Vertices Belonging to All Critical Independent Sets of a
Graph

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Abstract

Let \( G = (V, E) \) be a graph. A set \( S \subseteq V \) is independent if no two vertices from \( S \) are adjacent, and by \( \text{Ind}(G) = \Omega(G) \) we mean the set of all (maximum) independent sets of \( G \), while \( \text{core}(G) = \cap \{S : S \in \Omega(G)\} \). [13]. The neighborhood of \( A \subseteq V \) is \( N(A) = \{v \in V : N(v) \cap A \neq \emptyset\} \). The independence number \( \alpha(G) \) is the cardinality of each \( S \in \Omega(G) \), and \( \mu(G) \) is the size of a maximum matching of \( G \).

The number \( \text{id}_c(G) = \max\{|I| - |N(I)| : I \in \text{Ind}(G)\} \) is called the critical independence difference of \( G \), and \( A \in \text{Ind}(G) \) is critical if \( |A| - |N(A)| = \text{id}_c(G) \). [22]. We define \( \text{ker}(G) = \cap \{S : S \text{ is a critical independent set}\} \).

In this paper we prove that if a graph \( G \) is non-quasi-regularizable (i.e., there exists some \( A \in \text{Ind}(G) \), such that \( |A| > |N(A)| \)), then:

• \( \text{ker}(G) \subseteq \text{core}(G) \)
• \( |\text{ker}(G)| > \text{id}_c(G) \geq \alpha(G) - \mu(G) \geq 1 \).

Keywords: independent set, critical set, critical difference, maximum matching

1 Introduction

Throughout this paper \( G = (V, E) \) is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set \( V = V(G) \) and edge set \( E = E(G) \). We consider only graphs without isolated vertices.

If \( X \subseteq V \), then \( G[X] \) is the subgraph of \( G \) spanned by \( X \). By \( G - W \) we mean either the subgraph \( G[V - W] \), if \( W \subseteq V(G) \), or the partial subgraph \( H = (V, E - W) \) of \( G \), for \( W \subseteq E(G) \). In either case, we use \( G - w \), whenever \( W = \{w\} \). If \( X, Y \subset V \) are non-empty and disjoint, then we denote \( (X, Y) = \{xy : xy \in E, x \in X, y \in Y\} \).

The neighborhood of a vertex \( v \in V \) is the set \( N(v) = \{w : w \in V \text{ and } vw \in E\} \), while the closed neighborhood of \( v \in V \) is \( N[v] = N(v) \cup \{v\} \); in order to avoid ambiguity, we use also \( N_G(v) \) instead of \( N(v) \). In particular, if \( |N(v)| = 1 \), then \( v \) is a pendant vertex.
of $G$, and $\text{pend}(G) = \{v \in V(G) : v$ is a pendant vertex in $G\}$. The neighborhood of $A \subseteq V$ is denoted by $N(A) = N_G(A) = \{v \in V : N(v) \cap A \neq \emptyset\}$, and $N[A] = N(A) \cup A$.

A set $S \subseteq V(G)$ is independent if no two vertices from $S$ are adjacent, and by $\text{Ind}(G)$ we mean the set of all the independent sets of $G$. An independent set of maximum size will be referred to as a maximum independent set of $G$, and the independence number of $G$ is $\alpha(G) = \max\{|S| : S \in \text{Ind}(G)\}$. A graph $G$ is quasi-regularizable if one can replace each edge of $G$ with a non-negative integer number of parallel copies, so as to obtain a regular multigraph of degree $\geq 0$, [2]. For instance, $K_4 - e, e \in E(K_4)$, is quasi-regularizable, while $P_3$ is not quasi-regularizable. It is clear that a quasi-regularizable graph can not have isolated vertices.

**Theorem 1.1** For a graph $G$ the following assertions are equivalent:

(i) quasi-regularizable;

(ii) $|S| \leq |N(S)|$ holds for every $S \in \text{Ind}(G)$;

(iii) $G$ has a perfect 2-matching, i.e., $G$ contains a system of vertex-disjoint odd cycles and edges covering all its vertices.

Let $\Omega(G) = \{S : S$ is a maximum independent set of $G\}$ and $\xi(G) = |\text{core}(G)|$, where $\text{core}(G) = \cap \{S : S \in \Omega(G)\}$, [13].

Similarly, let $\text{corona}(G) = \cup \{S : S \in \Omega(G)\}$, and $\zeta(G) = |\text{corona}(G)|$, [9].

A matching is a set of non-incident edges of $G$; a matching of maximum cardinality $\mu(G)$ is a maximum matching, and a perfect matching is a matching covering all the vertices of $G$.

In the sequel we need the following characterization of a maximum independent set of a graph, due to Berge.

**Theorem 1.2** [2] An independent set $S$ belongs to $\Omega(G)$ if and only if every independent set $A$ of $G$, disjoint from $S$, can be matched into $S$.

$G$ is called a König-Egerváry graph provided $\alpha(G) + \mu(G) = |V(G)|$ [8, 20]. It is known that each bipartite graph satisfies this property.

**Theorem 1.3** [12] If $G$ is a König-Egerváry graph, $M$ is a maximum matching, then $M$ matches $V(G) - S$ into $S$, for every $S \in \Omega(G)$, and $\mu(G) = |V(G) - S|$.

In Boros et al. [8] it has been proved that if $G$ is connected and $\alpha(G) > \mu(G)$, then $\xi(G) = |\text{core}(G)| > \alpha(G) - \mu(G)$. This strengthened the following finding stated in [13]: if $\alpha(G) > (|V(G)| + k - 1)/2$, then $\xi(G) \geq k + 1$; moreover, $\xi(G) \geq k + 2$ is valid, whenever $|V(G)| + k - 1$ is an even number. For $k = 1$, the previous inequality provides us with a generalization of a result of Hammer et al. [8] claiming that if a graph $G$ has $\alpha(G) > |V(G)|/2$, then $\xi(G) \geq 1$. In [12] it was shown that if $G$ is a connected bipartite graph with $|V(G)| \geq 2$, then $\xi(G) \neq 1$. Jamison [9], Zito [29], and Gunther et al. [7] proved independently that $\xi(G) \neq 1$ is true for any tree $T$.

In Chlebík et al. [5] it has been found that if there is some $S \in \text{Ind}(G)$, such that $|S| > |N(S)|$, then $|\text{core}(G)| > \max\{|I| - |N(I)| : I \in \text{Ind}(G)\}$. It strengthens the inequality $|\text{core}(G)| > \alpha(G) - \mu(G)$ [8], since $\max\{|I| - |N(I)| : I \in \text{Ind}(G)\} \geq \alpha(G) - \mu(G)$ [17, 19].
The number \( d(X) = |X| - |N(X)| \) is called the difference of the set \( X \subseteq V(G) \), and
\( d_c(G) = \max\{d(X) : X \subseteq V(G)\} \) is the critical difference of \( G \). A set \( U \subseteq V(G) \) is critical if \( d(U) = d_c(G) \) \[22\]. The number \( id_c(G) = \max\{d(I) : I \in \text{Ind}(G)\} \) is called the critical independence difference of \( G \). If \( A \subseteq V(G) \) is independent and \( d(A) = id_c(G) \), then \( A \) is called critical independent \[22\].

For a graph \( G \) let us denote \( \ker(G) = \cap \{S : S \text{ is a critical independent set} \} \) and \( \varepsilon(G) = |\ker(G)| \).

For instance, the graph \( G_1 \) in Figure 1 has \( \ker(G_1) = \text{core}(G_1) = \{a, b\} \). The graph \( G_2 \) from Figure 1 has \( X = \{x, y, z, p, q\} \) as a critical non-independent set, because
\( d(X) = 1 = d_c(G_2) \), while \( \ker(G_2) = \{x, y\} \subset \text{core}(G_2) = \{x, y, z\} \). The graph \( G_3 \) from Figure 1 has \( \{t, u, v\} \) as a critical set, \( \ker(G_3) = \{u, v\} \), while \( \text{core}(G_3) = \{t, u, v, w\} \) is not a critical set.

![Figure 1: Non-quasi-regularizable graphs.](image)

Clearly, \( d_c(G) \geq id_c(G) \) is true for every graph \( G \).

**Theorem 1.4** \[22\] The equality \( d_c(G) = id_c(G) \) holds for every graph \( G \).

If \( A \in \Omega(G[N[A]]) \), then \( A \) is called a local maximum independent set of \( G \) \[14\].

It is easy to see that all pendant vertices are included in every maximum critical independent set. It is known that the problem of finding a critical independent set is polynomially solvable \[11, 22\].

**Theorem 1.5**
(i) \[18\] Each local maximum independent set is included in a maximum independent set.
(ii) \[17\] Every critical independent set is a local maximum independent set.
(iii) \[4\] Each critical independent set is contained in some maximum independent set.
(iv) \[10\] There is a matching from \( N(S) \) into \( S \), for every critical independent set \( S \).

In this paper we prove that \( \ker(G) \subseteq \text{core}(G) \) and \( \varepsilon(G) \geq d_c(G) \geq \alpha(G) - \mu(G) \) hold for every graph \( G \).

## 2 Results

**Theorem 2.1** Let \( A \) be a critical independent set of the graph \( G \) and \( X = A \cup N(A) \).
Then the following assertions are true:
(i) \( H = G[X] \) is a König-Egerváry graph;
(ii) \( \alpha(G[V - X]) \leq \mu(G[V - X]) \);
(iii) \( \mu(G[X]) + \mu(G[V - X]) = \mu(G) \); in particular, each maximum matching of \( G[X] \) can be enlarged to a maximum matching of \( G \).
Proof. (i) By Theorem 1.5(ii), $A$ is a local maximum independent set, which ensures that $\alpha(H) = |A|$, while Theorem 1.5(iv) implies $\mu(H) = |N(A)|$. Consequently, we get that

$$\alpha(H) + \mu(H) = |A \cup N(A)| = |X| = |V(H)|,$$

i.e., $H$ is a König-Egerváry graph.

(ii) According to Theorem 1.5(iii), there exists a maximum independent set $S$ such that $A \subseteq S$. Suppose that $|B| > |N(B)|$ holds for some $B \subseteq S \setminus A$. Then, it follows that

$$|A| - |N(A)| < (|A| - |N(A)|) + (|B| - |N(B)|) \leq |A \cup B| - |N(A \cup B)|,$$

which contradicts the hypothesis on $A$, namely, the fact that $|A| - |N(A)| = d_e(G)$. Hence $|B| \leq |N(B)|$ is true for every $B \subseteq S \setminus A$. Consequently, by Hall’s Theorem there exists a matching from $S \setminus A$ into $V \setminus S \setminus N(A)$ that implies $|S \setminus A| \leq \mu(G[V \setminus X])$.

It remains to show that $\alpha(G[V \setminus X]) = |S \setminus A|$. By way of contradiction, assume that

$$\alpha(G[V \setminus X]) = |D| > |S \setminus A|$$

for some independent set $D \subseteq V \setminus X$. Since $D \cap N[A] = \emptyset$, the set $A \cup D$ is independent, and

$$|A \cup D| = |A| + |D| > |A| + |S \setminus A| = \alpha(G),$$

which is impossible.

(iii) Let $M_1$ be a maximum matching of $H$ and $M_2$ be a maximum matching of $G[V \setminus X]$. We claim that $M_1 \cup M_2$ is a maximum matching of $G$.

![Figure 2: $S \in \Omega(G)$ and $A$ is a critical independent set of $G$.](image)

The only edges that may enlarge $M_1 \cup M_2$ belong to the set $(N(A), V \setminus S \setminus N(A))$. The matching $M_1$ covers all the vertices of $N(A)$ in accordance with Theorem 1.3 and part (i). Therefore, to choose an edge from the set $(N(A), V \setminus S \setminus N(A))$ means to loose an edge from $M_1$. In other words, no matching different from $M_1 \cup M_2$ may overstep $|M_1 \cup M_2|$.

Consequently, each maximum matching of $G[X]$ can find its counterpart in $G[V \setminus X]$ in order to build a maximum matching of $G$. ■

Theorem 2.1 allows us to give an alternative proof of the following inequality due to Lorentzen.
Theorem 2.4
For a graph $G$

In addition, \( \text{core}(N) \)

Proof. Let $A, B \subseteq V(G)$, we get $\alpha(G[V-X]) - \mu(G[V-X]) \leq 0$. Hence it follows that $\alpha(G[X]) - \mu(G[X]) \geq (\alpha(G[X]) + \alpha(G[V-X])) - (\mu(G[X]) + \mu(G[V-X])).$

Theorem 2.4(iii) claims that $\mu(G[X]) + \mu(G[V-X]) = \mu(G)$.

Since $A$ is a critical independent set, there exists some $S \in \Omega(G)$ such that $A \subseteq S$, and $\alpha(G[X]) = |A|$, by Theorem 2.4(i). Hence we have

$$\alpha(G[X]) + \alpha(G[V-X]) = |A| + |S - A| = \alpha(G).$$

In addition, Theorem 2.4(i) and Theorem 1.3 imply that $\mu(G[X]) = |N(A)|$.

Finally, we obtain

$$d_e(G) = \max \{|I| : I \in \text{Ind}(G)| = |A| - |N(A)| =$$

$$= \alpha(G[X]) - \mu(G[X]) \geq \alpha(G) - \mu(G),$$

and this completes the proof. ■

Applying Theorem 2.4 and Theorem 1.3 we get the following.

Corollary 2.3 [17] Let $J$ be a maximum critical independent set of $G$, and $X = J \cup N(J)$. Then the following assertions are true:

(i) $\alpha(G) = \alpha(G[X]) + \alpha(G[V-X])$;

(ii) $\alpha(G) = \alpha_e(G) + \alpha(G[V-X])$;

(iii) $G[X]$ is a König-Egerváry graph.

The graph $G$ from Figure 3 has $\ker(G) = \{a, b, c\}$. Notice that $\ker(G) \subseteq \text{core}(G)$; $S = \{a, b, c, v\}$ is a largest critical independent set, and neither $S \subseteq \text{core}(G)$ nor $\text{core}(G) \subseteq S$. In addition, $\text{core}(G)$ is not a critical independent set of $G$.

Figure 3: $G$ is a non-quasi-regularizable graph with $\text{core}(G) = \{a, b, c, u\}$.

Theorem 2.4 For a graph $G = (V, E)$ of order $n$, the following assertions are true:

(i) the function $d$ is supermodular, i.e., $d(A \cup B) + d(A \cap B) \geq d(A) + d(B)$ for every $A, B \subseteq V(G)$;

(ii) if $A$ and $B$ are critical in $G$, then $A \cup B$ and $A \cap B$ are critical as well;

(iii) $\ker(G) = \cap \{B : B$ is a critical set of $G\}$. 

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Proof. (i) Let us notice that \( N(A \cup B) = N(A) \cup N(B) \) and \( N(A \cap B) \subseteq N(A) \cap N(B) \). Further, we obtain
\[
d(A \cup B) = |A \cup B| - |N(A \cup B)| = |A \cup B| - |N(A) \cup N(B)| =
\]
\[
= |A| + |B| - |A \cap B| - |N(A)| - |N(B)| + |N(A) \cap N(B)| =
\]
\[
= (|A| - |N(A)|) + (|B| - |N(B)|) + |N(A) \cap N(B)| - |A \cap B| =
\]
\[
= d(A) + d(B) - (|A \cap B| - |N(A \cap B)|) + |N(A) \cap N(B)| - |A \cap B| =
\]
\[
= d(A) + d(B) - |A \cap B| + |N(A) \cap N(B)| - |N(A \cap B)| \geq
\]
\[
\geq d(A) + d(B) - d(A \cap B).
\]

(ii) By part (i), we have that
\[
d(A \cup B) + d(A \cap B) \geq d(A) + d(B) = 2d_c(G).
\]
Consequently, we get that \( d(A \cup B) = d(A \cap B) = d_c(G) \), i.e., both \( A \cup B \) and \( A \cap B \) are critical sets.

(iii) Let \( \Gamma_{c_i} \) be the family of all critical independent sets of \( G \), while \( \Gamma_c \) denotes the family \( \{B : B \text{ is a critical set in } G\} \).

By part (ii), both sets
\[
\ker(G) = \cap \{S : S \in \Gamma_{c_i}\} \text{ and } Q_c = \cap \{B : B \in \Gamma_c\}
\]
are critical. Theorem 1.4 implies that \( \Gamma_{c_i} \subseteq \Gamma_c \), and therefore, \( Q_c \subseteq \ker(G) \). On the other hand, \( Q_c \) is independent, because by Theorem 1.4, one of the critical sets from \( \Gamma_c \) is independent. Consequently, we obtain \( \ker(G) \subseteq Q_c \), and this completes the proof. ■

Theorem 2.5 For a graph \( G = (V, E) \) of order \( n \), the following assertions are true:
(i) \( V \supseteq \text{corona}(G) \supseteq S \supseteq \text{core}(G) \supseteq \ker(G) \), for every \( S \in \Omega(G) \);
(ii) \( n \geq \xi(G) \geq \alpha(G) \geq \xi(G) \geq \varepsilon(G) \geq d_c(G) \geq \alpha(G) - \mu(G) \);
(iii) \( \xi(G) \geq \alpha(G) - \mu(G) + \varepsilon(G) - d_c(G) \).

Proof. (i) Clearly, \( \text{core}(G) \subseteq S \subseteq \text{corona}(G) \subseteq V \) hold for each \( S \in \Omega(G) \). The set \( \ker(G) \) is independent by definition. According to Theorem 2.4(ii), \( \ker(G) \) is critical. Consequently, by Theorem 2.4(iv), there exists a matching \( M_L \) from \( N(\ker(G)) \) into \( \ker(G) \). Figure 4 will accompany us all the way to the end of the proof.

Let \( S \in \Omega(G) \), and \( A_1 = \ker(G) \cap S \). Since \( \ker(G) - A_1 \) is stable and disjoint from \( S \), Theorem 2.2 ensures that there is a matching \( M_B \) from \( \ker(G) - A_1 \) into \( S \), covering some subset \( A_2 \) of \( S - A_1 \). Let \( S \in \Omega(G) \), and \( A_1 = \ker(G) \cap S \). Since \( \ker(G) - A_1 \) is stable and disjoint from \( S \), Theorem 2.2 ensures that there is a matching \( M_B \) from \( \ker(G) - A_1 \) into \( S \), covering some subset \( A_2 \) of \( S - A_1 \). Clearly, we have
\[
|\ker(G) - A_1| = |A_2|, A_1 \cap A_2 = \emptyset, \text{ and } A_2 \subseteq N(\ker(G) - A_1) \cap S.
\]

Assume that there is some \( v \in (N(\ker(G) - A_1) \cap S) - A_2 \). The vertex \( v \) must be matched with some vertex from \( \ker(G) - A_1 \) by \( M_L \), because \( \{v\} \cup A_1 \subseteq S \). Hence \( M_L \) matches the set \( N(\ker(G) - A_1) \cap S \) into \( \ker(G) - A_1 \), which is impossible, since
\[
|N(\ker(G) - A_1) \cap S| \geq |\{v\} \cup A_2| > |A_2| = |\ker(G) - A_1|.
\]
Corollary 2.7

If \( G \) is a non-quasi-regularizable graph, then

\( n \geq \zeta (G) \geq \alpha (G) \geq \xi (G) \geq \varepsilon (G) \geq d_c(G) \geq \alpha (G) - \mu (G) \geq 1; \)

\( \xi (G) > \alpha (G) - \mu (G) + \varepsilon (G) - d_c(G). \)

Proof. According to Theorem 2.4, \( G \) is non-quasi-regularizable if and only if \( \ker(G) \neq \emptyset \), i.e., \( |\ker(G)| \geq 2 \). The fact that \( G \) has no isolated vertices implies \( N(\ker(G)) \neq \emptyset \), and consequently, it follows \( \varepsilon (G) = |\ker(G)| > |\ker(G)| - |N(\ker(G))| = d_c(G) \). Further, using Theorem 2.5 we get both \((i)\) and \((ii)\).

Corollary 2.7 \[ 7 \] If there is some \( S \in \text{Ind}(G) \) with \( |S| > |N(S)| \), then \( \xi (G) > d_c(G) \).

Figure 4: \( S \in \Omega(G) \), \( \ker(G) \), and \( A_1 = S \cap \ker(G) \).

Consequently, we get that \( N(\ker(G) - A_1) \cap S = A_2. \) Thus \( M_L \) matches the set \( N(\ker(G) - A_1) \cap S \) onto \( \ker(G) - A_1 \), and \( N(A_1) \) into \( A_1 \). Clearly, we have

\[ |\ker(G) - A_1| = |A_2|, \ A_1 \cap A_2 = \emptyset, \text{ and } A_2 \subseteq N(\ker(G) - A_1) \cap S. \]

Assume that there is some \( v \in (N(\ker(G) - A_1) \cap S) - A_2 \). The vertex \( v \) must be matched with some vertex from \( \ker(G) - A_1 \) by \( M_L \), because \( \{v\} \cup A_1 \subseteq S \). Hence \( M_L \) matches the set \( N(\ker(G) - A_1) \cap S \) into \( \ker(G) - A_1 \), which is impossible, since

\[ |N(\ker(G) - A_1) \cap S| \geq |\{v\} \cup A_2| > |A_2| = |\ker(G) - A_1|. \]

Consequently, we get that \( N(\ker(G) - A_1) \cap S = A_2. \) Thus \( M_L \) matches the set \( N(\ker(G) - A_1) \cap S \) onto \( \ker(G) - A_1 \), and \( N(A_1) \) into \( A_1 \).

In conclusion, we may assert that \( |\ker(G)| - |N(\ker(G))| = |A_1| - |N(A_1)| \). Hence, we infer that \( \ker(G) - A_1 = \emptyset \), otherwise we have that \( A_1 \) is a critical independent set of \( G \) with \( |A_1| < |\ker(G)| \), in contradiction with the hypothesis on minimality of \( \ker(G) \).

This ensures that \( \ker(G) \subseteq S \) for every \( S \in \Omega(G) \), which means that \( \ker(G) \subseteq \text{core}(G) \).

\((ii)\) Using part \((i)\), Theorem 2.4\((iii)\), and Corollary 2.2, we deduce that

\[ n \geq \zeta (G) \geq \alpha (G) \geq \xi (G) \geq \varepsilon (G) = |\ker(G)| \geq |\ker(G)| - |N(\ker(G))| = d_c(G) \geq \alpha (G) - \mu (G), \]

which completes the proof.

\((iii)\) It follows immediately from part \((ii)\). ■

Notice that \( \xi (K_{2,3}) = \varepsilon (K_{2,3}) > d_c(K_{2,3}) = 1 = \alpha (K_{2,3}) - \mu (K_{2,3}) \), while the graph \( G_2 \) is from Figure 4 satisfies \( \xi (G_2) > \varepsilon (G_2) > d(G_2) = 1. \)

Corollary 2.6 If \( d_c(G) > 0 \) or, equivalently, \( G \) is a non-quasi-regularizable graph, then

\((i)\) \( n \geq \zeta (G) \geq \alpha (G) \geq \xi (G) \geq \varepsilon (G) \geq d_c(G) \geq \alpha (G) - \mu (G) \geq 1; \)

\((ii)\) \( \xi (G) > \alpha (G) - \mu (G) + \varepsilon (G) - d_c(G). \)

Proof. According to Theorem 2.4, \( G \) is non-quasi-regularizable if and only if \( \ker(G) \neq \emptyset \), i.e., \( |\ker(G)| \geq 2 \). The fact that \( G \) has no isolated vertices implies \( N(\ker(G)) \neq \emptyset \), and consequently, it follows \( \varepsilon (G) = |\ker(G)| > |\ker(G)| - |N(\ker(G))| = d_c(G) \). Further, using Theorem 2.5 we get both \((i)\) and \((ii)\). ■
3 Conclusions

Writing this paper we have been motivated by the inequality

\[ \xi(G) = |\text{core}(G)| > \alpha(G) - \mu(G), \]

which is true for every graph \( G \) without isolated vertices, such that \( \alpha(G) > \mu(G) \) [3].

What we have found is that there exists a subset of \( \text{core}(G) \), which is a real obstacle to its nonemptiness. The cardinality of this subset, namely, \( \varepsilon(G) = |\text{ker}(G)| \) stands out above \( \alpha(G) - \mu(G) \) on its own.

The problem of whether there are vertices in a given graph \( G \) belonging to \( \text{core}(G) \) is \( \text{NP} \)-hard [3]. On the other hand, it has been noticed that for some families of graphs \( \text{core}(G) \) may be computed in polynomial time.

We conclude with the following question.

**Problem 3.1** Is it true that for any fixed positive integer \( k \), to decide if \( \varepsilon(G) > k \) is \( \text{NP} \)-complete?

References


