Robustness analysis of uncalibrated eye-in-hand visual servo system in the presence of parametric uncertainty

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Abstract
Purpose – The purpose of this paper is to present the methodology to the robust stability analysis of a vision-based control loop in an uncalibrated environment. The type of uncertainties considered is the parametric uncertainties. The approach adopted in this paper utilizes quadratic Lyapunov function to determine the composite Jacobian matrix and ensures the robust stability using linear matrix inequality (LMI) optimization. The effectiveness of the proposed approach can be witnessed by applying it to two-link robotic manipulator with the camera mounted on the end-effector.

Design/methodology/approach – The objective of this research is the analysis of uncertain nonlinear system by representing it in differential-algebraic form. By invoking the suitable system representation and Lyapunov analysis, the stability conditions are described in terms of linear matrix inequalities.

Findings – The proposed method is proved robust in the presence of parametric uncertainties.

Originality/value – Through a differential-algebraic equation, LMI conditions are devised that ensure the stability of the uncertain system while providing an estimate of the domain of attraction based upon quadratic Lyapunov function.

Keywords Stability (control theory), Control systems, Uncertainty management, Eye-in-hand, Linear matrix inequality, Uncalibrated

Paper type Research paper

Introduction
Eye-in-hand image-based visual servoing is a well-known technique that computes control values on the basis of image features directly, observed by a camera rigidly attached to the robot end-effector (Hutchinson et al., 1996). The objective of all vision-based control schemes is the minimization of error \( \xi \) between initial features \( s \) and desired features \( s_d \), i.e. \( \xi = s - s_d \). A large error \( \xi \) may produce unnecessary motion of the robot that leads to the failure of the task (Chaumette, 1998). This undesired motion can be caused by the parametric uncertainties present in the system. Therefore, it is essential to determine the robust stability analysis of a vision-based control loop in the presence of parametric uncertainties.

Over the last two decades, various methods came forward for analysis and synthesis of linear systems (Boyd et al., 1993; Zhang et al., 2010). These methods primarily focus upon determining stability for unstable linear systems by means of Lyapunov theory and linear matrix inequality (LMI) framework. An optimization scheme based upon Lyapunov is established which uses circle and Popov criteria for the analysis of linear systems (Hindi and Boyd, 1998). Other methods make use of complex Lyapunov functions, such as polyhedral (Gomes da Silva and Tarbouriech, 1999) and piece-wise quadratic (Johansson, 2002). However, the stability analysis for nonlinear systems by employing the Lyapunov theory and LMI framework has not been explored much.

The stability and performance analysis of uncertain nonlinear systems has recently been addressed by a few researchers. Some prominent contribution in this domain includes the work of Trofino (2000), Johansen (2000) and Coutinho et al. (2002) that considers mainly quadratic and polynomial Lyapunov functions. More often, LMI is also employed along with Lyapunov for the stability analysis of uncertain nonlinear systems. LMI-based approach is considered more effective; since, it can treat parametric uncertainties, equality and inequality constraints and so on in a numerical tractable way (Boyd et al., 1994; Khan et al., 2011b).

It is in common practice that behaviour of more general nonlinear dynamics can be represented by linear models using local approximations. Though, many industrial applications today are based upon linear models, the conclusions taken from a linear model can be imprecise or erroneous when the stability analysis is concerned. Hence, many dynamic systems in the present world are defined using differential-algebraic equations (DAEs) in a more natural way. Furthermore, use of structure-preserving ability of differential-algebraic form proved useful in various situations (Chiang et al., 1995). The main benefits of representing the system dynamics using DAEs are to make use of the well-established linear robust control theory to control and filter design (Boyd et al., 1994; Coutinho et al., 2008, 2009). In this paper, overall vision-based control loop including
robot dynamics is described by a set of differential and algebraic nonlinear equations.

Our main accomplishment is the extension of image-based visual servo control in an uncalibrated environment (Khan et al., 2011a). In this approach, the visual servo control is established upon transpose Jacobian control. An LMI optimization scheme based upon Lyapunov theory is proposed for the estimation of composite Jacobian matrix in an uncalibrated environment without considering any kind of uncertainty involved. Thanks to this extension, the composite Jacobian matrix is determined using LMI optimization for the uncertain nonlinear system. The proposed methodology is meant for determining the stability of vision-based uncertain nonlinear system in the presence of parametric uncertainties using Lyapunov function and LMI framework by transforming the original system into differential-algebraic nonlinear system. The main contributions of the paper are in twofolds. First, the original system into differential-algebraic nonlinear system. The proposed methodology is supposed to be bounded in this control law. In order to guarantee that system dynamics is described by a set of differential and algebraic nonlinear equations.

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\( V(x) = x^T P x + x^T P x \leq -\varepsilon_1 x^T x \)  
\( \mathbf{B}_1 \subset \{ x : V(x) \leq 1 \} \subset \mathbf{B}_3 \)

For all \( x(0) \in \mathbf{B}_1 \) and \( \delta(0) \in \mathbf{B}_3 \), the trajectory \( x(t) \) approaches the origin when \( t \to \infty \) for all \( \delta(t) \in \mathbf{B}_3 \).

The main impetus of the paper is to analyse the stability of system (1) and estimate its stability region using Lemma 1 through convex optimization. Furthermore, the proposed method has to devise the scheme as how to transform the system in a suitable form for describing DAEs.

### Preliminaries

Consider that the uncertain nonlinear system in equation (1) can be rewritten using DAE and is defined as:

\[
\begin{align*}
\dot{x} &= A_1(x, \delta)x + A_2(x, \delta)p + A_3(x, \delta)u(x) \\
0 &= \Omega_1(x, \delta)x + \Omega_2(x, \delta)p + \Omega_3(x, \delta)u(x)
\end{align*}
\]

where \( \pi \in \mathbb{R}^{n_1} \) is an auxiliary vector containing all functions of \( (x, \delta) \) which are not affine on \( (x, \delta) \) and \( A_1 \in \mathbb{R}^{2n \times 2n}, A_1 \in \mathbb{R}^{2n \times 2n}, A_1 \in \mathbb{R}^{2n \times 2n}, A_1 \in \mathbb{R}^{2n \times 2n}, A_1 \in \mathbb{R}^{2n \times 2n} \).

To simplify the notation, we use \( A_1(\cdot), A_2(\cdot), A_3(\cdot), \Omega_1(\cdot), \Omega_2(\cdot) \) without their respective dependence on \( (x, \delta) \).

Similarly to the system representation in equation (5), the control law can be represented by:

\[
\begin{align*}
u(x) &= B_1(x)x + B_2(x, \xi)\phi \\
0 &= \Phi_1(x)x + \Phi_2(x, \xi, \delta)\phi
\end{align*}
\]

where \( \phi \in \mathbb{R}^{n_1} \) contains all functions of \( x \) and \( \xi \) which are not affine in \( x \), and \( B_1(x) \in \mathbb{R}^{n_3 \times 2n}, B_2(x, \xi) \in \mathbb{R}^{n_3 \times n_1}, \Phi_1(x) \in \mathbb{R}^{n_3 \times 2n}, \Phi_2(x, \xi) \in \mathbb{R}^{n_3 \times n_1} \).

The control law defined using equation (6) has the robot manipulator states \( x(t) \) as well as the feature error vector \( \xi \) is included in this control law. In order to guarantee that system defined in equations (5) and (6) are well-posed, it is assumed that:

A2. The matrices \( \Omega_2 \) and \( \Phi_2 \) in equations (5) and (6) have full column rank, means that they are well-posed.

The above assumption implies that the auxiliary variables \( \pi \) and \( \phi \) can be eliminated from equations (5) and (6) to recover the original system representation in equation (1), i.e. one can return to the original system representation by defining \( \pi \) and \( \phi \) as follows:

\[
\pi = -\Omega_2^{-1}(\Omega_1x + \Omega_3u) \quad \phi = -\Phi_2^{-1}\Phi_1x
\]

The well posedness is guaranteed if the matrix \( \Omega_2 \) and \( \Phi_2 \) are non-singular, i.e. if rank \( \Omega_2 = n_3 \) and rank \( \Phi_2 = n_1 \).

Remark 1. It must be kept in mind that the choice of matrices \( A_1, \ldots, B_2 \) in equations (5) and (6) is not unique; since, there is no defined procedure for their selection. Consequently, an improper selection of these matrices can cause poor stability region estimate or even fail to produce system stability (Huang and Lu, 1996). To be able to make system less dependent upon these

\[
V(x) = x^T P x + x^T P x \leq -\varepsilon_1 x^T x
\]

\[
\mathbf{B}_1 \subset \{ x : V(x) \leq 1 \} \subset \mathbf{B}_3
\]
matrices, a remedy is forwarded in Trofino (2000) using different approaches, i.e. to add free multipliers to the problem. This will help in reducing the potential conservativeness. This approach is further apprehended using Lemma 2.

**Stability analysis**

In this section, we develop stability conditions in terms of LMIs to the robust analysis of closed-loop nonlinear systems. Consider a quadratic Lyapunov function:

\[ V(x) = x^TPx, \quad P = P^T > 0, \quad P \in \mathbb{R}^{n \times n} \]  

(8)

The overall closed-loop system can be obtained by substituting the control law from equation (6) into equation (5):

\[ \dot{x} = A_1x + A_2v + A_3[B_1x + B_2\phi] \]  

(9)

By considering the Lyapunov function (8) and by computing its time derivative along the trajectories of equation (9), one gets:

\[ \dot{V}(x) = (x^TA_1^T + x^TB_1A_3^T + \pi^TA_2^T + \phi^TB_2A_3^T)Px + x^TP(A_1x + A_3[B_1x + A_2\pi + A_1B_2\phi]) \]  

(10)

Taking into account the DAE, equation (10) can be represented in its matrix form as:

\[ \dot{V}(x) = \begin{bmatrix} x^T \\ \pi \\ \phi \end{bmatrix} \begin{bmatrix} A_1^TP + B_1^TA_3^TP + PA_1 + PA_3B_1 & PA_2 & PA_3B_2 \\ \pi & 0 & 0 \\ \phi & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \pi \\ \phi \end{bmatrix} \]  

(11)

Notice that the vector \([x^T \quad \pi^T \quad \phi^T]^T\) is not completely free, due to the relations between the variables \(x\), \(\pi\) and \(\phi\). Such relations expressed in equations (5) and (6) can be summarized on the following equality constraint:

\[ \begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 & 0 \\ \Phi_1 & 0 & 0 & \Phi_2 \end{bmatrix} \begin{bmatrix} x \\ \pi \\ \phi \end{bmatrix} = 0 \]  

(12)

To handle the above constrained inequality, we can apply the following version of the Finsler’s lemma (Kiyama and Iwasaki, 2000).

**Lemma 2.** Given matrices \(S_i = S_i^T \in \mathbb{R}^{n \times n}\) and \(M_i \in \mathbb{R}^{n \times n}\) then \(\lambda^2S_iA > 0, \forall \lambda \in \mathbb{R}^n, MA = 0, \lambda \neq 0, i = 1, \ldots, r\) if there exists a matrix \(L \in \mathbb{R}^{n \times n}\) such that:

\[ S_i + LM_i + (LM_i)^T < 0, \quad i = 1, \ldots, r \]  

(13)

Applying the above lemma, we get the following sufficient condition for equation (11) subject to equation (12) which holds:

\[ \Psi(x, \delta) + L \Xi(x, \delta) + (L \Xi(x, \delta))^T < 0, \quad \forall (x, \delta) \in B_\alpha \times B_\delta \]  

(14)

where:

\[ \Psi(x, \delta) = \begin{bmatrix} A_1^TP + B_1^TA_3^TP + PA_1 + PA_3B_1 & PA_2 & PA_3B_2 \\ \pi & 0 & 0 \\ \phi & 0 & 0 \end{bmatrix}, \quad \Xi(x, \delta) = \begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 & 0 \\ \Phi_1 & 0 & 0 & \Phi_2 \end{bmatrix} \]  

(15)

and:

\[ \Xi(x, \delta) = \begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 & 0 \\ \Phi_1 & 0 & 0 & \Phi_2 \end{bmatrix} \]  

(16)

As the matrix inequality in equation (14) is affine in \((x, \delta)\), it can be tested only in a finite number of points. More precisely, at the vertices of the polytope \(B_\alpha \times B_\delta\).

**Stability domain**

In case, where it is difficult or impossible to ascertain the global asymptotic stability, it is useful to find an estimate of the domain of attraction (DOA). Lyapunov theory is employed to determine the stability of the origin and to estimate its DOA. Now, consider the following set as an estimate of stability region:

\[ \mathcal{R} = \{x : x^TPx \leq 1\} \]  

(17)

whose boundary forms a level surface of the Lyapunov function candidate. The condition \(x \in B_\alpha\) can be written as:

\[ 2 - x^TPx - a_kx \geq 0, \quad k = 1, \ldots, n_c \]  

(18)

where \(n_c\) is the number of edges of \(B_\alpha\). From Lemma 1, the region \(\mathcal{R} = \{x : V(x) \leq 1\}\) will be an invariant set if \(\mathcal{R}\) is constrained in \(B_\alpha\). We can recast this condition, i.e. \(\mathcal{R} \subset B_\alpha\) as indicated below:

\[ a_kx \leq 1, \quad \forall x \in B_\alpha, \quad k = 1, \ldots, n_c \]  

(19)

Applying the S-procedure (Boyd et al., 1994), the above condition \(\mathcal{R} \subset B_\alpha\), takes the form:

\[ 2(1 - a_kx) + x^TPx - 1 \geq 0, \quad \forall x \in B_\alpha, \forall k \]  

(20)

Theorem 1 presents a convex characterization of Lemma 1 (Rohr et al., 2009).

**Theorem 1.** Consider system (1) and its DAE as in equation (5). Let \(B_\alpha\) be a given polytope describing the set of allowable uncertainties set. Suppose there exists matrices \(P = P^T > 0\) and \(L\) satisfying the following LMIs at all vertices of \(B_\alpha \times B_\delta\):

\[ \Psi(x, \delta) + L \Xi(x, \delta) + (L \Xi(x, \delta))^T < 0 \]  

(21)

where:

\[ \Psi(x, \delta) = \begin{bmatrix} A_1^TP + B_1^TA_3^TP + PA_1 + PA_3B_1 & PA_2 & PA_3B_2 \\ \pi & 0 & 0 \\ \phi & 0 & 0 \end{bmatrix} \]  

and:

\[ \Xi(x, \delta) = \begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 & 0 \\ \Phi_1 & 0 & 0 & \Phi_2 \end{bmatrix} \]  

(16)

Then \(V(x) = x^TPx\) is a Lyapunov function in \(B_\alpha\). Moreover, for all \(x(0) \in \mathcal{R}\) and \(\delta(t) \in B_\delta\), the trajectory \(x(t)\) approaches the origin when \(t \to \infty\), where \(\mathcal{R} = \{x : V(x) \leq 1\}\).

Proof. If the LMIs (21) and (22) are feasible, then by convexity they are also satisfied for all \((x, \delta) \in B_\alpha \times B_\delta\). Let \(\varepsilon_1\) and \(\varepsilon_2\) be, respectively, the smallest and largest eigenvalues of \(P\). Then, the following holds using Lemma 1:
\[ e_1 x^T x \leq x^T P x \leq x^T e_2 \]

Define the vector \( \psi := [x^T \quad \pi^T \quad u^T \quad \phi^T]^T \). Pre- and post-multiplying the LMI in equation (21) by \( \psi^T \) and \( \psi \), respectively, leads to:

\[
\psi^T \Psi \psi + \psi^T L \Xi \psi + (\psi^T L \Xi \psi)^T < 0
\]  

(23)

An interesting fact is that the product \( \Xi \psi \) results in a vector of zeros, leaving equation (23) only with \( \psi^T \Psi \psi < 0 \), which equals to \( V(x) \). As the elements of \( \Psi \) and \( \phi \) are bounded, there exists a sufficiently small positive scalar \( e_3 \), such that:

\[ V(x) \leq -e_3 x^T x \]

Pre- and post-multiplying equation (22) by \([1 \quad -x^T]\) and its transpose leads to:

\[
\begin{bmatrix}
1 & a_k \\
-x & P
\end{bmatrix}
\begin{bmatrix}
1 \\
-x
\end{bmatrix} \succeq 0, \quad \forall k
\]  

(24)

Then, \( \mathcal{R} \) is a positively invariant set, i.e. for all \( x(0) \in \mathcal{R} \) the trajectory \( x(t) \in \mathcal{R} \) approaches the origin as \( t \to \infty \).

**Remark 2.** From Theorem 1, a symmetric estimation of the DOA will be generated, due to the fact the LMI defined in equation (22) is tested for each specific \( a_k \) at all vertices of \( B_2 \). To be able to achieve a non-symmetric estimation from equation (22) we have to consider, for each \( a_k \), only the vertices of \( B_2 \) that belongs to the edge that corresponds to \( a_k \), i.e. the hyperplane \( \{x : a_k x = 1\} \).

**Optimization issues**

An estimate \( \mathcal{R} \) of the system DOA is achieved using Theorem 1, but we are mostly interested in finding the largest estimate inside \( B_2 \). To this end, we may solve the following convex optimization problem:

\[
\min_{P} \quad \text{trace}(P) \quad \text{s.t.} \quad \Psi(x, \delta) + L \Xi(x, \delta) + (L \Xi(x, \delta))^T < 0
\]

and

\[
\begin{bmatrix}
1 & a_k \\
\star & P
\end{bmatrix} \succeq 0, \quad \forall k
\]

The objective function corresponds to the minimization of the sum of the squared semi-axis lengths of the ellipsoid \( \mathcal{R} \).

**Computational issues**

The LMI defined using equation (21) involves matrix \( L \). For the two-link robotic manipulator case the size of the matrix \( L \) is \( 27 \times 21 \). Therefore, to reduce the computational cost and complexity, the variable \( L \) is eliminated using Finsler’s lemma as:

\[
\begin{cases}
\Psi - \sigma \Xi^T \Xi < 0 \\
\Psi - \sigma I < 0
\end{cases}
\]  

(25)

We can certainly overlook the second inequality, because if we find \( \sigma \) satisfying the first inequality, we can always find one which satisfies the second.

**Case study: a two-link robot manipulator**

In this section, the controller scheme developed in the previous section is tested on an academic example and is simulated using MATLAB 7.5 with Simulink, CVX Toolbox and Video and Image Processing Blockset (Grant and Boyd, 2009). Figure 1 shows the scheme of the 2-degree-of-freedom (DOF) planar robot considered. It is made up of two links and two revolute joints. Each link is characterized by the following parameters: mass \( (m_i) \), length \( (l_i) \), mass center position \( (d_i) \), and inertia \( (I_i) \), where \( i = 1, 2 \). Table I gives the robot parameters involved in deriving the dynamics, and are also employed for controller synthesis.

**System description/nonlinear mechanical control system**

The following Euler-Lagrange equation of motion is used to describe the behaviour of a \( n \)-DOF robotic manipulator (Lewis et al., 2004):

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + g(q) = \tau
\]  

(26)

where \( q = [q_1, \ldots, q_n]^T \in \mathbb{R}^n \) represents joint variables (joint positions) and \( \dot{q} \in \mathbb{R}^n \) is the time derivative (joint speeds). Vector \( \tau \in \mathbb{R}^n \) (generalized torques applied on the joint axes) is the torque input vector. \( M(q) \in \mathbb{R}^{n \times n} \) represents inertia associated with the distribution of mass which is also symmetric and positive definite. \( C(q, \dot{q}) \in \mathbb{R}^{n \times n} \) represents interaxis velocity coupling due to centrifugal and Coriolis forces and \( g(q) \in \mathbb{R}^n \) denotes the vector of gravitational forces.

**Vision-based control law**

The control input \( \tau \) fed to the manipulator is adopted from Khan et al. (2011a):

**Figure 1 Scheme of the 2-DOF robot arm**

<table>
<thead>
<tr>
<th>Table I Nominal parameters of two-link manipulator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td>Lengths</td>
</tr>
<tr>
<td>Position of mass center</td>
</tr>
<tr>
<td>Masses</td>
</tr>
<tr>
<td>Inertia moments</td>
</tr>
</tbody>
</table>
\[ \tau = -M J^T \dot{q} K \xi + g \]  

(27)

where \( K \in \mathbb{R}^{n \times n} \) is a diagonal positive-definite proportional matrix and \( k \) represents the number of feature points extracted from the image. \( J^T \) is calculated using the LMI optimization algorithm fulfilling constraints as well as satisfying performance criterion.

The system behavior can be written in terms of the state vector \([q^T \quad q^T] \in \mathbb{R}^{2n}\) using equations (26) and (27) as (Khan et al., 2011a):

\[
\frac{d}{dt} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = M^{-1} \left(-M J^T K \xi - C \dot{q} \right)
\]

(28)

The assumption made here is that there exists a robot desired joint configuration \( \xi_d \) for which the feature error vanishes, i.e. \( \xi = \xi_d(q_d) \). Therefore, we have \([q^T \quad q^T] = [q_d^T \quad 0]^T\) as an equilibrium point.

Through the tangent half-angle formulae, trigonometric nonlinearities can be embedded into the presentation (5) with no conservatism as applied for instance in Danes and Bellot (2006) for robotic systems. The basic idea is to apply the change of variable:

\[ q = 2 \arctan(r) \]  

(29)

In other words, trigonometric functions on the variable \( q \) can be transformed into rational ones in the variable \( r \).

The change of variables (29) is used to transform the trigonometric nonlinearities in rational ones, obtaining the equivalent model:

\[ F = 2r(I + r^2)^{-1}r^2 - 0.5J^T K \xi(I + r^2) - M^{-1}C r \]  

(30)

Defining \( x_1 = r \) and \( x_2 = \dot{r} \), the system (30) can be represented in the state-space form:

\[
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = 2x_1x_2^T x_2 (I + x_2^T x_1)^{-1} - 0.5J^T K \xi(I + x_2^T x_1) - M^{-1}C x_2
\]

(31)

Let us consider that the uncertain terms \((I + \delta_1)\) and \((I + \delta_2)\) are, respectively, associated to the parameters \( M \) and \( C \) as follows:

\[
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = 2x_1x_2^T x_2 (I + x_2^T x_1)^{-1} - 0.5J^T K \xi(I + x_2^T x_1) - M^{-1}(I + \delta_1)^{-1}C(I + \delta_1)x_2
\]

(32)

The system can be represented in its differential-algebraic form equation (5) by defining:

\[
\phi = \begin{bmatrix} x_1^T x_2 (I + x_2^T x_1)^{-1} \\ x_2 (I + x_2^T x_1)^{-1} \\ x_1 (I + x_2^T x_1)^{-1} \\ x_1^T (I + x_2^T x_1)^{-1} \\ (I + \delta_1)^{-1} x_2 \end{bmatrix}
\]

The state-space defined in equation (32) is in the closed-loop form. The control law terms included in this state-space has the form:

\[
u = \begin{bmatrix} 0 \\ -0.5J^T K \xi(I + x_2^T x_1) \end{bmatrix}
\]

which can be represented in a form equivalent to equation (6), by defining:

\[
\dot{\xi} = \begin{bmatrix} \xi \\ \xi^T \xi \\ 1_{1 \times 1} \\ x_1 \\ x_1^T x_1 \end{bmatrix}
\]

and:

\[
B_1 = \begin{bmatrix} 0_{n \times n} & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} -0.5J^T K \xi & 0_{n \times n} & 0_{n \times n} & -0.5J^T K \xi \end{bmatrix}
\]
unknown variables, i.e. $P$ and $Q$. Once $P$ is determined the Jacobian matrix can be recovered from $Q$ using the inverse of $P$. The plot for feature error norm can be shown through Figure 2. It can be examined easily that the initial error norm is quite large in magnitude. But the visual servo controller manages to drag down the error norm to almost 33 per cent of its initial value only after one iteration. The error norm reaches less than one after eight iterations. Figure 3 shows the features error plot. The controller manages to converge the feature errors to zero at a very quick rate. Also, both the error plots exhibit exponential stability as expected. The trajectories made by features between initial and final points are plotted in Figure 4. The gains used for positive diagonal matrix $K_p$ are $(20,20,30,15,25,45,40,10)$ considered for the case.

Let the state-space domain is bounded by the following parameterized set $B_\varepsilon$: as:

$$B_\varepsilon := \{ x \in \mathbb{R}^2 : |x_1| \leq 0.15, |x_2| \leq 0.15 \}$$

**Figure 2** Error norm (pixels) plot for features

**Figure 3** Feature errors trajectory (pixels) plot
The bounds for the uncertainties $\delta_1$ and $\delta_2$ can be determined by performing the gridding search in the uncertainties $\delta_1$ and $\delta_2$. Following are the bounds that are obtained for the uncertain parameters:

\[ |\delta_1| \leq 0.097 \quad \text{and} \quad |\delta_2| \leq 0.99. \]

Figures 5 and 6 show the polytope of admissible states $B_x$, the estimate of the DOA $\theta$ and the stable system trajectories for $q$ and $\dot{q}$. $B_x$ is the state polytope which holds the allowable values of the states. Several simulations have been performed for a set of initial conditions and at the extremum values of $\delta_1$ and $\delta_2$, that leads to four trajectories for each initial conditions. These initial conditions are chosen based upon the selection of 20 different points (almost equidistant) on the boundary. The multiple trajectories originated as a result, which are plotted in Figures 5-8. All stable trajectories must converge to the origin, if the trajectories tend to move away from the origin or outside the stability region, it will be considered as an unstable trajectory.

The initial state is bound to start from inside the region, i.e. $x(0) \in B_x$. By looking at the Figures 5 and 6, we can easily
deduce that the trajectory $x(t)$ in both the cases approaches the origin as $t \to \infty$. Conservativeness test is also applied by increasing the bound on $\delta_1$ by determining that whether the system trajectories converge to the equilibrium point or not. Through the test, it is revealed that for $\delta_1 > 0.11$ all the system trajectories do not tend towards the equilibrium point under analysis as shown through Figures 7 and 8, where the dotted lines represent unstable trajectories. It can be witnessed that the uncertainty level is raised by 14 per cent then its initial bound, means that the robust stability of the system has improved.

**Conclusion**

This paper has presented an LMI-based approach to the stability analysis of vision-based control loop in the presence of parametric uncertainties. Through a DAE, we devise LMI conditions that ensure the stability of the uncertain system while providing an estimate of the DOA based upon quadratic Lyapunov functions. The vision-based control includes composite Jacobian matrix which is calculated using LMI optimization while ensuring stability. Simulation results for 2-DOF robotic arm using eight feature points were presented to illustrate the stability of an uncertain nonlinear system.

**References**


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