

The Quantum Cheshire Cat effect: Theoretical basis and observational implications

Q. Duprey,¹ S. Kanjilal,² U. Sinha,³ D. Home,² and A. Matzkin¹

¹*Laboratoire de Physique Théorique et Modélisation (CNRS Unité 8089),
Université de Cergy-Pontoise, 95302 Cergy-Pontoise cedex, France*

²*Center for Astroparticle Physics and Space Science (CAPSS),
Bose Institute, Kolkata 700 091, India*

³*Raman Research Institute, Sadashivanagar, Bangalore, India*

Abstract

The Quantum Cheshire Cat (QCC) is an effect introduced recently within the Weak Measurements framework. The main feature of the QCC effect is that a property of a quantum particle appears to be spatially separated from its position. The status of this effect has however remained unclear, as claims of experimental observation of the QCC have been disputed by strong criticism of the experimental as well as the theoretical aspects of the effect. In this paper we clarify in what precise sense the QCC can be regarded as an unambiguous consequence of the standard quantum mechanical formalism applied to describe quantum pointers weakly coupled to a system. In light of this clarification, the raised criticisms of the QCC effect are rebutted. We further point out that the limitations of the experiments performed to date imply that a loophole-free experimental demonstration of the QCC has not yet been achieved.

I. INTRODUCTION

Weak measurements were introduced [1] in 1988 as a theoretical scheme for minimally perturbing non-destructive quantum measurements. It is indeed well-known that a standard quantum measurement irremediably destroys the premeasurement state of a system by projecting it to an eigenstate of the measured observable, say B . It is therefore impossible, according to this “standard view”, to know anything about a given system property, represented by an observable A , as the system evolves from a prepared initial state towards the final eigenstate obtained after measuring B .

The weak measurement scheme introduced by Aharonov, Albert and Vaidman [1] aims to measure A without appreciably modifying the system evolution relative to the case without the measurement. This is achieved by means of a weak coupling between the system observable A and a dynamical variable of an external degree of freedom (that we will designate here by the term “quantum pointer”). This weak coupling entangles the quantum pointer with the system, until the final projective measurement of B correlates the obtained system eigenstate with the quantum state of the weak pointer. The resulting state of the weak pointer has picked up a shift (relative to its initial pre-coupled state) proportional to the real part of a quantity known as the *weak value* of A .

While the meaning of weak values has been debated since their inception [2–5], several experimental implementations of the weak measurement protocol have been carried out: weak values have thus been measured for different observables in many quantum systems [6–13]. Concurrently, theoretical schemes based on weak measurements have been proposed with practical [14–16] or foundational [17–21] aims. Among the latter, Aharonov et al [22] introduced an interferometric based scheme baptized the Quantum Cheshire Cat (QCC). A QCC situation takes place when at some location (say, region I) the weak value of an observable representing a system property (eg, polarization) vanishes while the weak value of the system’s spatial projector is non-zero. At some other location (say region II) the opposite takes place (the weak values of the spatial projector and system property are zero and non-zero respectively). Echoing the features of Lewis Carroll’s eponym character, Aharonov et al. loosely described this Cheshire cat situation in Ref. [22] as seeing the grin

(the polarization) without the cat (the photon in region I)¹; they further wrote in [22] that the QCC scheme implies “*physical properties can be disembodied from the objects they belong to*”. A tentative experimental realisation of the QCC effect employing neutrons published soon after [24] concluded “*that the system behaves as if the neutrons go through one beam path, while their magnetic moment travels along the other*”. Very recently, an experiment with single photons similar to the neutron one was reported [25].

The Quantum Cheshire cat phenomenon reported in Refs [22, 24] has raised a string of criticisms, in particular in the published works [5, 26–29]. The common trend in many of these criticisms is to view the QCC effect as a false paradox, an illusion that would have an arguably simpler interpretation. Unfortunately, rather than making their point using unambiguous conceptual and technical terms, most of the works criticizing the Quantum Cheshire Cat effect did not analyze the weak measurements framework (generally even avoiding to mention weak values). Interpretational claims (that ultimately depend on the interpretation of the standard quantum formalism, not on the specificities of weak values) were not always distinguished from practical pointer readouts, and the theoretical QCC effect was not carefully discriminated from the tentative experimental implementations.

Hence, instead of demystifying the Quantum Cheshire Cat, the criticism that has been made may have brought additional confusion. To be fair, it must be pointed out that some of the well-known works dealing with weak measurements, such as the QCC paper [22] put the stress on delivering a simple take-home message without providing the detailed formal arguments that would explain and justify the main message.

In this work, our aim will be to carefully scrutinize the Quantum Cheshire Cat effect in order to lift the confusion on what this effect is really about. We will discriminate the theoretical, ideal QCC from the shortcomings that can inevitably appear in any experimental realization. We will disentangle the interpretational aspects from the technical terms in which the effect is couched. We will in particular argue through the analysis of the meaning of vanishing weak values that spatial separation between a particle and its properties can be consistently defined, provided one is willing to relax ascribing properties to a quantum system solely through the eigenstate-eigenvalue link.

¹ As we will see below, the property (the grin) is not the polarization itself but a specific polarization component. A generalization that would apply to any polarization component was suggested in Ref. [23].

In order to do so we will first recall the aspects of weak measurements that will be relevant to describe the Quantum Cheshire Cat (Sec. 2). We will then define, in Sec. 3, the QCC effect from the weak values obtained when precise coupling conditions are met, detailing the weak measurement process that was omitted in the original QCC paper [22] or in subsequent QCC related works [30]. Indeed, a proper account of this detailed process appears to be one of the two crucial ingredients necessary in order to dispel the confusion that has emerged around the QCC effect. Sec. 4 will be devoted to the description of the tentative experimental implementation of the QCC with neutrons and with single photons. We will in particular highlight several essential differences with the ideal theoretical scheme. The published criticism of the QCC effect will be examined in Sec. 5 and compared to the precisely defined QCC introduced in Sec. 3. In the Discussion section (Sec. 6) we will detail the technical and conceptual aspects of the arguments given in the criticisms; we will also explain under which assumptions it is legitimate to interpret the effect as “disembodiment”. This will be seen to depend on the general interpretation of the quantum formalism that is favoured. We will finally give our conclusions in Sec. 7; anticipating on our assessments, we will conclude that: (i) the Quantum Cheshire Cat effect is a well defined quantum feature derived from the standard quantum formalism that can be interpreted as a spatial separation of a particle from one of its properties if some assumptions regarding property ascription are made; (ii) the QCC effect as predicted theoretically has not yet been experimentally observed; (iii) most of the works rebuking the QCC effect produced a substantial criticism of the experimental attempts to observe the Quantum Cheshire Cat, but a criticism of the ideal QCC can only be undertaken within a proper conceptual framework able to account for the relation between weakly coupled pointers and the properties of the measured quantum system. This is in our view necessary in order to analyze the issue of spatial separation of a quantum particle from one of its properties in a pre and postselected situation.

II. WEAK MEASUREMENTS AND WEAK VALUES

A. Protocol: Preselection, Unitary coupling, Postselection

The basic idea underlying the weak measurement (WM) approach is to give an answer to the question: “*what is the value of a property of a quantum system at some intermediate*

time while the system evolves from an initial state $|\psi(t_i)\rangle$ to a final state $|\chi(t_f)\rangle$?”, where $|\chi(t_f)\rangle$ is the result of a standard projective measurement made at time t_f . As we are only interested here in applying WM to derive the Quantum Cheshire Cat effect, we will restrict our discussion to the property corresponding to a bivalued observable A , with eigenstates and eigenvalues denoted by $A|a_k\rangle = a_k|a_k\rangle$, $k = 1, 2$.

Suppose that initially (at $t = t_i$) the system is prepared (preselected) into the state $|\psi(t_i)\rangle$. Let $|\varphi(t_i)\rangle$ designate the initial state of the quantum pointer. The total initial quantum state is the product state

$$|\Psi(t_i)\rangle = |\psi(t_i)\rangle |\varphi(t_i)\rangle. \quad (1)$$

We assume the pointer is local (its wavefunction has compact support in configuration space), and that the system and the pointer will interact during a brief time interval τ centered around $t = t_w$ (physically corresponding to the time during which the system and the quantum pointer interact). The interaction between the system and the quantum pointer is given by the Hamiltonian

$$H_{int} = g(t)AP. \quad (2)$$

A is the system observable that couples to the momentum P of the pointer. $g(t)$ is a smooth function non-vanishing only in the interval $t_w - \tau/2 < t < t_w + \tau/2$ and such that $g \equiv \int_{t_w - \tau/2}^{t_w + \tau/2} g(t) dt$ appears as the effective coupling constant. Recall that the coupling (2) is the usual interaction employed to account for projective measurements of A (von Neumann’s impulsive model): in that case $g(t)$ is sharply peaked and each $|a_k\rangle$ is correlated with an orthogonal state of the strongly coupled pointer. In a projective measurement the collapse of the pointer projects the system state to a random eigenstate $|a_{k_0}\rangle$. Here instead g will be small, the weakly coupled quantum pointer does not collapse, and the system will undergo at postselection a genuine projective measurement that will consequently project the quantum pointer to a specific final state, as we now detail.

Let $U(t_w, t_i)$ be the unitary operator describing the system evolution between t_i and t_w (we disregard the self-evolution of the pointer state). After the interaction ($t > t_w + \tau/2$)

the initial uncoupled state (1) has become entangled ²:

$$|\Psi(t)\rangle = U(t, t_w) e^{-igAP} U(t_w, t_i) |\psi(t_i)\rangle |\varphi(t_i)\rangle \quad (3)$$

$$= U(t, t_w) e^{-igAP} |\psi(t_w)\rangle |\varphi(t_i)\rangle \quad (4)$$

$$= U(t, t_w) \sum_{k=1,2} e^{-iga_k P} \langle a_k | \psi(t_w)\rangle |a_k\rangle |\varphi(t_i)\rangle. \quad (5)$$

At time t_f the system undergoes a standard projective measurement: a given observable \hat{B} is measured and the system ends up in one of its eigenstates $|b_k\rangle$. We filter the results of this projective measurement by keeping only a chosen outcome, say b_f . The system is thus postselected in the corresponding state $|b_f\rangle$. We will denote the postselected state by $|\chi_f(t_f)\rangle \equiv |b_f\rangle$; in most of the paper we will be dealing with a single postselected state and drop the second label, writing $|\chi(t_f)\rangle$ or $|\chi_f\rangle$ instead. After postselection, the state of the pointer, given by Eq. (5), becomes

$$|\varphi(t_f)\rangle = \sum_{k=1,2} [\langle \chi(t_w) | a_k\rangle \langle a_k | \psi(t_w)\rangle] e^{-iga_k P} |\varphi(t_i)\rangle, \quad (6)$$

where we have used $\langle \chi(t_w) | = \langle \chi(t_f) | U(t_f, t_w)$. $\varphi(x, t_f)$ is then given by a superposition of shifted initial states

$$\varphi(x, t_f) = \sum_{k=1,2} [\langle \chi(t_w) | a_k\rangle \langle a_k | \psi(t_w)\rangle] \varphi(x - ga_k, t_i). \quad (7)$$

This expression is similar to the usual von Neumann projective measurement: the first step of a projective measurement (the *premeasurement*, in which each eigenstate $|a_k\rangle$ of the measured observable is correlated with a given state $\varphi(x - ga_k)$ of the pointer) is identical here, but in a projective measurement the second step is a projection to an eigenstate $|a_{k_f}\rangle$ of A.

Let us now assume the coupling g is sufficiently small so that $e^{-iga_k P} \approx 1 - iga_k P$ holds

² In Eq. (3) the interaction appears to take place precisely at t_w ; this “midpoint rule” holds provided τ is small relative to the system evolution timescale (see Appdx A in the Supp. Mat. of Ref. [17]).

for each k^3 . Eq. (6) becomes

$$|\varphi(t_f)\rangle = \langle\chi(t_w)|\psi(t_w)\rangle \left(1 - igP \frac{\langle\chi(t_w)|A|\psi(t_w)\rangle}{\langle\chi(t_w)|\psi(t_w)\rangle}\right) |\varphi(t_i)\rangle \quad (8)$$

$$= \langle\chi(t_w)|\psi(t_w)\rangle \exp(-igA^wP) |\varphi(t_i)\rangle \quad (9)$$

where

$$A^w = \frac{\langle\chi(t_w)|A|\psi(t_w)\rangle}{\langle\chi(t_w)|\psi(t_w)\rangle} \quad (10)$$

is the weak value of the observable A given pre and postselected states $|\psi\rangle$ and $|\chi\rangle$ respectively (we will sometimes employ instead the full notation $A_{\langle\chi|\psi\rangle}^w$ to specify pre and postselection). For a localized pointer state, expanding to first order the terms $\varphi(x - ga_k, t_i)$ in Eq. (7) leads to Eq. (9): when A^w is real, the overall shift $\varphi(x - gA^w, t_i)$ is readily seen to result from the interference due to the superposition of the slightly shifted terms $\varphi(x - ga_k, t_i)$, as shown early on in Ref. [31]. Note that the shift gA^w is very small so it needs to be evaluated from the statistics over a large number of trials.

Summing up, we see that a weak measurement contains 4 steps: preselection, weak coupling through the Hamiltonian (2), postselection and readout of the quantum pointer.

B. Weak values

1. Observable average

A^w is in general a complex quantity. Following Eq. (9) the real part of the weak value A^w appears as the shift brought to the initial pointer state $|\varphi(t_i)\rangle$ by the interaction with the system via the coupling Hamiltonian (2). The weak values are generally different from the eigenvalues, but obey a similar relation with regard to the computation of expectation values. Indeed, the expectation value of A in state $|\psi(t_w)\rangle$, given in terms of eigenvalues by

$$\langle\psi(t_w)|A|\psi(t_w)\rangle = \sum_f |\langle a_f|\psi(t_w)\rangle|^2 a_f, \quad (11)$$

³ The exact condition for the asymptotic expression to hold implies that the higher order terms of order m obey $\|g^m P^m \langle\chi(t_w)|A^m|\psi(t_w)\rangle\| \ll \|gP \langle\chi(t_w)|A|\psi(t_w)\rangle\| < 1$ for Eq. (8) and $\|gPA^w\| \ll 1$ for Eq. (9). These conditions take precise forms for specific pointer wavefunctions, in particular when $\varphi(x, t_i)$ is Gaussian, see eg Ref. [31].

can also be written in terms of weak values as

$$\langle \psi(t_w) | A | \psi(t_w) \rangle = \langle \psi(t_w) | U^\dagger(t_f, t_w) U(t_f, t_w) A | \psi(t_w) \rangle \quad (12)$$

$$= \langle \psi(t_w) | U^\dagger(t_f, t_w) \sum_f | b_f \rangle \langle b_f | U(t_f, t_w) A | \psi(t_w) \rangle \quad (13)$$

$$= \sum_f |\langle \chi_f(t_f) | \psi(t_f) \rangle|^2 \frac{\langle \chi_f(t_w) | A | \psi(t_w) \rangle}{\langle \chi_f(t_w) | \psi(t_w) \rangle} \quad (14)$$

$$= \sum_f |\langle \chi_f(t_f) | \psi(t_f) \rangle|^2 A_{\langle \chi_f |, | \psi \rangle}^w \quad (15)$$

where we have used $\langle \chi_f(t_f) | = \langle b_f |$ and $\langle \chi_f(t_w) | = \langle \chi_f(t_f) | U(t_f, t_w)$ denotes the postselected state $\langle b_f |$ evolved backward in time up to $t = t_w$. Eq. (15) is expressed in terms of the probabilities of obtaining a postselected state $|b_f\rangle$ (instead of the probability of obtaining an eigenstate), with the rationale that the probabilities of obtaining the postselected states are not modified by the weak coupling.

2. Vanishing weak values

Let us start by first looking at the case of null *eigenvalues*. In a standard projective measurement, a vanishing eigenvalue implies that the state of the measurement pointer is left, untouched, remaining in the initial state: the coupling has no effect on the pointer. But the system state does change, as it is projected to the eigenstate associated with the null eigenvalue for the measured observable. For example imagine a beam of atoms incoming on a beamsplitter, after which the quantum state of each atom can be described by the superposition $|u\rangle + |l\rangle$ of atoms traveling along the *upper* or *lower* paths; if a measurement of the projector onto the upper path $\Pi_u \equiv |u\rangle \langle u|$ yields 0, then the quantum state has collapsed to $|l\rangle$ and indeed one is certain to find the atom on the lower path. If the atom has some integer spin, then measuring the spin projection along some direction can yield a null eigenvalue. The atom spin state is projected to the corresponding eigenstate (as can be verified by making subsequent measurements) corresponding to no spin component along that direction.

For weak measurements, the main difference is that the system's state is not projected after the weak coupling. Instead, the overall state evolves according to $e^{-igAP} |\psi(t_w)\rangle |\varphi(t_i)\rangle \approx |\psi(t_w)\rangle |\varphi(t_i)\rangle - iA(g|\psi(t_w)\rangle) P |\varphi(t_i)\rangle$. The system state appears as essentially unperturbed

except for the small fraction $g|\psi(t_w)\rangle$ that couples to the quantum pointer. $A(g|\psi(t_w)\rangle)$ is precisely the post-interaction (or premeasured) part of the system state, corresponding to the slight change in the system state produced by the weak coupling to the quantum pointer. The weak value appears as the imprint of this coupling left on the pointer, conditioned on the final projective measurement. A *vanishing weak value* correlates successful postselection with the quantum pointer having been left unchanged despite the interaction with the system. The reason is that the quantity $\langle\chi(t_w)|A|\psi(t_w)\rangle$, which is the numerator in the definition (10) of A^w vanishes. We will adopt here Feynman’s terminology [32] and designate this quantity by the term “transition element”⁴.

Hence when a weak value vanishes the coupling has no effect (though the pointer is left unchanged) as in the case of vanishing eigenvalues, but the implication is not relative to eigenvectors but to the transition elements between $A(g|\psi(t_w)\rangle)$ and the postselected state $|\chi(t_w)\rangle$. Put differently, if the postselected state is obtained, then whenever $A^w = 0$ the property represented by A cannot be detected by the weakly coupled quantum pointer. For example when the weak value of a projector $(\Pi_u)_{\langle\chi_f|\psi\rangle}^w = 0$, this implies (i) no effective action of the coupling on the quantum pointer (that is left unchanged) with respect to the postselected state $|\chi_f\rangle$ and (ii) that the state $|\chi_f\rangle$ cannot be reached from the initial state $|\psi(t_i)\rangle$ by the part of the system wavefunction that has interacted with the pointer in the region where Π_u was weakly measured⁵. If instead some spin observable S_γ is weakly measured in some region and $\langle S_\gamma \rangle_{\langle f|i\rangle}^w = 0$ this implies again that (i) there is no effective action on the quantum pointer of the coupling between S_γ and the pointer variable, and (ii) the final spin state $|f\rangle$ cannot be obtained by premeasuring S_γ on the initial spin state $|i\rangle$, i.e. the fraction of the spin state that is perturbed by the interaction with the pointer and thereby transforms as $S_\gamma|i\rangle$ does not reach $|f\rangle$. We will discuss in Sec. VI how these statements can be further interpreted. We will for the moment note that the argument encapsulated by (ii) can be logically restated as: since the final state is reached when postselection is successful, a null weak value (i.e., a vanishing transition element) implies that for this postselected system state, the property represented by the weakly measured observable cannot have been detected by the quantum pointer. Hence in some loose sense

⁴ In the literature, this term is also called “transition matrix element” or “transition amplitude”

⁵ Note that this reasonings hold for asymptotically weak couplings, those that do not affect the postselection probabilities $|\langle\chi_f(t_f)|\psi(t_f)\rangle|^2$; otherwise terms beyond the linear expansion (8) would contribute.

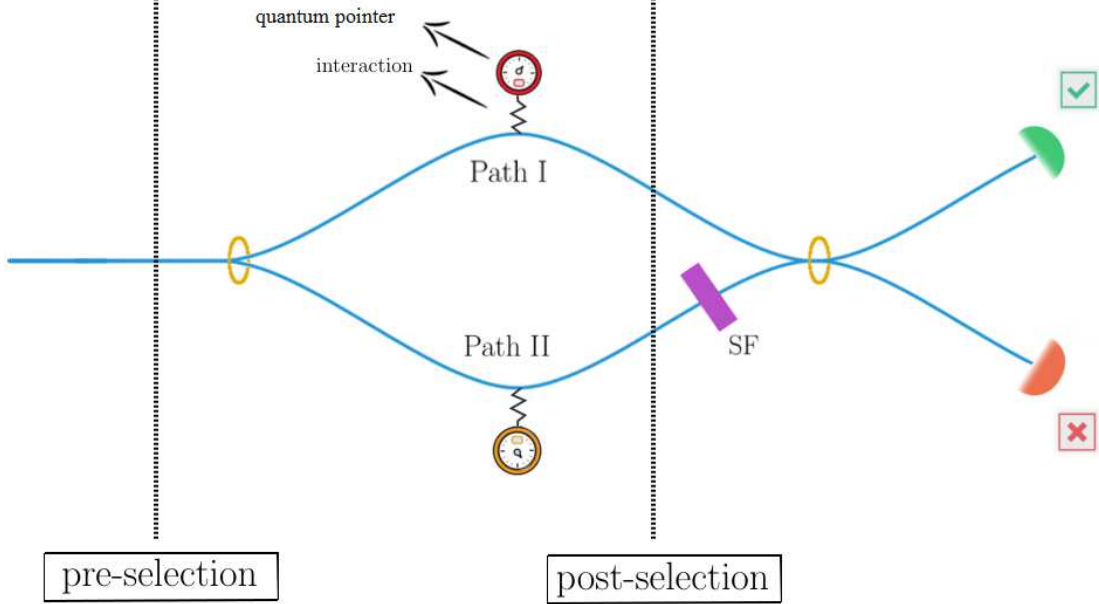


FIG. 1: The ideal Quantum Cheshire Cat setup based on a 2-path interferometer. A quantum pointer is placed on each path. Each pointer interacts locally with the particle, coupling the particle observable $A_j = \Pi_j$ or $A_j = (\sigma_x)_j$ (where $j = I, II$) to the pointer dynamical variable P_I or P_{II} resp. (g is the coupling strength). The particle then travels along the interferometer essentially unperturbed until it gets detected (SF is a spin flipper, required for postselection of the spin state). Successful postselection (detection in the upper port) is correlated with pointers in final states $\exp(-igA_j^w P_j) |\varphi_j(t_i)\rangle$ with the weak values $\Pi_I^w = 1, (\sigma_x)_I^w = 0$ and $\Pi_{II}^w = 0, (\sigma_x)_{II}^w = 1$ for pointers I and II respectively.

(to be refined below), that property “was not there”, ie in the region where the system and weak pointer interacted.

III. THE QUANTUM CHESHIRE CAT EFFECT

The Quantum Cheshire Cat effect, as introduced in Ref. [22] is based on the Mach-Zehnder interferometer shown in Fig. 1. Rather than elaborating on the original argument [22], we will instead give a proper account of the QCC in terms of weak measurements, by filling the gaps left in the theoretical account given in [22]. We will mostly remain here at a technical level (interpretations will be discussed later in Sec. VI).

Assume that a quantum particle with spin 1/2 enters the interferometer shown in Fig. 1 in

state $|\psi_i\rangle |s_i\rangle$ where $|\psi_i\rangle$ is a localized wavepacket and $|s_i\rangle$ the spin state in which the system was prepared. Let us label by I and II the upper and lower arms of the interferometer, and assume a quantum pointer in state $|\varphi_I(t_i)\rangle$ sits on arm I . Hence the total system-pointer initial state is

$$|\Psi(t_i)\rangle = |\psi_i\rangle |s_i\rangle |\varphi_I(t_i)\rangle. \quad (16)$$

The system will then enter the interferometer, evolving to

$$U(t, t_i) |\psi_i\rangle |s_i\rangle = \frac{1}{\sqrt{2}} (|\psi_I\rangle + |\psi_{II}\rangle) |s_i\rangle \quad (17)$$

and then locally interact along arm I through the interaction Hamiltonian (2) given here by $g(t)A_IP_I$ where index I emphasizes the coupling takes place only on branch I and P_I is the pointer's linear momentum. A is a system observable that will be taken to be either the spatial projector or some spin component. After the interaction the state vector is given by Eq. (4) that takes here the form

$$|\Psi(t)\rangle = U(t, t_w) (e^{-igA_IP_I} |\psi_I\rangle |s_i\rangle + |\psi_{II}\rangle |s_i\rangle) |\varphi_I(t_i)\rangle / \sqrt{2}. \quad (18)$$

Assume finally that the system is postselected after exiting the interferometer ($|\psi_I\rangle$ and $|\psi_{II}\rangle$ thus overlap) in the state

$$|\chi_f\rangle = (|\psi_I\rangle |s_i\rangle + |\psi_{II}\rangle |-s_i\rangle) / \sqrt{2}. \quad (19)$$

To be definite we will choose the initial state to be prepared in state $|s_i\rangle = |+z\rangle$ so $|-s_i\rangle = |-z\rangle$.

Let us now examine the final state of the quantum pointer, that will depend on the observables to which it was weakly coupled. Then, after postselection, the pointer state is given by Eq. (9):

$$|\varphi_I(t_f)\rangle = \frac{1}{\sqrt{2}} \langle \chi_f | U(t_f, t_w) (|\psi_I\rangle + |\psi_{II}\rangle) |s_i\rangle \exp(-igA_I^w P_I) |\varphi_I(t_i)\rangle \quad (20)$$

$$= \frac{1}{2} \exp(-igA_I^w P_I) |\varphi_I(t_i)\rangle. \quad (21)$$

Hence the final state of the quantum pointer that was weakly coupled to A_I depends on the weak value

$$A_I^w = \frac{\langle \chi_f | U(t_f, t_w) A_I U(t_w, t_i) |\psi_i\rangle |s_i\rangle}{\langle \chi_f | U(t_f, t_i) |\psi_i\rangle |s_i\rangle} \quad (22)$$

$$= [\langle \psi_I | \langle +z |] A_I [|\psi_I\rangle | +z \rangle]. \quad (23)$$

For $A_I = \Pi_I$ (projector to the spatial wavefunction of arm I , $\Pi_I \equiv |I\rangle\langle I|$) and $A_I = (\sigma_x)_I$ (the spin component along the x axis, where I recalls the coupling takes place along path I), we obtain by using Eq. (23):

$$\Pi_I^w = 1 \quad (\sigma_x)_I^w = 0. \quad (24)$$

Similarly, a quantum pointer coupled to A_{II} can be placed along path II . Its final quantum state $|\varphi_{II}(t_f)\rangle$ is obtained after postselection (to the same state $|\chi_f\rangle$) exactly as above, yielding

$$|\varphi_{II}(t_f)\rangle = \frac{1}{2} \exp(-igA_{II}^w P_{II}) |\varphi_{II}(t_i)\rangle \quad (25)$$

with

$$A_{II}^w = [\langle\psi_{II}|\langle -z|] A_{II} [|\psi_{II}\rangle|+z\rangle]. \quad (26)$$

This gives us for $A_{II} = \Pi_{II}$ and $A_{II} = (\sigma_x)_{II}$ respectively

$$\Pi_{II}^w = 0 \quad (\sigma_x)_{II}^w = 1. \quad (27)$$

The conjunction of Eqs. (24) and (27) defines the quantum Cheshire Cat effect. Indeed, the quantum pointer coupled to the system on path I only detects the spatial wavefunction, but the interaction with the spin component σ_x has no effect. Conversely along arm II , a quantum pointer picks up a shift due to the coupling with the system's spin component σ_x but the coupling to the spatial wavefunction along path II has no effect on the pointer.

Following our discussion on null weak values in Sec. II B 2, the fact that the particle cannot be seen on path II means that the slight change brought to the spatial wavefunction of the system by the interaction with the weakly coupled pointer on path II cannot be postselected in the state $|\chi_f\rangle$ (the transition element $\langle\chi_f|\Pi_{II}|\psi_i\rangle$ vanishes). Similarly the spin component σ_x is not detected on arm I because the postselected spin state $|+z\rangle$ cannot be reached by the part $\sigma_x|+z\rangle$ of the spin state that has been modified by coupling (the transition element $\langle+z|\sigma_x|+z\rangle$ vanishes). Note that in principle different weak measurements can be made jointly, on both arms, or subsequently, on the same arm (since for an asymptotically weak interaction, to lowest order, only the unperturbed part of the system state is taken into account when the system interacts with a subsequent weak pointer).

IV. EXPERIMENTAL IMPLEMENTATIONS OF THE QUANTUM CHESHIRE CAT

A. Experiment in a neutron interferometer

A few months after the publication of the Quantum Cheshire cat paper by Aharonov et al [22], an experimental realization of the QCC effect was implemented [24] with single neutrons in a Mach-Zehnder-like triple-Laue interferometer. Due to strong experimental constraints, the scheme employed in the experiment was significantly different from the ideal QCC scheme described in Sec. III. In particular, it was not experimentally feasible to couple quantum pointers to the neutron inside the interferometer; instead, the weak values were inferred from the intensities of the detected neutron signal after postselection. While the weak values determined experimentally were in excellent agreement with Eqs. (24) and (27) defining the QCC effect, the fact that weak measurements were not made (the weak values were instead inferred from intensities measured after making transformations) has implications concerning the validity of the observation of the QCC effect, as we now detail below.

Rather than summarizing the experiment as described in Ref. [24], we will highlight the differences with the theoretical QCC proposal. For simplicity we will employ the same pre and postselected states employed in the theoretical proposal [22] and in Sec. III (in the experiment pre and postselected states were inverted relative to the theoretical proposal, but this has no consequence for our present discussion).

The crucial difference between the QCC theoretical proposal and the neutron experiment is the lack of quantum pointers interacting with the neutron (or the lack of additional degrees of freedom of the neutron that would act as such pointers). Hence there is no such thing as the state vector $|\varphi_I(t_i)\rangle$ in equation (16) or in any of the equations below equation (16). This is of course a radical departure from the weak measurement formalism introduced in Sec. II. Instead of relying on weak measurements, the neutron QCC experiment is based on introducing external potentials along the arms. These interactions modify the postselected intensity (ie, the neutrons detected in the postselected state) relative to the intensity obtained without the interaction in that arm.

The spatial projector weak values are obtained by inserting an absorber on arms I or

II. The absorber is not a quantum pointer: it is modeled by an external decay potential $V_j = e^{-iM_j}$ where j stands for either arm I or II ; M is the absorption coefficient. For M small, some simple manipulations lead to [24]

$$\mathcal{I}_j^{abs} = |\langle \chi_f | U(t, t_i) |\psi_i\rangle | +z \rangle|^2 (1 - 2M_j \Pi_j^w) = \mathcal{I}_0 (1 - 2M_j \Pi_j^w) \quad (28)$$

where $\mathcal{I}_0 = |\langle \chi_f | U(t, t_i) |\psi_i\rangle | +z \rangle|^2$, obtained from Eqs. (17) and (19), defines the detected reference intensity after postselection. Hence the experimental observation of \mathcal{I}_j^{abs} when the absorber is placed on arm j allows to extract the weak value Π_j^w . But no weak measurement has been made: instead of a weak interaction, we have imposed a strong interaction that happens with a small probability; instead of a pointer whose state would reflect the effect of the coupled observable on the quantum state of the pointer, we are inferring the presence of neutrons along path I by the fact that the relative number of postselected neutrons decreases. Hence a postselected neutron does not carry any signature of the interaction (precisely because the neutrons that have interacted with the absorber on path I have been absorbed and have thus vanished). This does not entail that one cannot conclude from Eq. (28) that the neutrons can only reach postselection by going through path I : indeed when an absorber is placed along path I this is reflected by the fact that $\mathcal{I}_I^{abs}/\mathcal{I}_0 < 1$ while an absorber along path II has no effect on the intensity, $\mathcal{I}_2^{abs}/\mathcal{I}_0 = 1$. However this conclusion does not rely on weak measurements nor on weak values but only on the relative intensities, consistent with the fact that no coupling to a quantum pointer has taken place: no observable has been weakly measured. Instead, the conclusion $\Pi_{II}^w = 0$ is made from the lack of backaction of the strong interaction with the absorber on the postselected neutron intensity (see Fig. 2).

For the spin component weak value the same objection can be made, now with more serious consequences. The external potential employed is the one for a magnetic moment in a magnetic field B_j oriented along the x axis, $U_j = -\gamma (\sigma_x)_j B_j/2$ where γ is the gyromagnetic ratio and the index j means that B_j is non-zero only in a region along arm j (and thus affects only the magnetic moment $\gamma (\sigma_x)_j$ on arm j). Since $\int dt U_j = \alpha (\sigma_x)_j/2$, where α is the precession angle induced by the magnetic field, we have for small α

$$e^{-i \int dt U_j/\hbar} U(t_w, t_i) |\psi_i\rangle |s_i\rangle = \left[1 + i \frac{\alpha}{2} (\sigma_x)_j - \frac{\alpha^2}{8} \Pi_j \right] (|\psi_I\rangle |s_i\rangle + |\psi_{II}\rangle |s_i\rangle) / \sqrt{2}. \quad (29)$$

Note that the second order term is needed because postselection leads here to intensities,

not to pointer shifts [compare with Eq. (18)]. Postselection indeed yields

$$\mathcal{I}_j^{mag} = |\langle \chi_f | U(t, t_i) |\psi_i\rangle | +z \rangle|^2 \left(1 + \frac{\alpha^2}{4} |(\sigma_x)_j^w|^2 - \frac{\alpha^2}{4} \Pi_j^w \right) \quad (30)$$

where $\mathcal{I}_0 = |\langle \chi_f | U(t, t_i) |\psi_i\rangle | +z \rangle|^2$ is again the reference intensity and \mathcal{I}_j^{mag} the detected intensity when a magnetic field is applied on arm j .

In the neutron experiment, Eq. (30) is employed to compute $|(\sigma_x)_j^w|$ from the observed relative intensities $\mathcal{I}_j^{mag}/\mathcal{I}_0$. The term Π_j^w was taken from the value inferred experimentally from Eq. (28), though it would also have been consistent to use $\delta_{I,j}$ instead. Indeed $(\sigma_x)_j^2 = 1_j$ (the identity along arm j) but upon postselection $1_j (|\psi_I\rangle |s_i\rangle + |\psi_{II}\rangle |s_i\rangle)$ vanishes for $j = II$. The upshot is that while the theoretical prediction $(\sigma_x)_I^w = 0$ can be recovered experimentally from \mathcal{I}_I^{mag} by fitting Eq. (30), the magnetic field along arm I nevertheless has an effect on the spin σ_x through the last term $-\alpha^2/4$ of Eq. (30), a point that was made in Ref. [29]. In this particular experiment this effect can be claimed to be systematic (in the sense that it doesn't depend on the field orientation – ie, which spin component couples with the field), but it is there nonetheless: the appearance of higher order terms is generic when inferring weak values from intensity measurements, rather than making genuine weak measurements.

B. Experiment with single photons

A QCC experiment with photons was carried out very recently [25]. The authors make clear from the start that their setup is based on the one implemented in the neutron experiment, where the photon polarization takes the place of the neutron spin (the polarization state along the interferometer arms is $|H\rangle$ or $|V\rangle$ instead of the spin states $|+z\rangle$ and $|-z\rangle$), and the circular polarisation weak value replaces $(\sigma_x)_j^w$. Indeed, leaving aside the specificities of the photon experiment (use of a Sagnac interferometer, coincidence detection with the heralded photon), the scheme employed in Ref. [25] in order to display the QCC effect is identical to the neutron experiment: the presence weak value Π_j^w is inferred from the photon counts with/without an absorber placed along the arms, and the polarization weak value is inferred from photon counts with/without a rotation of the polarization induced by half wave plates. This similarity can be seen directly by comparing Eqs. (3) and (6) of Ref. [25] with Eqs. (28) and (30) given above for the neutron experiment respectively.

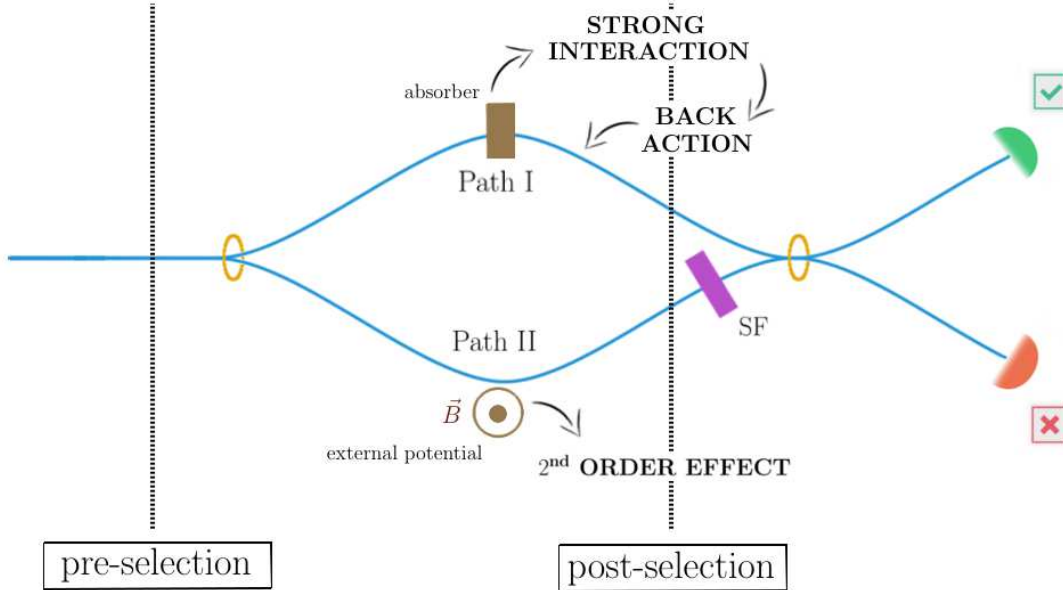


FIG. 2: Schematic representation of the differences between the ideal Quantum Cheshire Cat setup proposed theoretically and the experimental implementation with neutrons [24]. These differences also apply to a similar experiment performed with photons in a Sagnac-like interferometer [25] (see text).

Hence, our remarks made above for the neutron experiment also hold for this photon experiment: no weak measurements were made, and inferring weak values from measured intensities does not give an unambiguous demonstration of the Cheshire cat effect. It is noteworthy that the authors of Ref. [25] point out in their Conclusion that their observed measurement statistics can be understood without recourse to weak values.

V. CRITICISM OF THE QCC IN THE LITERATURE

A. General remarks

The Quantum Cheshire Cat scheme has been criticized by several groups [5, 26–29]. At the basis of the criticism there is the underlying idea that claiming separation of a property from a particle is preposterous. However the criticism was not always carefully formulated. Most of the works did not clearly discriminate the original theoretical proposal from the neutron experimental implementation (which as we have shown above are very different). They were also not rigorous when making assertions concerning the disembodiment, often

using terms in plain English (which can be at best an interpretation of the QCC effect) rather than scrutinizing the technical definition of the QCC effect in terms of weak measurements; actually most of these works with the exception of Ref. [5] did not base their arguments on the characteristics of the weak measurements framework, that was generally ignored.

As a result, the criticism did not deliver clarification. We study below the main arguments that were given in the criticisms and assess their relevance. The first class of arguments involves the detected intensities, a second type of arguments invokes interference, while other arguments attempt to discard the possibility of disembodiment.

B. Intensities

The work by Stuckey et al [29] criticizes the QCC neutron experiment only, on the basis that the detected intensities when the spin weak value is $(\sigma_x)_I^w = 0$ still shows the effect of the magnetic field (they do accept that the absorber's effect on the intensities is a proof that the particle did not take path II). Their argument starts from the identity

$$e^{-i\alpha\sigma_x/2} = \cos \frac{\alpha}{2} I - i \sin \frac{\alpha}{2} \sigma_x. \quad (31)$$

Stuckey et al point out that the magnetic field has an effect on the intensities through the $\cos \frac{\alpha}{2}$ term: to lowest order this term is quadratic, exactly as the term containing the weak value contribution. So they conclude that this contradicts the statement according to which the spin component σ_x did not travel through arm I . This argument is technically correct as far as intensities are concerned; it is equivalent to the observation we made below Eq. (30). However this argument does not involve weak measurements (although we know from our discussion above that the QCC is defined from weak measurements that leave the post-selected intensities of the system undisturbed), nor does Ref. [29] propose an alternative framework for a non-destructive measurement that would ascribe a non-zero value to the spin component on arm I , despite the fact that $(\sigma_x)_I^w = 0$. Still it is valuable to remark that Stuckey et al [29] leave open the possibility of observing a genuine QCC.

In a very different work Atherton et al [28] performed a classical optics experiment based on a Mach-Zehnder type interferometer. They start with polarized beams and compare the intensities obtained on one of the output ports with or without an absorber or a polarization rotating plate inserted on either arms I or II (a polarizer ensures postselection of the

detected beam). This setup indeed mimicks the neutron experiment described in Sec. IV A. The observed electric field intensities display the same behavior as the neutron counts. On this basis, Atherton et al conclude that there is nothing quantum about the QCC effect, that in their view is an “illusion”. This classical physics experiment has the merit of highlighting the fact that the observed intensities in the neutron experiment do not need to be interpreted with the weak measurements formalism, since no quantum weak values can be obtained with classical beams. However attempting to draw conclusions on the value of quantum observables from a classical optics experiment is an impossible task: the classical beams travel through both arms, so what could the Cheshire cat effect mean in this context? Instead a single neutron obeys the standard quantum rule of projective measurements and cannot be found simultaneously on both arms. Atherton et. al. do not define precisely the analogue of the quantum observable σ_x and there is no mention of anything that can play the role of weak values in their work, although they are necessary to precisely define the QCC effect. Therefore the findings of Ref. [28] are only relevant to the specific implementation of the neutron experiment summarized in Sec. IV A, not to the QCC effect itself.

C. Interference

In an interesting paper, Correa et al [26] assert that the QCC effect arises from “simple quantum interference”. They mean by that statement that a quantum pointer remaining unchanged (as this happens for vanishing weak values) results from the interference of almost perfectly overlapping final pointer states. This remark is uncontroversial – this is of course the way weak measurements work in general [see Eq. (6)], as shown early in a 1989 work [31]. Significantly, Correa et al. explicitly introduce the quantum pointer states (our states $|\varphi(t_i)\rangle$ of Secs. II and III) but treat the pointer-system interactions in an ad-hoc way, by stating in words how these pointer states are transformed, rather than introducing an interaction Hamiltonian such as our equation (2). Hence weak values do not explicitly appear in their treatment, but the motion of the weakly coupled pointer states are recovered by the superposition of the quantum state after postselection, see our Eq. (7) above.

However in our view the Quantum Cheshire Cat effect does not depend on the underlying mathematical formulation of the pointers motion, but on giving a physical meaning to these motions, as we now discuss for the most salient physical property claimed to characterize

the QCC effect, disembodiment.

D. Disembodiment

The main objective of the criticisms seems to undermine the claim made in the original proposal by Aharonov et al. that Eqs. (24)-(27) could imply disembodiment of the particle from one of its properties. Correa et al [26] write that their interference argument allows them to interpret the QCC phenomenon without appealing to disembodiment, but they do not produce a full reasoning that would support this claim. Indeed, the interference account of the quantum pointer dynamics does not endorse nor disprove the “disembodiment” claim: the superposition of state vectors is a Hilbert space feature, that at least according to standard quantum mechanics is only a mathematical description aimed to compute probabilities. The authors of Ref. [26] do not elaborate on whether their argument implies going beyond this standard view, for example by endowing the state vectors with some ontological features that would then propagate along both arms. Instead, whether the motion of the quantum pointers weakly coupled to the particle’s position or spin component is indicative of disembodiment depends on a framework ascribing properties to a quantum system in the absence of a projective measurement. The weak measurements formalism constitutes such a framework, and discarding the possibility of disembodiment implies either replacing the WM formalism with some alternative proposition, or refuse that quantum properties can be defined beyond projective measurements. This point will be further discussed in Sec. VI.

Michielsen et al [27] also objected on disembodiment by running numerical experiments. Strictly speaking their work only applies to the neutron experiment. They simulate the observed interferences with/without absorbers/spin rotators on arms I and II . The simulation is based on a discrete event learning model, in which particles act as messengers in such a way that the particle counts in the outgoing ports of a Mach-Zehnder interferometer quickly converge towards the quantum probabilities. Such a model was previously employed to reproduce intensities in neutron interferometric experiments [33], so Ref [27] is an extension of that previous work so as to include the absorber/rotator interactions. In any case Michielsen et al. do not consider quantum pointers and weak measurements in their model, and while it would be interesting to investigate if the Quantum Cheshire Cat can be properly formulated within the discrete event learning model by including the coupled quantum pointers

explicitly, their results are not relevant to the QCC effect as we have properly defined it.

VI. DISCUSSION

A. General remarks

The main property characterizing the Quantum Cheshire Cat seen here is that for a fixed initial and final state of the system (photon, neutron), the state of a quantum pointer weakly coupled to the system's spatial wavefunction is modified for a pointer placed along path I but not for a pointer placed along path II . If the quantum pointer is coupled to the spin component σ_x instead, the opposite behavior is obtained. Since neither the neutron nor the single photon experiments have recourse to weak pointers, criticizing the experiments done so far as not having realized the QCC is legitimate. However we have seen that some authors of the criticism did not clearly discriminate the experimental implementations from the ideal QCC. For instance Atherton et al [28] and Michielsen et al [27] criticize the neutron experiment, but from there cast suspicion on the ideal theoretical scheme as being an “illusion”.

While doing so is strictly speaking inconsistent, the common element underlying the criticism of the Quantum Cheshire Cat formulated in Refs [26–29] is to refute the idea of a spatial separation between the particle and one of its properties. Unfortunately, rather than starting from the technical definition of the QCC and from there prove that this definition cannot imply a spatial separation, Refs [26–29] do not specify the assumptions they make concerning the possibility of ascribing properties to a quantum system in the absence of a projective measurement. The weak measurements formalism proposes a framework accounting for such properties, and from there a technical definition for spatial separation is obtained in terms of weak values. This is arguably different than taking the term “spatial separation” in a literal sense, that would disregard the well-known conceptual difficulties of the standard quantum formalism in giving an unambiguous account of the physical state and properties of a system. These difficulties are ultimately due to the fact that contrary to classical mechanics or classical optics, the relationship between the theoretical terms of quantum mechanics and physical reality are unknown, and most often denied. Hence relying on pointers to assess the value of the property of a quantum system is crucial; this in turn

hinges on employing an explicit conceptual and interpretative framework.

B. Technical and conceptual aspects

The Quantum Cheshire Cat is technically defined by Eqs. (24) and (27) in the context of weak measurements with postselection that do not affect the coherence of the system. So the first question is whether one can have Eqs. (24) and (27) (in the context of weak measurements) while still being able to assess the particle can be found on path II , or that spin component σ_x can be found along path I (conditioned on successful postselection). In order to answer this question, the authors of Refs. [26–29] do not take into account the fact that some type of measurement or interaction (presumably different from the weak measurement protocol) needs to be proposed. Otherwise it is impossible to assert anything about the system properties. In particular, relying on the state vector, as done in [26, 29], is insufficient: every quantum physicist agrees that the total state vector is in a superposition state along both paths, and that postselection will imply a certain correlation due to interference. But the quantum axiomatics remain silent on the meaning of the state vector, that may be taken as a simple computational tool (this is the standard view conveyed in textbooks), or as an element of a more elaborate ontology, but clearly not in a literal manner as representing a classical field that would propagate simultaneously along both arms of the interferometer.

C. Interpretations

This leads us to the second question: does the technical definition of the QCC recalled in the preceding paragraph necessarily imply some form of spatial separation, or disembodiment? The answer depends on how the quantum pointer's motions are interpreted. Indeed, the pointers measure weak values, and null weak values are null transition elements. And transition elements are well-defined quantities in standard quantum mechanics.

Now having pointers measuring transition elements does not fit with the eigenstate-eigenvalue link (by which the value of the property represented by the observable is associated with a quantum state of the system). This would be impossible given the aim of the WM scheme, as recalled at the beginning of Sec. II: a given weak value is not associated with a given state of the system, but with the transition of the (time-evolved) preselected

state to the postselected state induced by the weak coupling between system and pointer observables. As we have seen in Sec. II B 2, vanishing transition amplitudes imply that the final postselected state cannot be reached by the part of the system state that has been perturbed by the weak interaction with the quantum pointer.

Hence $\Pi_{II}^w = 0$ implies that the transformation generated on the preselected state by premeasuring Π_{II} does not reach the postselected state. If one upholds an interpretation in which a property relies on projection to an eigenstate, then measuring a vanishing transition element has no bearing on a statement concerning property ascription. This is the criticism developed by Sokolovski [5]. In the terminology of Ref. [5], the transition amplitudes belong to “virtual paths”, that do not describe the real path of a particle. Indeed, according to this view, only a strong projective measurement can tell us if the particle is or not in arm II . A projective measurement creates a “real path” that precludes the possibility of measuring any other property on the arm in which the particle is not found (otherwise the uncertainty principle would be violated), so that by that account a real path cannot accommodate the idea of spatial separation. If we do not perform a projective measurement, but measure instead a null weak value, then we know that the transition element $\langle \chi_f | \Pi_{II} | \psi_i \rangle$ vanishes, but this only characterizes the reaction of the system to a small perturbation and should not be taken as a measurement of the position of the particle.

While this point of view is consistent, it is also possible to go further in interpreting transition elements as characterizing system properties. The null weak value operationally means that the pointer state corresponding to the particles detected in the postselected state is unaffected by the presence of weak interactions. Accordingly $\Pi_{II}^w = 0$ implies that the particle’s presence cannot be detected by a quantum pointer weakly interacting with the spatial wavefunction on path II and be detected in the chosen postselection state $|\chi_f\rangle$ [Eq. (19)]: such a correlation is forbidden by standard quantum mechanics, as the transition element $\langle \chi_f | \Pi_{II} | \psi_i \rangle$ vanishes. On this basis it is possible to uphold that the final state cannot have been reached by taking path II (otherwise the weakly coupled pointer along arm II would have been displaced upon postselection). In a crude sense, we can say that the particle has not been in arm II .

The same reasoning can be made concerning the spin component. $(\sigma_x)_I^w = 0$ because the transition element $\langle +z | (\sigma_x)_I | +z \rangle$ vanishes. A vanishing transition element means that the transformed part of the spin state resulting from coupling σ_x weakly to a quantum

pointer on path I is orthogonal to the postselected state (hence the postselected cannot be reached by the fraction of the system state coupled to the pointer). Therefore σ_x cannot be found on path I in this sense: the premeasurement of σ_x on path I slightly changes the spin state, but for the stipulated postselection, this slight change has no effect, leaving the state of the quantum pointer undisturbed. Note that this is very different from the projective measurement process yielding a null eigenvalue, given that a vanishing eigenvalue is associated with a particular eigenstate (see Sec. II B 2).

It should be stressed that relying on transition elements to assess the value of properties in pre/post-selected systems does not necessarily call for paradoxes. This can be understood as the confirmation that the effect of the superposition principle (or sum over paths) can be observed by a local weak coupling of a system observable with a quantum pointer. In general, the transition element on either path will be not vanishing, yielding an observable effect on the pointers placed on both arms of the interferometer. However for specific choices of pre/postselected states, a given system observable may generate a transition to the final state only along a given arm, while such a transition cannot take place along the other.

VII. CONCLUSION

In this paper, we have bridged the gap, both at a theoretical and at a conceptual level, between the original Quantum Cheshire Cat proposal [22], and the various experimental realizations and criticisms that have been formulated. In doing so, we have clarified the meaning of the QCC effect and dispelled the considerable degree of confusion that was seen to arise from the raised criticisms of the Quantum Cheshire Cat proposal.

We have argued that “disembodiment” can be said to hold if it is defined in terms of transition elements for the system observables. As we have seen, a quantum pointer after postselection detects the system observable to which it is weakly coupled only when the relevant transition element for that observable does not vanish. By a suitable choice of pre and post-selected states, the spatial wavefunction can only be detected by a quantum pointer placed on one of the paths (and not the other) while the spin component is only seen on the other path.

We conclude by summarizing our main results:

- The Quantum Cheshire Cat effect is a well defined quantum feature derived from

the standard quantum formalism for pre- and post-selected states of a system; the interpretation of the effect in terms of spatial separation of a particle from one of its properties hinges on the issue of the relation between property ascription and weakly coupled pointers;

- The QCC effect as predicted theoretically has not yet been experimentally observed, as the experimental realizations done so far have not been able to properly implement the weak measurement protocol;
- Most of the works criticizing the QCC effect did not introduce a proper framework in order to analyze the issue of spatial separation of a quantum particle from one of its properties in a pre and postselected situation, so their criticism is incomplete.

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- [1] Y. Aharonov, D. Z. Albert, and L. Vaidman. *Phys. Rev. Lett.*, 60:1351, 1988.
 - [2] A. J. Leggett. *Phys. Rev. Lett.*, 62 2325, 1989.
 - [3] A. Peres. *Phys. Rev. Lett.*, 62 2326, 1989.
 - [4] Y. Aharonov and L. Vaidman *Phys. Rev. Lett.*, 62 2327, 1989.
 - [5] D. Sokolovski, *Phys. Lett. A* 380 1593 2016.
 - [6] G. J. Pryde, J. L. O'Brien, A. G. White, T. C. Ralph, and H. M. Wiseman. *Phys. Rev. Lett.*, 94 220405, 2005.
 - [7] David J. Starling, P. B. Dixon, A. N. Jordan, and J. C. Howell *Phys. Rev. A* 80, 041803, 2009
 - [8] J. S. Lundeen, B. S. A. Patel, C. Stewart, and C. Bamber. *Nature*, 474 188, 2011.
 - [9] L. J. Salazar-Serrano, D. Janner, N. Brunner, V. Pruneri, and J. P. Torres. *Phys. Rev. A*, 89 012126, 2014.

- [10] S. Sponar, T. Denkmayr, H. Geppert, H. Lemmel, A. Matzkin, J. Tollaksen, and Y. Hasegawa. *Phys. Rev. A*, 92 062121, 2015.
- [11] C. Fang, J.-Z. Huang, Y. Yu, Q. Li and G. Zeng, *J. Phys. B.* 49 175501, 2016.
- [12] M. Cormann, M. Remy, B. Kolaric, and Y. Caudano, *Phys. Rev. A* 93, 042124 2016
- [13] L. Vaidman, A. Ben-Israel, J. Dziewior, L. Knips, M. Weissl, J. Meinecke, C. Schwemmer, R. Ber, and H. Weinfurter, *Phys. Rev. A* 96, 032114 2017
- [14] D. Sokolovski and E. Akhmatkaya, *Ann. Phys.* 339, 307, 2013.
- [15] Y. J. Zhang, H. Han, H. Fan, and Y. J. Xia, *Ann. Phys.* 354, 203, 2015.
- [16] I. Esin, A. Romito and Y. Gefen, *Quantum Stud.: Math. Found.* 3, 265, 2016.
- [17] A. Matzkin. *Phys. Rev. Lett.*, 109 150407, 2012.
- [18] U. Singh and A. K. Pati, *Ann. Phys.* 343, 141 2014
- [19] Y. Aharonov, E. Cohen, and A. C. Elitzur, *Ann. Phys.* 355, 258 2015
- [20] H. Hofmann *Eur. Phys. J. D* 70, 118 2016
- [21] Q. Duprey and A. Matzkin *Phys. Rev. A* 95, 032110 2017
- [22] Y. Aharonov, D. Rohrlich, S. Popescu, and P. Skrzypczyk. *New. J. Phys.*, 15:113015, 2013.
- [23] Y. Guryanova, N. Brunner and S. Popescu, arXiv:1203.4215, 2012.
- [24] T. Denkmayr, H. Geppert, S. Sponar, H. Lemmel, A. Matzkin, J. Tollaksen, and Y. Hasegawa. *Nature Communications*, 5, 4492, 2014.
- [25] J. M. Ashby, P. D. Schwarz, and M. Schlosshauer. *Phys. Rev. A*, 94, 012102, 2016.
- [26] R. Correa, M. F. Santos, C. H. Monken, and P. L. Saldanha. *New J. Phys.*, 17, 053042, 2015.
- [27] K. Michielsen, T. Lippert, and H. D. Raedt. *Proc. SPIE*, 9570, 957000, 2015.
- [28] D. P. Atherton, G. Ranjit, A. A. Geraci, and J. D. Weinstein. *Opt. Lett.*, 40, 879, 2015.
- [29] W. Stuckey, M. Silberstein, and T. McDevitt. *Int. J. Quantum Found.*, 2, 17, 2016.
- [30] A. Matzkin and A. K. Pan. *J. Phys. A: Math. Theor.*, 46 315307, 2013.
- [31] I. M. Duck, P. M. Stevenson, and E. C. G. Sudarshan. *Phys. Rev. D*, 40, 2112, 1989.
- [32] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw Hill, New York, 2005).
- [33] H. D. Raedt, F. Jin, and K. Michielsen. *Quantum Matter*, 1,1, 2012.