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Approximate solutions of Schrodinger equation and expectation values of Inversely Quadratic Hellmann-Kratzer (IQHK) potential

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Abstract The study presents approximate solutions of Schrodinger equation with the Inversely quadratic Hellmann-Kratzer (IQHK) potential. The energy eigenvalues and corresponding wavefunctions are obtained analytically using the Nikiforov-Uvarov (NU) method. The expectation values of inverse position r^{-1}, square of inverse position, r^{-2}, kinetic energy, *T*, potential energy, *V*, and square of momentum, p^2, and their respective numerical values are evaluated via the Hellmann–Feynman theorem. Special cases of IQHK potential are also reported.

1 Introduction

The exact solution of Schrodinger equation and other relativistic wave equations for known solvable potentials are very essential in many fields of physics since they contain vital information corresponding to any system of interest. However, exact solutions of Schrodinger equation have only been established in rare cases [1–3]. Recently, some authors have tried to solve Schrodinger equation with different potentials [4–8], using different methods such as Nikiforov-Uvarov (NU) method [9], factorization method [3], Ansatz method [10], supersymmetry technique [11–13], Formular method [14], etc.

The interaction of two or more potentials in describing a physical system is interesting due to increased applications. Recently, some studies on combination of different potentials have been reported under the Schrodinger equation [15–19]. As a motivation, in the present study, a superposition of the Inversely quadratic Hellmann and Kratzer (IQHK) potentials is studied within the framework of non-relativistic Schrodinger equation. The proposed IQHK potential is of the form

$$V(r) = -\frac{A}{r} + \frac{Be^{-\alpha r}}{r^2} - 2D_e\left(\frac{r_e}{r} - \frac{1}{2}\frac{r_e^2}{r^2}\right),\ r \in [0, \infty), \tag{1}$$

where A and B are respectively the strengths of the coulomb and the Yukawa potentials and are connected to the height of the potential. α, D_e, r_e, r are respectively, the screening parameter, dissociation energy, equilibrium internuclear distance, and internuclear separation. The screening parameter accounts for nuclear screening or shielding within a multi-electron system. The dissociation energy is the energy that is required to break a bond and form two atomic or molecular daughter particles having an electron of the original shared pair each. It is simply a measure of the strength of a bond. Descriptively, the dissociation energy is the vertical separation between the limit of dissociation and the minimum point on a potential curve and it coincides with $r = r_e$. The equilibrium internuclear distance is the internuclear distance at which the potential energy has a minimum value and it defines the equilibrium length of a bond. It should be noted that A and α can only be positive, while B can be positive or negative. Very recently, the inversely quadratic Hellmann potential was applied to the study of the optical properties of GaAs quantum dots [20]. The Kratzer potential itself has been studied in various forms [21–23]. The potential models discussed in this work are useful in some branches of physics such as chemical physics, atomic physics, and nuclear physics. Therefore, it is important to find solutions to Schrödinger equation for a particle under such potentials.

In quantum mechanics, the expectation values of quantum observables are very fundamental in that they can be applied to predict the most probable position of an electron confined in a nucleus in a range of distance [24]. Several authors have studied the expectation values of different systems [25–30].

With reference to previous studies on the expectation values of various potential models and the potential application of the proposed IQHK potential, we are therefore motivated in this work to apply the Hellmann–Feynman theorem to the eigenvalues of IQHK potential and obtain the expectation values. Thus, the objectives of this work are two-fold: first to obtain bound state solutions of the Schrodinger equation for the IQHK potential and then to determine the expectation values of the potential using the Hellman–Feynman theorem.

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2 Parametric NU method

The equation for the parametric NU method in its parametric form reads [31]

$$\frac{d^2\psi(s)}{ds^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{d\psi(s)}{ds}+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{s^2(1-\alpha_3 s)^2}\psi(s)=0,\ s\in[0,\infty) \tag{2}$$

α_1, ..., α_n and ξ_1, ξ_2, ξ_3 are real parametric constants and $\psi(s)$ is the wave function. For $\psi(s)$ to be an acceptable wavefunction, it must be single-valued, continuous and vanish at infinite distances.

The energy eigenvalues equation and eigenfunctions respectively satisfy the following equations

$$(\alpha_2-\alpha_3)n+\alpha_3 n^2-(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0, \tag{3}$$

$$\psi(s)\ =\ N_{nl}\,s^{\alpha_{12}}(1-\ \alpha_3 s)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)}\ P_n^{\left(\alpha_{10}-1,\ \frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1\ -\ 2\alpha_3 s), \tag{4}$$

where

$$\left.\begin{aligned}&\alpha_4=\frac{1}{2}(1-\alpha_1),\ \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),\ \alpha_6=\alpha_5^2+\xi_1,\ \alpha_7=2\alpha_4\alpha_5-\xi_2,\\&\alpha_8=\alpha_4^2+\xi_3,\ \alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6,\ \alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8},\\&\alpha_{11}=\alpha_2-2\alpha_5+2\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right),\ \alpha_{12}=\alpha_4+\sqrt{\alpha_8},\ \alpha_{13}=\alpha_5-\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right).\end{aligned}\right\}. \tag{5}$$

$P_n^{(a,b)}$ are the orthogonal Jacobi polynomial.

3 Solution of Schrodinger equation with IQHK potential

To obtain the energy eigenvalues of the IQHK potential, we consider Schrodinger equation of the form:

$$\frac{d^2R_{nl}(r)}{dr^2}+\frac{2}{r}\frac{dR_{nl}(r)}{dr}\left[\frac{2\mu}{\hbar^2}(E_{nl}-V(r))-\frac{l(l+1)}{r^2}\right]R_{nl}(r)=0. \tag{6}$$

The parameters in Eq. (6) are defined as follows: R_{nl} is the radial wave function, μ is the mass of the system, *V(r)* is the potential function, E_{nl} is the energy, $\hbar$ is the Planck's constant, *n* is the radial quantum number, and *l* is the orbital angular momentum quantum number.

If we expand the potential in Eq. (1) by Taylor series and neglect terms above r^2 (valid for small values of α), the potential in Eq. (1) becomes

$$V(r)=\left(-\frac{A}{r}-\frac{\alpha B}{r}-\frac{2D_e r_e}{r}\right)+\left(\frac{B}{r^2}+\frac{D_e r_e^2}{r^2}\right)+\frac{\alpha^2 B}{2}. \tag{7}$$

The potential in Eq. (7) behaves properly at the limits of r, i.e., $V_{r\to 0}(r)=\infty$ due to internuclear repulsion and $V_{r\to\infty}$ tends to a constant $\left(\frac{\alpha^2 B}{2}\right)$. The potential has an attractive long range part and a repulsive part as shown in Fig. 1. The terms proportional to $\left(\frac{1}{r}\right)$ is attributed to the coulombic part of the potential and the terms in $\left(\frac{1}{r^2}\right)$ are attributed to the electronic kinetic energy.

Substituting Eq. (7) in (6) and simplifying gives

$$\frac{d^2R_{nl}(r)}{dr^2}+\frac{2}{r}\frac{dR_{nl}(r)}{dr}+\frac{-Qr^2+Xr-T}{r^2}R_{nl}(r)=0, \tag{8}$$

where

$$Q=\frac{-2\mu}{\hbar^2}\left(E_{nl}-\frac{\alpha^2 B}{2}\right);\ X=\frac{2\mu}{\hbar^2}(A+\alpha B+2D_e r_e);\ T=\frac{2\mu}{\hbar^2}\left(B+D_e r_e^2\right)+l(l+1). \tag{9}$$

Comparing Eqs. (8) and (2), we obtain the following:

$$\left.\begin{aligned}&\alpha_1=2,\ \alpha_2=\alpha_3=0,\ \alpha_4=-\tfrac{1}{2},\ \alpha_5=0,\ \alpha_6=Q,\ \alpha_7=-X,\alpha_8=\tfrac{1}{4}+T\\&\alpha_9=Q,\ \alpha_{10}=1+2\sqrt{\tfrac{1}{4}+T},\ \alpha_{11}=2\sqrt{Q},\ \alpha_{12}=-\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+T},\ \alpha_{13}=\sqrt{Q}\end{aligned}\right\}. \tag{10}$$

Substituting the values of Eq. (10) into Eqs. (3) and (4), the energy eigenvalues and corresponding wave function becomes

$$E_{nl}=\frac{\alpha^2 B}{2}-\frac{\hbar^2}{2\mu}\left(\frac{\frac{\mu}{\hbar^2}(A+\alpha B+2D_e r_e)}{\frac{1}{2}+n+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu}{\hbar^2}\left(B+D_e r_e^2\right)}}\right)^2, \tag{11}$$

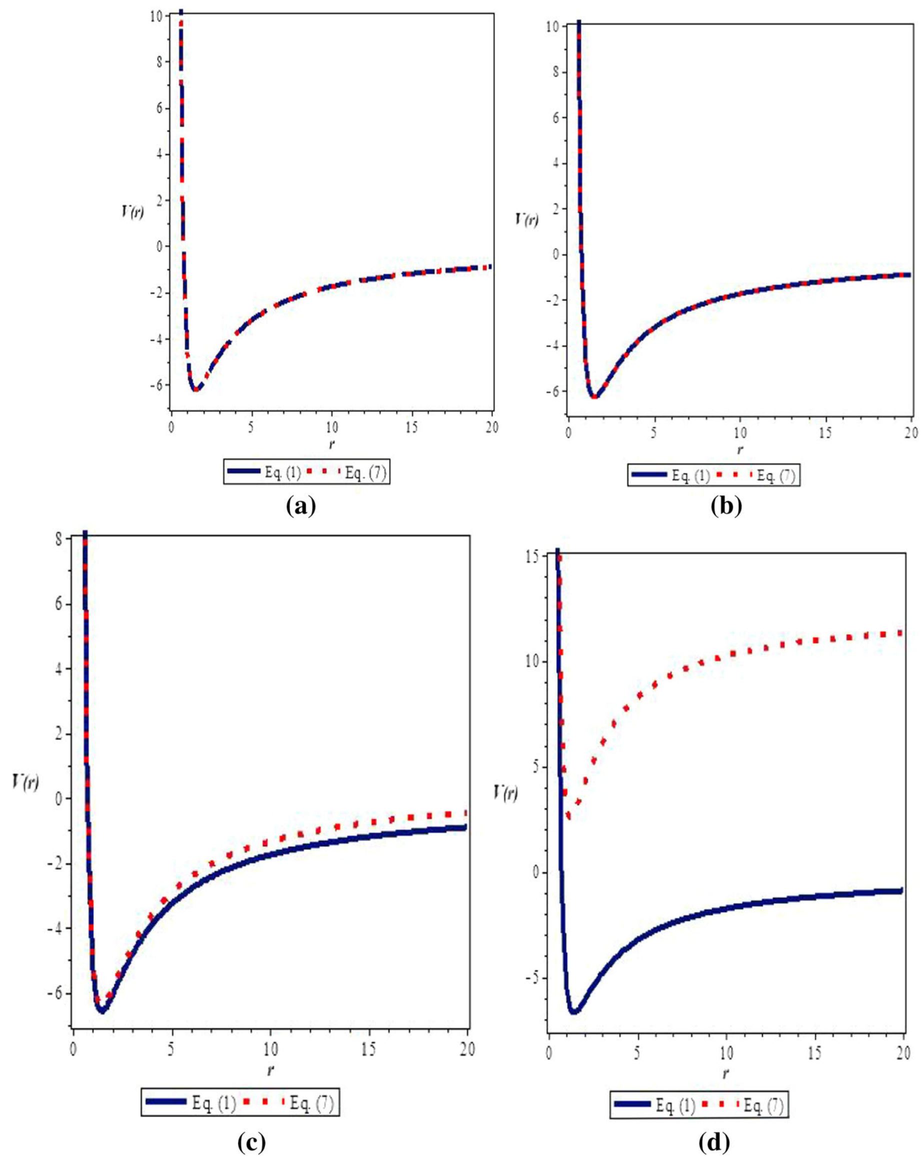

Fig. 1 Variation of the Inversely quadratic Hellmann-Kratzer potential, Kratzer potential and the Inversely quadratic Hellmann potential with internuclear distance: **a** $\alpha = 0.01$, **b** $\alpha = 0.1$, **c** $\alpha = 1$, and **d** $\alpha = 5$

and

$$R_{nl}(r) = N_{nl}\, r^{\chi} e^{-r\sqrt{Q}} L_n^{2\chi+1}\left(2\sqrt{Q}r\right), \tag{12}$$

where

$$\chi = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4T}.$$

The Normalization constant is obtained as follows

$$N_{nl} = \sqrt{\frac{\left(2\sqrt{-Q}\right)^{2c_4+2}}{\sum_{i=0}^{n} \frac{\Gamma(2c_4+3+i)}{\Gamma(1+i)(\Gamma(2-n+i)\Gamma(n-i+1))^2}}}. \tag{13}$$

The wave function is well behaved and vanishes at the limits of r, i.e., $R_{nl}(r) \to 0$ as $r \to 0$ and $r \to \infty$.

3.1 Special cases of IQHK potential

After studying the nonrelativistic solutions of IQHK potential, we examine some cases:

Case 1*:* The first case corresponds to the Kratzer potential *($A = B = 0$)*:

$$V(r) = -2D_e\left(\frac{r_e}{r} - \frac{1}{2}\frac{r_e^2}{r^2}\right). \tag{14}$$

The energy Eq. (11) reduces to that of Kratzer potential

$$E_{nl} = -\frac{2\mu}{\hbar^2}D_e^2 r_e^2\left(\frac{1}{2} + n + \sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2\mu}{\hbar^2}D_e r_e^2}\right)^{-2}. \tag{15}$$

Equation (15) is identical to Eq. (18) in ref. [32], and Eq. (115) in ref. [33] when $l = L$. If we set $r_e =$ A, $B = {r_e}^2$ and $\sigma = \frac{1}{2}\left(1 + \sqrt{(1+2l)^2 + \frac{8\mu}{\hbar^2}D_e B}\right)$, then Eq. (15) will be exactly the same as Eq. (46) in ref. [34].

Case 2*:* Case two is the Inversely quadratic Hellman potential ($D_e = 0$):

$$V(r) = -\frac{A}{r} + \frac{Be^{-\alpha r}}{r^2}. \tag{16}$$

Here, we obtain the energy spectrum as:

$$E_{nl} = \frac{\alpha^2 B}{2} - \frac{\hbar^2}{2\mu}\left(\frac{\frac{\mu}{\hbar^2}(A + \alpha B)}{\frac{1}{2} + n + \sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2\mu B}{\hbar^2}}}\right)^2. \tag{17}$$

Equation (17) is the same as Eq. (16) in ref. [35]. Equation (17) is also similar to Eq. (34) in Ref. [36] under the condition $\frac{\alpha^2}{2} = \delta^2$, $B = b$, $A = a$.

Case 3: The third case is the Inversely quadratic Yukawa potential ($A = 0$, $D_e = 0$):

$$V(r) = \frac{Be^{-\alpha r}}{r^2}. \tag{18}$$

The corresponding energy equation is obtained as

$$E_{nl} = \frac{\alpha^2 B}{2} - \frac{\hbar^2}{2\mu}\left(\frac{\frac{\mu\alpha B}{\hbar^2}}{\frac{1}{2} + n + \sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2\mu B}{\hbar^2}}}\right)^2. \tag{19}$$

Equation (19) is consistent with Eq. (33) in ref. [36] under the condition $\frac{\alpha^2}{2} = \delta^2$, $B = -2V_0$.

Case 4: The fourth case is the coulomb potential ($D_e = B = 0$):

$$V(r) = -\frac{A}{r}. \tag{20}$$

And the corresponding energy spectrum as

$$E_{nl} = -\frac{\mu A^2}{2\hbar^2(n + l + 1)^2} \tag{21}$$

Equation (21) is in agreement with Eqs. (36), (50), (161) and (61) in refs. [33, 34, 37, 38], respectively.

4 Expectation values of the IQHK potential

Using the Hellmann–Feynman theorem [39, 40], the expectation values of the IQHK potential are obtained. The theorem states that the derivative of the energy of a system with respect to a parameter, υ, is equal to the expectation value of the derivative of the Hamiltonian of the system with respect to the same parameter. The equation governing the Hellmann–Feynman theorem is given by

$$\frac{\partial E_{nl}(\upsilon)}{\partial \upsilon} = \left\langle \psi_{nl}\right|\frac{\partial H(\upsilon)}{\partial \upsilon}\left|\psi_{nl}\right\rangle \tag{22}$$

The effective Hamiltonian for the IQHK potential is

$$H = \frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2\mu r^2} - \frac{A}{r} + \frac{Be^{-\alpha r}}{r^2} - 2D_e\left(\frac{r_e}{r} - \frac{1}{2}\frac{r_e^2}{r^2}\right). \tag{23}$$

By setting υ to be *l, A,* and μ, the expectation values for r^{-2}, r^{-1} and T, respectively are obtained. The expectation value p^2 is obtained using the formula $\langle T\rangle = \frac{\langle p^2\rangle}{2\mu}$. The expectation value of V is obtained from $\langle V\rangle = \langle H\rangle - \langle T\rangle$.

Expectation value of r^{-2}

$$\langle r^{-2}\rangle = \frac{\Omega^2}{\left(\frac{1}{2}+n+\sqrt{\frac{1}{4}-P}\right)^3\sqrt{\frac{1}{4}-P}} \tag{24}$$

Expectation value of r^{-1}

$$\langle r^{-1}\rangle = \frac{\Omega}{\left(\frac{1}{2}+n+\sqrt{\frac{1}{4}-P}\right)^2} \tag{25}$$

Expectation value of T

$$\langle T\rangle = \frac{\hbar^2\Omega^2}{2\mu\left(\frac{1}{2}+n+\sqrt{\frac{1}{4}-P}\right)^2} - \frac{\Omega\left(D_e r_e^2+B\right)}{\left(\frac{1}{2}+n+\sqrt{\frac{1}{4}-P}\right)^3\sqrt{\frac{1}{4}-P}} \tag{26}$$

Expectation value of p^2

$$\langle p^2\rangle = \frac{\hbar^2\Omega^2}{\left(\frac{1}{2}+n+\sqrt{\frac{1}{4}-P}\right)^2} - \frac{2\mu\Omega\left(D_e r_e^2+B\right)}{\left(\frac{1}{2}+n+\sqrt{\frac{1}{4}-P}\right)^3\sqrt{\frac{1}{4}-P}}. \tag{27}$$

Expectation value of V

$$\langle V\rangle = \gamma - \frac{\hbar^2\Omega^2}{\mu\left(\frac{1}{2}+n+\sqrt{\frac{1}{4}-P}\right)^2} + \frac{\Omega^2\left(\frac{(D_e r_e^2+B)}{\sqrt{\frac{1}{4}-P}} - \frac{\hbar^2}{2}\left(\frac{1}{2}+n+\sqrt{\frac{1}{4}-P}\right)\right)}{\mu\left(\frac{1}{2}+n+\sqrt{\frac{1}{4}-P}\right)^3}. \tag{28}$$

5 Discussion of results

Figure 1 is a plot of IQHK potential as given in Eq. (1) and Eq. (7) versus internuclear distance, r, are compared for various values of α. The shape of the IQHK potential is more similar to the Kratzer potential than the Hellman potential. The implication of this is that a particle confined within the IQHK potential vibrates more within the field of the Kratzer potential than the Hellman potential due to the fact that the effect of the Kratzer potential is more prominent than that of the Hellman potential. As stated before, the potential has an attractive and repulsive part and tends to a constant value as r increases, but approaches a constant value as r approaches zero. From Fig. 1a–d it is obvious that the approximation in Eq. (7) only holds true for small values of the screening parameter, α. The plots were made using the parameters $\alpha = 0.01$, $D_e = 6$, $A = 1$, $B = 1$, $r_e = 1.5$.

In Fig. 2, the variation of the energy of the IQHK potential with the adjustable screening parameter, α, is shown for values of quantum numbers and. The energy is seen to increase as α increases. This implies that electrons in higher energy states will

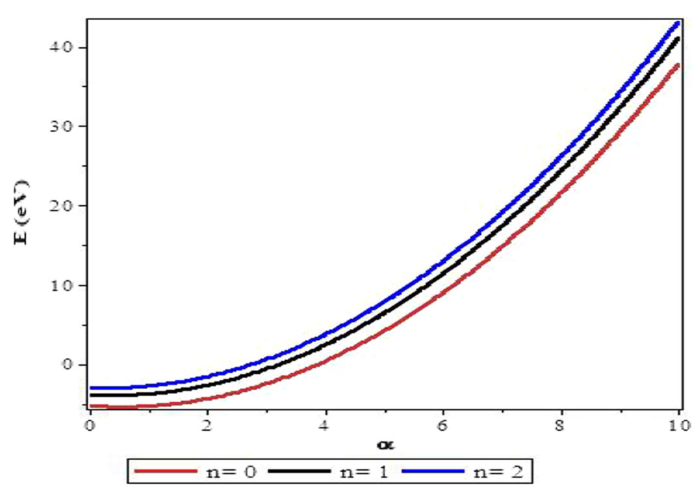

Fig. 2 Variation of the energy of the IQHK potential with the screening parameter, α

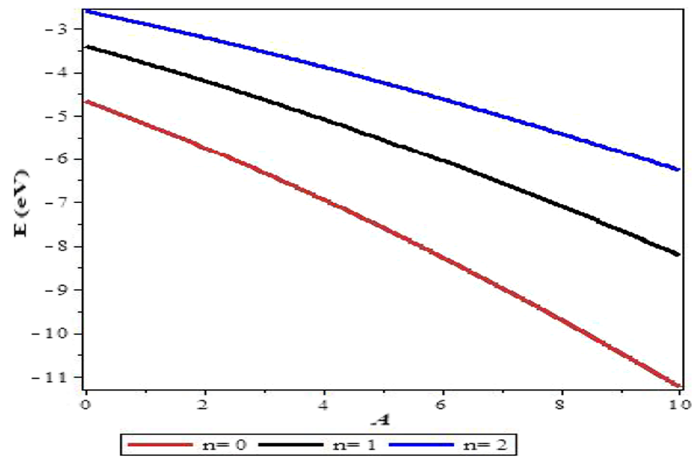

Fig. 3 Variation of the energy of the IQHK potential with the potential parameter, A

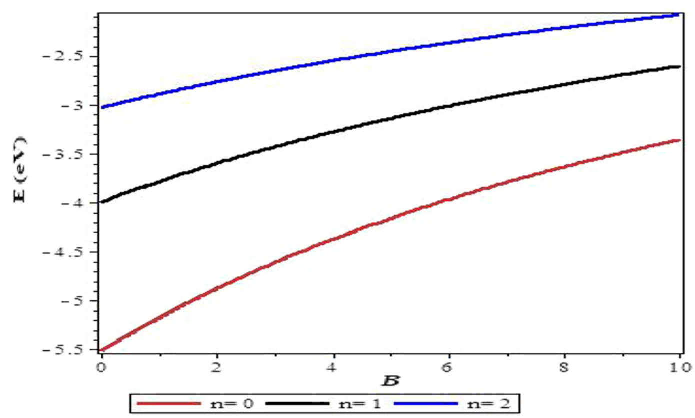

Fig. 4 Variation of the energy of the IQHK potential with the potential parameter, B

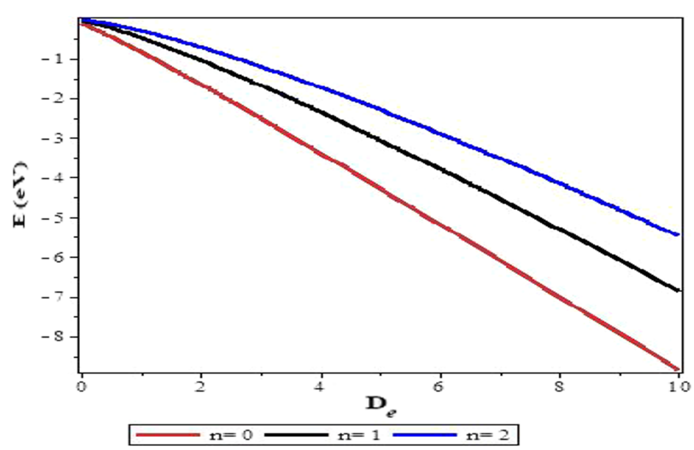

Fig. 5 Variation of the energy of the IQHK potential with the dissociation energy, D_e

experience more shielding than those in the lower states. In Fig. 3, the energy is observed to decrease linearly with an increase in the potential parameter, A. The reverse trend is observed in Fig. 4, as energy increases with parameter, B. Figure 5 shows the variation of energy with the dissociation energy, D_e. It can be observed that the energy decreases with an increase in dissociation energy, D_e. The plots were made with $l = 0$, $D_e = 6$, $A = 1$, $B = 1$, $r_e = 1.5$.

Table 1 shows results of the IQHK potential for some values of screening parameter, α. In all the cases, the energy is observed to increase with the quantum numbers n and l. This is a direct verification of the assertion made from Fig. 2. Table 2 presents the energy eigenvalues of IQH potential. The energy is observed to increase with quantum numbers n and l for all values of α. Table 3 presents the results of the energy of the inversely quadratic Yukawa potential for arbitrary.

quantum numbers n and l for $\alpha = 0.01$, 0.05, and 0.1. The energy is also observed to increase as n and l increase. In Table 4, the numerical values of the expectation values of the IQHK potential for arbitrary quantum numbers n and l are presented. $\langle r^{-2} \rangle$ and $\langle r^{-1} \rangle$ are observed to increase as n and l increase. $\langle T \rangle$ and $\langle V \rangle$ are observed to increase as l increases, but decrease as n increases. This

Table 1 Energy eigenvalues of the inversely quadratic Hellman-Kratzer (IQHK) potential with $D_e = 6$; $r_e = 1.5$; $A = 1$, $B = 2$; $\mu = \hbar = 1$

n	l	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
0	0	− 4.876669242	− 4.915379866	− 4.959510796
1	0	− 3.598037370	− 3.625969268	− 3.656563172
2	0	− 2.763503540	− 2.666916735	− 2.806159008
	1	− 2.647001960	− 2.784400411	− 2.687441916
3	0	− 2.188906365	− 2.204959446	− 2.220634714
	1	− 2.106490249	− 2.121848571	− 2.136651290
	2	− 1.961236202	− 1.975370048	− 1.988634946
4	0	− 1.776518042	− 1.789094734	− 1.800404000
	1	− 1.716087403	− 1.728154672	− 1.738824150
	2	− 1.608732197	− 1.619894474	− 1.629427370
	3	− 1.474207334	− 1.484235581	− 1.492344246
5	0	− 1.470567948	− 1.480565515	− 1.488635650
	1	− 1.424949166	− 1.434562172	− 1.442149336
	2	− 1.343375184	− 1.352300531	− 1.359024061
	3	− 1.240154422	− 1.248209630	− 1.253840350
	4	− 1.128651574	− 1.135766824	− 1.140217050

Table 2 Energy eigenvalues of the inversely quadratic Hellman (IQH) potential with; $A = -1$, $B = 1$, $\mu = \hbar = 1$

n	l	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
0	0	− 0.12246250000	− 0.1115625000	− 0.09625000000
1	0	− 0.05440000000	− 0.04888888890	− 0.04000000000
2	0	− 0.03057812500	− 0.02695312500	− 0.02031250000
	1	− 0.02350130510	− 0.02043661652	− 0.01446388851
3	0	− 0.01955200000	− 0.01680000000	− 0.01120000000
	1	− 0.01579339544	− 0.01333898520	− 0.008093715240
	2	− 0.01201842267	− 0.009862898140	− 0.004973903035
4	0	− 0.01356250000	− 0.01128472222	− 0.006250000000
	1	− 0.01133222420	− 0.009231029840	− 0.004406796865
	2	− 0.008966471095	− 0.007052586640	− 0.002451629005
	3	− 0.007106704375	− 0.005340068050	− 0.000914631715
5	0	− 0.009951020410	− 0.007959183675	− 0.003265306120
	1	− 0.008520741840	− 0.006642148265	− 0.002083257720
	2	− 0.006941216765	− 0.005187683020	− 0.000777865095
	3	− 0.005646660125	− 0.003995623675	0.000292016424
	4	− 0.004644554752	− 0.003072860589	0.001120202685

Table 3 Energy eigenvalues of the Inversely quadratic Yukawa (IQY) potential with $B = 0.5$; $\mu = \hbar = 1$

n	l	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
0	0	0.00002022542486	0.0005056356216	0.002022542486
1	0	0.00002317627458	0.0005794068645	0.002317627458
2	0	0.00002404508497	0.0006011271243	0.002404508497
	1	0.00002432483082	0.0006081207704	0.002432483082
3	0	0.00002441386755	0.0006103466888	0.002441386755
	1	0.00002455546751	0.0006138866878	0.002455546751
	2	0.00002467403841	0.0006168509603	0.002467403841
4	0	0.00002460395796	0.0006150989490	0.002460395796
	1	0.00002468533654	0.0006171334134	0.002468533654
	2	0.00002475837586	0.0006189593965	0.002475837586
	3	0.00002481135064	0.0006202837660	0.002481135064
5	0	0.00002471460128	0.0006178650319	0.002471460128
	1	0.00002476561269	0.0006191403172	0.002476561269
	2	0.00002481376197	0.0006203440492	0.002481376197
	3	0.00002485037216	0.0006212593041	0.002485037216
	4	0.00002487769977	0.0006219424943	0.002487769977

Table 4 Expectation values of the Inversely quadratic Hellman-Kratzer (IQHK) potential with $D_e = 6$; $r_e = 1.5$; $A = 1$, $B = 1$; $\alpha = 0.01$; $\mu = \hbar = 1$

N	l	$<r^{-2}>$	$<r^{-1}>$	$<T>$	$<p^2>$	$<V>$
0	0	0.3239722014	0.5445692098	0.478533420	0.957066840	− 5.654613752
1	0	0.2026668213	0.3983238904	0.847399669	1.694799338	− 4.633418245
2	0	0.1350975450	0.3039575178	0.930201804	1.860403608	− 3.819268006
	1	0.1220861193	0.2904469549	0.990449578	1.980899156	− 3.751097883
3	0	0.09451763801	0.2395464971	0.906383704	1.812767408	− 3.183223157
	1	0.08606433454	0.2300584088	0.938772325	1.877544650	− 3.125427500
	2	0.07242011224	0.2134511328	0.978761390	1.957522780	− 3.007564405
4	0	0.06869107565	0.1936339311	0.8444699193	1.688939839	− 2.684910433
	1	0.06292797847	0.1867175566	0.8622946871	1.724589374	− 2.636995059
	2	0.05353481069	0.1745084720	0.8824482709	1.764896542	− 2.541101294
	3	0.04317896476	0.1593414491	0.8884454851	1.776890970	− 2.402935959
5	0	0.05147861623	0.1597591816	0.7720710867	1.544142173	− 2.290532107
	1	0.04739413699	0.1545630626	0.7819069236	1.563813847	− 2.250978832
	2	0.04068448522	0.1453265237	0.7914035723	1.582807145	− 2.172682176
	3	0.03319995966	0.1337331218	0.7897339069	1.579467814	− 2.060817231
	4	0.02625267175	0.1213175725	0.7724597856	1.544919571	− 1.925533310

implies that the average value of the measurement of the kinetic energy of a particle confined in the IQHKP is positively influenced by the rotational quantum number, l, and negatively by the vibrational quantum number, n. For a particle in a state (n, l), the expected value of the result of the measurement of the kinetic energy of the particle will be higher if it occupies a high-lying state. In the case of $\langle V \rangle$, an increase is observed with increasing quantum numbers n and l. This implies that a measurement of the potential energy of a particle confined in the IQHKP will produce higher values if such a particle occupies a state that is high-lying and produces lower values for a state that is low-lying.

6 Conclusions

In this paper, we solved the Schrodinger equation for the newly proposed IQHK potential using the NU method. The expectation values of the potential are evaluated with the aid of Hellmann–Feynman theorem. The numerical results are also obtained and all the special cases of the potential are found to be consistent with the literature, thus, confirming the accuracy of our method. Our results could have applications in mathematical physics, atomic physics, molecular and nuclear physics, and other related fields.

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Data Availability Statement This manuscript has associated data in a data repository. [Authors' comment: All data included in the manuscript are available upon request by contacting with the corresponding author.]

Declarations

Conflict of interest The authors declare that they have no known conflict of interest.

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