditions for fuzzy BAM neural networks with norm-bounded uncertainties by employing the same approach used in Theorem 3.4.

**Theorem 3.5.** Given scalars $l, h \geq 1$ the fuzzy BAM neural networks (8) with Markovian jumping parameters are robustly passive in the sense of Definition 2.1, if there exist symmetric positive definite matrices $P_{1k} > 0$, $P_{2k} > 0$, $Q_j > 0$, $(j = 1, 2, \ldots, 6)$, $R_j > 0$, $(j = 1, 2)$ and $Z_j > 0$, $(j = 1, 2, 3, 4)$, and positive diagonal matrices $X_j$, $(j = 1, 2, 3, 4)$ and a scalars $\lambda > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, and any appropriately dimensioned matrices $L_1, L_2$ such that the following LMI hold for $k = 1, 2, \ldots, s$ and $i = 1, 2, \ldots, r$,

$$
\begin{bmatrix}
\Theta & W_{\xi_1}^T L_1 M_k & W_{\xi_2}^T L_2 M_k \\
* & -\epsilon_1 I & 0 \\
* & * & -\epsilon_2 I
\end{bmatrix} < 0,
$$

(34)

where

$$
\Theta = \Omega + \epsilon_1 N_1^T N_1 + \epsilon_2 N_2^T N_2,
$$

$$
N_1^T = \begin{bmatrix}
-E_{1i,k} & 0_{n,(l+6)n+(h+4)m} & E_{2i,k} & E_{3i,k} & 0_{n,m}
\end{bmatrix},
$$

$$
N_2^T = \begin{bmatrix}
0_{m,(l+4)n} & E_{5i,k} & E_{6i,k} & 0_{m,n} & -E_{4i,k} & 0_{m,(h+6)m}
\end{bmatrix},
$$

$$
W_{\xi_1} = \begin{bmatrix}
I_{ln} & 0_{ln,7n+(h+7)m} \\
0_{n,(l+1)n} & I_n & 0_{n,5n+(h+7)m} \\
0_{n,(l+3)n} & I_n & 0_{n,3n+(h+7)m}
\end{bmatrix},
$$

$$
W_{\xi_2} = \begin{bmatrix}
0_{h_m,(l+7)n} & I_{h_m} & 0_{h_m,7m} \\
0_{m,(l+7)n+(h+1)m} & I_m & 0_m \\
0_{m,(l+7)n+(h+3)m} & I_m & 0_{m,3m}
\end{bmatrix},
$$

and the remaining parameters are defined as in Theorem 3.4.

**Proof:** By considering parameter uncertainties in (8) and replacing $A_{i,k}$, $B_{i,k}$, $C_{i,k}$, $D_{i,k}$, $E_{i,k}$, $F_{i,k}$ by $A_{i,k} + \Delta A_{i,k}(t)$, $B_{i,k} + \Delta B_{i,k}(t)$, $C_{i,k} + \Delta C_{i,k}(t)$, $D_{i,k} + \Delta D_{i,k}(t)$, $E_{i,k} + \Delta E_{i,k}(t)$, $F_{i,k} + \Delta F_{i,k}(t)$ in Theorem 3.4, we have

$$
\Omega + W_{\xi_1}^T L_1 M_k Z_k(t) N_1 + \left[W_{\xi_1}^T L_1 M_k Z_k(t) N_1\right]^T \\
+ W_{\xi_2}^T L_2 M_k Z_k(t) N_2 + \left[W_{\xi_2}^T L_2 M_k Z_k(t) N_2\right]^T < 0.
$$

By using Lemma 3.3 the above inequality becomes

$$
\Omega + \epsilon_1^{-1} W_{\xi_1}^T L_1 M_k M_k^T L_1^T W_{\xi_1} + \epsilon_1 N_1^T N_1 \\
+ \epsilon_2^{-1} W_{\xi_2}^T L_2 M_k M_k^T L_2^T W_{\xi_2} + \epsilon_2 N_2^T N_2 < 0.
$$

(35)

Applying the Schur complement Lemma 3.1 to (35), we obtain (34). This completes the proof. $\square$
4. Numerical simulation

In this section, we present two numerical examples to demonstrate the effectiveness of the proposed methods.

**Example 4.1.** Consider the time delayed fuzzy BAM neural networks with Markovian jumping parameters and absence of parameter uncertainties together with the $i$-th rule as follows:

**Plant Rule 1:** IF \{\(u_1(t)\) and \(v_1(t)\) are \(\eta_1^i\)} \text{ THEN} \n
\[
\begin{align*}
\dot{x}(t) &= -A_{1,k}x(t) + B_{1,k}f(y(t)) + C_{1,k}f(y(t - \rho(t))) + U_1(t), \\
\dot{y}(t) &= -D_{1,k}y(t) + E_{1,k}g(x(t)) + F_{1,k}g(x(t - \tau(t))) + U_2(t),
\end{align*}
\]

**Plant Rule 2:** IF \{\(u_2(t)\) and \(v_2(t)\) are \(\eta_2^i\)} \text{ THEN} \n
\[
\begin{align*}
\dot{x}(t) &= -A_{2,k}x(t) + B_{2,k}f(y(t)) + C_{2,k}f(y(t - \rho(t))) + U_1(t), \\
\dot{y}(t) &= -D_{2,k}y(t) + E_{2,k}g(x(t)) + F_{2,k}g(x(t - \tau(t))) + U_2(t),
\end{align*}
\]

with the following parameters:

\[
\begin{align*}
A_{1,1} &= \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}, & A_{2,1} &= \begin{bmatrix} 5 & 0 \\ 0 & 4.8 \end{bmatrix}, & A_{1,2} &= \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, & A_{2,2} &= \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \\
B_{1,1} &= \begin{bmatrix} 0.4 & -0.6 \\ 0.3 & 0.5 \end{bmatrix}, & B_{2,1} &= \begin{bmatrix} 1 & 0.2 \\ -1 & 1 \end{bmatrix}, \\
B_{1,2} &= \begin{bmatrix} 0.4 & 0.2 \\ -0.5 & 0.1 \end{bmatrix}, & B_{2,2} &= \begin{bmatrix} 0.4 & 1 \\ -0.6 & 0.1 \end{bmatrix}, \\
C_{1,1} &= \begin{bmatrix} 0.1 & 0.2 \\ 1 & -0.3 \end{bmatrix}, & C_{2,1} &= \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}, \\
C_{1,2} &= \begin{bmatrix} 0.1 & 0.3 \\ 0.1 & 0.4 \end{bmatrix}, & C_{2,2} &= \begin{bmatrix} 0.2 & 0.4 \\ 0.2 & 0.5 \end{bmatrix}, \\
D_{1,1} &= \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}, & D_{2,1} &= \begin{bmatrix} 3 & 0 \\ 0 & 3.5 \end{bmatrix}, & D_{1,2} &= \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}, & D_{2,2} &= \begin{bmatrix} 8 & 0 \\ 0 & 5 \end{bmatrix},
\end{align*}
\]
The activation functions are described by 
\[
g_1(x) = \tanh(2x), \quad g_2(x) = \tanh(-4x), \
m_1(y) = \tanh(2y) \quad \text{and} \quad m_2(y) = \tanh(-4y),
\]
the membership functions for Rule 1 and Rule 2 are 
\[
\eta_1 = 1/e^{-2\eta_1} \quad \text{and} \quad \eta_2 = 1 - \eta_1.
\]
Clearly, it can be seen that the assumption (A1) is satisfied with 
\[
G_1 = 0, \quad G_2 = 2, \quad G_1 = -4, \quad G_2 = 0, \quad F_1 = 0, \quad F_2 = 2, \quad F_1 = -4, \quad F_2 = 0.
\]
Thus 
\[
G_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.
\]
If the delay-fractioning parameters \( l \) and \( h \), time delay lower bounds \( \tau_1 \) and \( \rho_1 \), and time derivative limits \( \mu_1 \) and \( \mu_2 \) are given, the LMI conditions in Theorem 3.4 can be readily solved by using MATLAB LMI toolbox. If we set \( l, h = 1, 2, 3, 4 \) and the lower bounds \( \tau_1, \rho_1 = 2 \) then the obtained time delay upper bounds for different values of \( \mu_1, \mu_2 \) are given in Table 1. It is clear that the calculated upper bounds \( \tau_2 \) and \( \rho_2 \) increase as the fractioning number \( l \) and \( h \) increase. It is noted that as \( l \) and \( h \) increase, the computational complexity also increases correspondingly, so the optimum result will be less conservative.

<table>
<thead>
<tr>
<th>( \mu_1 = \mu_2 )</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>( \geq 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l = 1, h = 1 )</td>
<td>( \tau_2, \rho_2 = 2.1346, 2.1346 )</td>
<td>2.1336, 2.1336</td>
<td>2.1334, 2.1334</td>
<td>2.1334, 2.1334</td>
</tr>
<tr>
<td>( l = 2, h = 2 )</td>
<td>( \tau_2, \rho_2 = 2.1348, 2.1348 )</td>
<td>2.1339, 2.1339</td>
<td>2.1337, 2.1337</td>
<td>2.1337, 2.1337</td>
</tr>
<tr>
<td>( l = 3, h = 3 )</td>
<td>( \tau_2, \rho_2 = 2.2593, 2.2593 )</td>
<td>2.2362, 2.2362</td>
<td>2.2340, 2.2340</td>
<td>2.2337, 2.2337</td>
</tr>
<tr>
<td>( l = 4, h = 4 )</td>
<td>( \tau_2, \rho_2 = 2.2836, 2.2836 )</td>
<td>2.2395, 2.2395</td>
<td>2.2375, 2.2375</td>
<td>2.2372, 2.2372</td>
</tr>
</tbody>
</table>
In particular, for the considered fuzzy BAM neural networks when $k = 1, 2$, for the given $l, h, \tau_1, \rho_1, \mu_1$ and $\mu_2$, by solving the LMI’s in Theorem 3.4 via MATLAB LMI toolbox, one can obtain the feasible solutions which are not presented here due to the page limit. Therefore, by Theorem 3.4, it is clear that the considered fuzzy BAM model in Example 4.1 with above given parameters is passive. If we choose our initial values of the state variables as $(x_1(t), x_2(t)) = (0.5, -1)$, $(y_1(t), y_2(t)) = (-0.5, -2)$ with the absence of external inputs $U_1(t) = U_2(t) = 0$, then the trajectories of the state variables are shown in Fig. 1 and Fig. 2. The simulation results reveal that both $x(t)$ and $y(t)$ are converging to the equilibrium point zero, so we conclude that the considered fuzzy BAM neural networks is internally stable.
EXAMPLE 4.2. Consider the uncertain fuzzy BAM neural networks with Markovian jumping parameters and time-varying delays together with the $i$-th rule as follows:

**Plant Rule 1:** IF \( \{u_1(t) \text{ and } v_1(t) \text{ are } \eta_1^i\} \), THEN

\[
\begin{align*}
\dot{x}(t) &= -[A_{1,k} + \Delta A_{1,k}(t)]x(t) + [B_{1,k} + \Delta B_{1,k}(t)]f(y(t)) \\
&\quad + [C_{1,k} + \Delta C_{1,k}(t)]f(y(t - \rho(t))) + U_1(t), \\
\dot{y}(t) &= -[D_{1,k} + \Delta D_{1,k}(t)]y(t) + [E_{1,k} + \Delta E_{1,k}(t)]g(x(t)) \\
&\quad + [F_{1,k} + \Delta F_{1,k}(t)]g(x(t - \tau(t))) + U_2(t),
\end{align*}
\]

**Plant Rule 2:** IF \( \{u_2(t) \text{ and } v_2(t) \text{ are } \eta_2^i\} \), THEN

\[
\begin{align*}
\dot{x}(t) &= -[A_{2,k} + \Delta A_{2,k}(t)]x(t) + [B_{2,k} + \Delta B_{2,k}(t)]f(y(t)) \\
&\quad + [C_{2,k} + \Delta C_{2,k}(t)]f(y(t - \rho(t))) + U_1(t), \\
\dot{y}(t) &= -[D_{2,k} + \Delta D_{2,k}(t)]y(t) + [E_{2,k} + \Delta E_{2,k}(t)]g(x(t)) \\
&\quad + [F_{2,k} + \Delta F_{2,k}(t)]g(x(t - \tau(t))) + U_2(t),
\end{align*}
\]

with the following parameters

\[
\begin{align*}
A_{1,1} &= \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, & A_{2,1} &= \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}, & A_{1,2} &= \begin{bmatrix} 9 & 0 \\ 0 & 8 \end{bmatrix}, & A_{2,2} &= \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}, \\
B_{1,1} &= \begin{bmatrix} 0.4 & -0.6 \\ 0.3 & 0.5 \end{bmatrix}, & B_{2,1} &= \begin{bmatrix} 1 & 0.2 \\ -1 & 1 \end{bmatrix}, \\
B_{1,2} &= \begin{bmatrix} -0.4 & 0.2 \\ -0.5 & 0.1 \end{bmatrix}, & B_{2,2} &= \begin{bmatrix} 0.4 & 1 \\ 0.6 & -0.1 \end{bmatrix}, \\
C_{1,1} &= \begin{bmatrix} 0.1 & 0.2 \\ 1 & -0.3 \end{bmatrix}, & C_{2,1} &= \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}, \\
C_{1,2} &= \begin{bmatrix} 0.1 & -0.3 \\ 0.1 & 0.4 \end{bmatrix}, & C_{2,2} &= \begin{bmatrix} 0.2 & 0.4 \\ 0.2 & 0.5 \end{bmatrix}, \\
D_{1,1} &= \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}, & D_{2,1} &= \begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix}, & D_{1,2} &= \begin{bmatrix} 9 & 0 \\ 0 & 5 \end{bmatrix}, & D_{2,2} &= \begin{bmatrix} 8 & 0 \\ 0 & 5 \end{bmatrix},
\end{align*}
\]
Further, in this example we consider the activation and membership functions same as in Example 4.1. Consequently, the assumption (A1) is satisfied. In particular, when \( k = 1, 2 \) and \( l, h = 1, 2, 3, 4 \), and the lower bounds \( \tau_1, \rho_1 = 2 \), then by solving the LMIs in Theorem 3.5, the obtained time delay upper bounds for different values of \( \mu_1, \mu_2 \) are given in Table 2. Therefore, it follows from Theorem 3.5 that

<table>
<thead>
<tr>
<th>( \mu_1 = \mu_2 )</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>( \geq 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l = 1, h = 1 )</td>
<td>( t_2, \rho_2=2.1710, 2.1710 )</td>
<td>2.1585, 2.1585</td>
<td>2.1557, 2.1557</td>
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</tr>
<tr>
<td>( l = 2, h = 2 )</td>
<td>( t_2, \rho_2=2.1782, 2.1782 )</td>
<td>2.1641, 2.1641</td>
<td>2.1603, 2.1603</td>
<td>2.1600, 2.1600</td>
</tr>
<tr>
<td>( l = 3, h = 3 )</td>
<td>( t_2, \rho_2=2.3692, 2.3692 )</td>
<td>2.2345, 2.2345</td>
<td>2.2225, 2.2225</td>
<td>2.2165, 2.2165</td>
</tr>
<tr>
<td>( l = 4, h = 4 )</td>
<td>( t_2, \rho_2=2.4007, 2.4007 )</td>
<td>2.2342, 2.2342</td>
<td>2.2223, 2.2223</td>
<td>2.2164, 2.2164</td>
</tr>
</tbody>
</table>
the considered fuzzy BAM neural network with Markovian switching is robustly passive in the sense of Definition 2.1.

5. Summary

In this paper, we have investigated the problem of passivity analysis for uncertain fuzzy BAM neural networks with Markovian jumping parameters and time varying delays. By constructing a novel Lyapunov functional and by utilizing the delay fractioning technique, new passivity conditions have been established to achieve the passivity performance. Moreover, in derivation of the passivity criteria, it is assumed that the description of the activation functions are more general than the commonly used Lipschitz conditions. Further, the result is extended to derive the robustness conditions for the passivity analysis by considering the norm bounded uncertainties in the model. These criteria have been developed in the frame of LMIs, which can be easily solved via standard numerical software. Finally, two examples are given to demonstrate applicability and usefulness of the developed theoretical results.

REFERENCES