Soft impact dynamics of a cantilever beam: equivalent SDOF model versus infinite-dimensional system

U Andreaus, L Placidi and G Rega

DOI: 10.1177/0954406211414484

The online version of this article can be found at:
http://pic.sagepub.com/content/225/10/2444

Published by:
http://www.sagepublications.com

On behalf of:

Institution of Mechanical Engineers

Additional services and information for Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science can be found at:

Email Alerts: http://pic.sagepub.com/cgi/alerts
Subscriptions: http://pic.sagepub.com/subscriptions
Reprints: http://www.sagepub.com/journalsReprints.nav
Permissions: http://www.sagepub.com/journalsPermissions.nav
Citations: http://pic.sagepub.com/content/225/10/2444.refs.html

>> Version of Record - Sep 30, 2011

What is This?
Soft impact dynamics of a cantilever beam: equivalent SDOF model versus infinite-dimensional system

U Andreaus¹, L Placidi², and G Rega¹*
¹Department of Structural and Geotechnical Engineering, University of Rome La Sapienza, via Eudossiana 18, 00184 Roma, Italy
²Faculty of Engineering, International Telematic University Uninettuno, Corso Vittorio Emanuele II 39, 00186 Roma, Italy

The manuscript was received on 2 March 2011 and was accepted after revision for publication on 1 June 2011.

DOI: 10.1177/0954406211414484

Abstract: Non-smooth dynamics of a cantilever beam subjected to a transverse harmonic force and impacting onto a soft obstacle is studied. Upon formulating the equations of motion of the beam, proper attention is paid to identifying the mechanical properties of an equivalent single-degree-of-freedom (SDOF) piecewise linear impacting model. A multi-degree-of-freedom (MDOF) model of the impacting beam is also derived via standard finite elements. An ‘optimal’ identification curve of the obstacle spring rigidities in the two models is obtained by comparing the relevant pseudo-resonance frequencies. The identification is then exploited in the non-linear dynamic regime to get hints on some main, mostly regular, features of non-linear dynamic response of the impacting beam by the actual investigation of the behaviour of the sole equivalent SDOF model, with a definitely lower computational effort. Sample regular and non-regular responses of the MDOF model are also presented where the identification does not work. Overall, useful points are made as regards the possibility and the limitations of referring to an SDOF impacting model to investigate the non-linear response of the underlying infinite-dimensional system.

PACS-1998 classification codes: 02.30.Hq, 46.10.+z, 46.30.Pa

Keywords: Impacting beam, equivalent SDOF model, finite element model, soft contact, numerical simulations, non-linear dynamics

1 INTRODUCTION

Vibro-impact motion characterizes a large class of engineering systems. In some of them, the associated non-linearities introduce desirable effects, e.g. when operating vibration hammers, driving machinery, milling, or in impact print hammers and shock absorbers; in others, like gearboxes, bearings, and fuel elements in nuclear reactors, non-linearities can introduce undesirable effects. The unwanted behaviour can be, e.g. a high-amplitude response leading to fatigue or high noise level.

The huge number of studies published on impact phenomena witnesses of their spread interest within the scientific community and highlights the importance of reliably modelling the impact phenomenon in technical applications (see, e.g. Jerrelind and Stensson [1] and the references therein). In fact, difficulties are often encountered in this respect, also depending on various physical conditions to be possibly realized. The two most commonly pursued approaches consider either (a) impact oscillators with an instantaneous contact modelled via a coefficient of restitution or (b) piecewise systems for which contact is modelled as a linear or Hertzian spring, and
separate equations of motion describe the in-contact and out-of-contact dynamics. Owing to its simplicity, the former approach exploiting the instantaneous contact assumption has been used in many theoretical and numerical studies. However, most real systems exhibit soft impacts of finite duration, whose actual occurrence has to be properly taken into account for a reliable description of system global dynamics.

In this article, the dynamics of a steel cantilever beam subject to harmonic forcing, with a motion-limiting constraint on one side, is considered. For systems which are linear away from the constraint, such as the elastic beam vibrating with small amplitude displacements, non-linearity is induced by the non-smooth nature of the impact, which can entail rich and complicated dynamics. For specific ranges of values of the control parameters (e.g. the forcing frequency), impacts between the beam and the constraint result in vibro-impact periodic motions of the beam, possible subharmonic responses, transition to chaos, and coexisting attractors [2].

A vibro-impact system is modelled as a spring-mass system with amplitude constraint since the pioneering work of Shaw and Holmes [3], who analytically determined the stability of periodic solutions and identified chaotic features such as period-doubling, horseshoes, and strange attractors. As far as impacting beams are concerned, their non-linear dynamics has been studied through single-degree-of-freedom (SDOF) models in a large number of works, with the attention being paid mostly to the regularity of the impact motion, which is of special interest for applications. Classical studies are those by Moon and Shaw [4], who considered an SDOF approach to model a vibro-impact cantilever beam experiment, by reducing the system to a single-mode model exhibiting piecewise linear dynamics, and Shaw [5], who made further analyses and experimental studies using a cantilever beam contacting a stiff stop to compare cases of moderate and large ratios between the stiffness of the beam’s first bending mode and the stiffness at the stop. Experiments in this area were also conducted by Fang and Wickert [6], who used a cantilevered beam with a tip mass to demonstrate period-one, period-two, and chaotic motion, and Bishop et al. [7], who focused on identifying zones which separate regular period-one impacting solutions from irregular, impacting and non-impacting, motions. In impact oscillators, systems with various types of clearances between springs and mass were considered by many investigators, with notable contributions made, e.g., by Peterka [8], Wiercigrzog and Sin [9], Danca and Codreanu [10]. As regards impacting beams, Emans et al. [11] investigated the combined effects of clearance and beam non-linearities, with the impact model being a one-sided spring with a clearance. Lin et al. [12] extended the work by Emans et al., investigating a both-sided impact model. Andreus et al. [13] considered a soft impact bilinear oscillator also aimed at representing the single-mode dynamics of an impacted cantilever beam and studied the transition between different responses.

All of the mentioned works modelled the impacting beam as an SDOF system and the corresponding experiments aimed at confirming this assumption as closely as possible. Studies on multi-degree-of-freedom (MDOF) models of the impacting beam are more limited in number. Two degrees of freedom have been considered in a few papers, see, e.g. the references in Blazejczyk-Okolewska et al. [14], who presented detailed investigations of the dynamical behaviour of a massless cantilever beam with two concentrated masses. Overall, among many works devoted to modelling multi-dimensional impact oscillators, a few of them that are more directly concerned with impacting beams include the use of lumped mass-type systems [15–17], Galerkin reduced models [18], and finite element (FE) models [19], and refer to continuous rods and beam elements [20–22], also in an experimental framework [23]. Interest has been devoted to creating reliable low-dimensional models of flexible vibro-impact systems [24, 25] and to showing how models with more than one degree of freedom are actually required for a good assessment of the long-term behaviour of the system [18, 19]. Specific issues like features of wave propagation [26, 27] and numerical procedures for reliably calculating contact forces [28, 29] have been also dealt with.

However, apart from a few works, the whole latter body of literature has been concerned mainly with modelling the impact event in itself rather than with investigating the global dynamics of the beam. Nonetheless, for engineering design purposes, the accurate mathematical modelling of their global dynamics is of major importance. This article aims at addressing the soft contact dynamics of an impacted cantilever beam via a somehow mixed approach. Indeed, on one side, advantage is taken of the possibility to use an equivalent SDOF model to reliably and systematically describe the system dynamics in some ranges of control parameter values, with low computational effort. On the other side, recourse is made to an FE-based MDOF model to get hints on some main response features of the actual infinite-dimensional system, wherein the minimal reduced order model does not reliably
work. Accordingly, the scope of the study is twofold: (a) identifying an SDOF model equivalent to the impacting cantilever beam, an issue which is not commonly pursued in literature and (b) comparatively simulating its soft contact dynamics against that of an FE model of the impacting beam. Sample results are also obtained via the MDOF model in other ranges of system control parameters.

The article is organized as follows. The mechanical system is presented in Section 2 by separately addressing the infinite-dimensional beam and the equivalent SDOF model. The identification procedure and its outcomes are presented in Section 3. Section 4 is devoted to comparing results of numerical simulations made with the SDOF and MDOF models (Section 4.1), discussing periodic solutions as reliably obtainable with the former upon exploiting the identification results (Section 4.2), and presenting sample responses of the latter where the identification does not work (Section 4.3). The article ends with some conclusions.

2 MECHANICAL SYSTEM

2.1 The infinite-dimensional impacting beam

An impacted cantilever beam is considered. The beam is clamped at the left-hand side and free at the right-hand side in the undeformed configuration. An impact is considered only if the tip displacement reaches a certain gap (Fig. 1). A transverse sinusoidal force is applied at the tip. If the right-hand side of the beam reaches the obstacle, a change in the configuration of the system is considered because of the obstacle. The partial differential equation that governs the transversal displacement field $u(x, t)$ is of second order in the time $t$ and of fourth order in the space $x$

$$\forall x \in [0, L], \forall t \in [0, \infty) \quad K_B u'''' + c_B \ddot{u} + \rho A \dddot{u} = 0$$

(1)

where $L$ is the length of the beam, $K_B$ its bending stiffness,* $A$ the cross-sectional area, $\rho$ the volumetric density, and the dot and prime the derivative with respect to $t$ and $x$, respectively. The Rayleigh attenuation coefficients $\alpha$ and $\beta$ will be used in the subsequent FE solution by relating them to the damping coefficient $c_B$ based on the discretization process. The first boundary–initial conditions between the initial time $t_0 = 0$ and the first impact at $t = t_1$ are

$$\forall t \in [0, t_1] \quad u(0, t) = u'(0, t) = u''(L, t) = 0$$

(2)

$$K_B u''''(L, t) = -F_0 \sin(\omega t)$$

(3)

$$\forall x \in [0, L] \quad u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = \dot{u}_0(x)$$

(4)

where $u_0(x)$ and $\dot{u}_0(x)$ are, respectively, the initial displacement and velocity fields with the condition, without loss of generality

$$u_0(L) > -\delta$$

(5)

so that the beam is initially out of the obstacle. $F_0$ and $\omega$ are, respectively, the amplitude and frequency of the external force applied at the right-hand side tip of the beam and $\delta$ (also called gap) the distance of the obstacle from the tip of the beam in its unstressed configuration. The first impact time $t = t_1$ is defined as follows

$$\begin{cases} u(x, t_1) = u_1(x), \quad u_1(L) = -\delta \\ \dot{u}(x, t_1) = \dot{u}_1(x), \quad \dot{u}_1(L) < 0 \end{cases}$$

(6)

where $u_1(x)$ and $\dot{u}_1(x)$ are the corresponding displacement and velocity fields, so that the second boundary–initial conditions between the first impact at $t = t_1$ and the time $t = t_2$ when the beam tip is again out of the obstacle are

$$\forall t \in [t_1, t_2] \quad u(0, t) = u'(0, t) = u''(L, t) = 0$$

(7)

$$K_B u''''(L, t) = k_B^d (u(L, t) + \delta) - F_0 \sin(\omega t)$$

(8)

$$\forall x \in [0, L] \quad u(x, t_1) = u_1(x), \quad \dot{u}(x, t_1) = \dot{u}_1(x)$$

(9)

where $k_B^d$ is the stiffness of the spring that is used to simulate the soft obstacle in the beam model. The second impact time $t = t_2$ is defined as

$$\begin{cases} u(x, t_2) = u_2(x), \quad u_2(L) = -\delta \\ \dot{u}(x, t_2) = \dot{u}_2(x), \quad \dot{u}_2(L) > 0 \end{cases}$$

(10)

*In the case of a beam with a cross-section having the inertia moment $I$ and made by an isotropic material with Young’s modulus $E$, $K_B = E I$.
where \( u^e(x) \) and \( i^e(x) \) are, respectively, the displacement and velocity fields at the second impact time, so that the third boundary–initial conditions between the second impact at \( t = t_2 \) and the time \( t = t_3 \) when the tip of the beam is again inside the obstacle are

\[
\forall t \in [t_2, t_3] \quad u(0, t) = u'(0, t) = u''(L, t) = 0
\]

\[
K_S u''(L, t) = -F_0 \sin(\omega t)
\]

\[
\forall x \in [0, L] \quad u(x, t_2) = u^e(x), \quad u(x, t_3) = i^e(x)
\]

Thus, formally, the set of boundary–initial conditions reads

\[
\forall t \in [t_i, t_{i+1}] \quad u(0, t) = u'(0, t) = u''(L, t) = 0
\]

\[
K_S u''(L, t) = k^p (u(L, t) + \delta) \times H(-u(L, t) - \delta) - F_0 \sin(\omega t)
\]

\[
\forall x \in [0, L] \quad u(x, t_i) = u_i(x), \quad u(x, t_i) = i^o_i(x)
\]

where \( H(y) \) is the Heaviside function

\[
H(y) = \begin{cases} 
1, & y \geq 0 \\
0, & y < 0 
\end{cases}
\]

and the sequence of increasing times \( t_i (i = 1, 2, \ldots) \) is defined as follows

\[
u(L, t_i) = -\delta, \quad u(x, t_i) = u_i(x), \quad i^o_i(x) = i^o_i(x)
\]

such that

\[
i = 0, \quad \implies \quad u_0(L) > 0
\]

\[
i = \text{odd}, \quad \implies \quad i^o_i(L) < 0
\]

\[
i = \text{even}, \quad \implies \quad i^o_i(L) > 0
\]

where the beam is initially out of the obstacle and the tip velocity is positive for even impact and negative for odd one. The grazing phenomenon, \( i^o_i(L) = 0 \) is not considered in this study.

The jump in the definition (17) of the Heaviside function makes the partial differential equation (1) with the conditions (14 to 16) a non-linear partial differential equation, that will be simulated using the numerical values of Table 1 and the commercial code ADINA 8.3™.

### 2.2 The equivalent SDOF impacting model

The impacting beam is an infinite-dimensional system and its numerical evaluation is time consuming. Thus, it is useful to identify an equivalent SDOF model suitable to describe the single-mode beam dynamics. The piecewise linear oscillator considered in Fig. 2 is a mass–spring–damper system governed by the following ordinary differential equation

\[
m\ddot{q} + c(q) \dot{q} + f(q) = F_0 \sin(\omega t)
\]

where \( q \) is the displacement, from the unstressed configuration of the spring with rigidity \( k_s \), of a material point with mass \( m \) at time \( t \), \( F_0 \sin(\omega t) \) the external sinusoidal force, \( c(q) \) the attenuation coefficient, and \( f(q) \) the opposite of the force exerted by the system springs. In order to simulate the occurrence of the impact event, \( c(q) \) and \( f(q) \) are two piecewise functions defined as follows

\[
f(q) = \begin{cases} 
k_s q + k^o_s (q + \delta) H(-q - \delta), & q > -\delta \\
k_s q + k^o_s (q + \delta), & q \leq -\delta
\end{cases}
\]

\[
c(q) = \begin{cases} 
c_s, & q > -\delta \\
c_s + c^o_s, & q \leq -\delta
\end{cases}
\]

where \( \delta \) is the amplitude of the gap. The subscript ‘s’ refers to the system composed of the spring with rigidity \( k_s \) and the damper with attenuation coefficient \( c_s \). The subscript ‘o’ refers to the obstacle, which is simulated by the spring with rigidity \( k^o_s \), whose unstressed configuration is shifted by the gap \( \delta \) from that of the spring with rigidity \( k_s \) and by the damper with attenuation coefficient \( c^o_s \). Both the
spring $k_S$ and the damper $c_S$ exert forces, respectively $-k_S(q + \delta)$ and $-c_S q$, only if $q \leq -\delta$ and their superscript ‘S’ refers to the SDOF model being defined. The numerical values used in the computer simulations will be set according to the identification procedure presented in the next section.

3 THE IDENTIFICATION OF THE SDOF MODEL

3.1 Spring rigidity and point mass

In order to do fast numerical simulations, the infinite-dimensional beam system should be studied through the simpler SDOF model. To do this, the parameters of the latter have to be identified. First, the mass $m$ of the SDOF point and its spring rigidity $k_s$ with the mass density $\rho A$ of the beam and its bending stiffness $K_B$ are identified. A static and a dynamic identification is pursued for the stiffness and the mass, respectively, without the obstacle. Thus, it is assumed that the displacement of the SDOF point is the same as the tip displacement of the beam under the same external static force, i.e. $k_s = 3K_B/L^3$, and that the resonance frequency of the SDOF point is the same as the first mode frequency $\omega_1$ of the beam, $m = k_s/\omega_1^2 = 3K_B/\omega_1^2 L^3$. Finally, the obstacle spring rigidity $k_o$ in the SDOF model is identified with the obstacle spring rigidity $k_o$ in the beam model via a method that is illustrated in the next section.

3.2 Obstacle spring rigidity

The two non-linear models do not possess resonant natural frequencies. However, a sinusoidal external force with varying frequency is applied to the SDOF model and, for a given obstacle spring rigidity, the maximum displacement for each driving frequency is evaluated through numerical simulations of equation (22). For example, in Fig. 3, the phase portrait of the point mass in the SDOF model for the values of the parameters showed in the caption is depicted: the maximum displacement is seen to be 1.95 mm. Thus, there are different values for the maximum displacement, one for every external force frequency and, for a given spring rigidity, the bell-shaped plot of the maximum displacement versus driving frequency, shown in Fig. 4 is obtained. The frequency relative to the largest maximum displacement is known as the pseudo-resonance frequency relative to the given spring rigidity. Thus, for the sake of simplicity, in Fig. 4, it is derived that the pseudo-resonance frequency for a spring rigidity equal to $10^{5.5}$ N/m = 316.228 N/m is 94 Hz. The same kind of simulation is implemented for a certain number of spring rigidities keeping the damping constant and the normalized results are shown in Fig. 5(a). Normalization is made with respect to the amplitude related to each pseudo-resonance frequency, in order to show that the difference in the slopes on the left- and right-hand sides of the peak is systematic.

The same work is done with the beam model through ADINA 8.3 TM (Section 4), for which the analogon of Fig. 5(a) is showed in Fig. 5(b). In comparing the two sets of curves, it must be observed that the underlying ones before normalization exhibit decreasing values of the pseudo-resonance peak for increasing obstacle rigidities, to an extent which is greater for the beam than for the SDOF as a result of the Rayleigh damping of the former increasing with the obstacle rigidity against the constant damping of the latter. This entails a flattening and widening of the beam’s bell-shaped curves, which is then reflected into higher tails towards lower frequencies (already occurring for the SDOF curves) when the curves are normalized with respect to the decreasing peaks. However, it is worth noting that this phenomenon does not affect the comparison between the pseudo-resonance peaks of the two sets of curves, which is indeed the tool actually governing the identification of the obstacle spring rigidity in the SDOF model.

Fig. 3 SDOF phase portrait ($k_o = 316.228$ N/m, $\omega = 90$ Hz)

Fig. 4 SDOF pseudo-resonance curve ($k_o = 316.228$ N/m)
Thus, two graphs of pseudo-resonance frequencies vs obstacle spring rigidities are built, which are reported in Fig. 6, one for the beam model and one for the SDOF model. For a given value of obstacle rigidity, the pseudo-resonance frequency of the latter is greater than that of the former since in the beam solution the effect of the higher modes is also captured. The spring rigidity of the obstacle \(k^S_o\) is identified in the SDOF model with the spring rigidity \(k^B_o\) in the beam model in such a way the pseudo-resonance frequencies are the same for both models, see, e.g. the horizontal line in Fig. 6 corresponding to 100 Hz frequency. In order to obtain an identification curve, the two resonance curves of Fig. 6 are first interpolated for the pseudo-resonance \(\nu_r\), by the following test function

\[
\nu_r = a + b \log k_o + c (\log k_o)^2
\]

(25)

where the best interpolating values for the two sets \((a_S, b_S, c_S)\) and \((a_B, b_B, c_B)\) are obtained via the least square method. Thus, the following equation is set

\[
\begin{align*}
  a_S + b_S \log k^S_o + c_S (\log k^S_o)^2 \\
  = a_B + b_B \log k^B_o + c_B (\log k^B_o)^2
\end{align*}
\]

(26)

to derive the numerical identification curve that is found to be represented, in the given range, as follows

\[
k^S_o = a_1 + a_2 \log k^B_o + a_3 (\log k^B_o)^2 + a_4 (\log k^B_o)^3
\]

(27)

with

\[
\begin{align*}
a_1 &= 4.0 \times 10^7, \quad a_2 = -9.1 \times 10^6 \\
a_3 &= 6.7 \times 10^5, \quad a_4 = -1.6 \times 10^4
\end{align*}
\]

(28)

The result is shown in Fig. 7 and called the obstacle rigidity identification curve. Note that the values of the pseudo-resonance frequencies in Fig. 5 were obtained by sampling the driving force frequency with a certain step (0.5 Hz). This entails some approximation in determining the pseudo-resonance values, which affects the identification of the obstacle rigidity. The error bars in Fig. 7 account for this approximation.
approximation. In the sequel, the best agreement between SDOF and beam results will be obtained using the upper bound threshold.

### 3.3 Damping coefficient

The optimal choice of the damping coefficient $c_s$ of the equivalent SDOF model (equation (24)) is made via a heuristic procedure. The identification target is the damping effect ensuing from the values $\alpha = 10^{-5}$ s and $\beta = 10^{-5}$ s$^{-1}$ of the Rayleigh damping coefficients considered in the FE model of the system comprised of the beam and the obstacle; accordingly, the coefficient $c_s^0$ is set equal to zero (Table 2). Identification is made by comparing the phase portraits of the responses provided by the MDOF model and by the equivalent SDOF, for various values of the SDOF damping coefficient and two different pairs of beam/SDOF obstacle rigidity values as obtained from Fig. 7, i.e. one pair ($k^B_o = 2.4 \times 10^5$ N/m, $k^S_o = 2.05 \times 10^5$ N/m) corresponding to a ‘mid-soft’ obstacle and the other ($k^B_o = 10^5$ N/m, $k^S_o = 1.2 \times 10^5$ N/m) to a definitely ‘soft’ obstacle.

In the former case, the parameter to be minimized is assumed to be the distance between the SDOF and beam responses in the phase portrait measured along the negative displacement axis at zero velocity (Fig. 8(b)). The relation between this distance and the SDOF damping coefficient exhibits a plateau for $c_s \leq 0.2$ Ns/m (Fig. 8(a)). In the latter case, the parameter to be minimized is assumed to be the distance between the SDOF and beam responses measured at the two closest points of the orbit (Fig. 9(b)). Its relation with $c_s$ exhibits again a plateau for $c_s \leq 1$ Ns/m (Fig. 9(a)).

The good agreement occurring in both cases between the dynamic response of the beam and that of the equivalent oscillator, as obtained with the value $c_s = 0.2$ Ns/m can be observed in Figs 8(b) and 9(b), respectively. Therefore, it is referred to in the following calculations. In addition, it is worth noting that for both cases (Figs 8(b) and 9(b)), the Fourier spectra of the acceleration of the beam tip and of the SDOF exhibit the same superharmonics of order three and two, besides the fundamental component.

### 4 NUMERICAL SIMULATIONS

As already said at the end of Section 2.1, the vibrations of the beam whose harmonically excited tip impacts a bilinear spring with a clearance are governed by the partial differential equation (1) with boundary-initial conditions (14 to 16). This non-linear problem was solved using ADINA 8.3TM. The structural assemblage sketched in Fig. 1 was modelled as a mono-dimensional system and discretized by means of Euler beam

---

**Table 2** Numerical values used in SDOF simulations

<table>
<thead>
<tr>
<th>$m$ (g)</th>
<th>$k_s$ (N/m)</th>
<th>$c_s$ (Ns/m)</th>
<th>$c^0_s$</th>
<th>$\delta$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>381</td>
<td>67200</td>
<td>0.2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Fig. 8** (a) Identification of the damping coefficient $c_s$. (b) comparison between phase portraits of beam (thick blue line) and SDOF (thin red line) for optimal choice of $c_s$.

Notes: $k^B_o = 2.4 \times 10^5$ N/m, $k^S_o = 2.05 \times 10^5$ N/m and $\omega = 34.2$ Hz

**Fig. 9** (a) Identification of the damping coefficient $c_s$. (b) comparison between phase portraits of beam (thick blue line) and SDOF (thin red line) for optimal choice of $c_s$.

Notes: $k^B_o = 10^5$ N/m, $k^S_o = 1.2 \times 10^5$ N/m and $\omega = 26.83$ Hz
finite elements with nodes having two degrees of freedom in the bending plane. In particular, 10 FEs and 11 nodes were implemented for the beam; the impact model is a one-sided spring with a clearance, and a piecewise linear stiffness was used to model the impact process; geometrical and material data were taken according to Table 1.

As far as the SDOF system is concerned, the technique based on the transcendental maps matching the in-contact and out-of-contact dynamics of an impacting bilinear oscillator [13] was used to analytically solve equation (22) with restoring force, equation (23), and viscous damping, equation (24).

Applying the procedure of Section 3 led to identify the parameters of the equivalent SDOF model: \( m = 0.381 \text{ kg, } k_s = 6.72 \times 10^4 \text{ N/m, } k_c = 2.05 \times 10^5 \text{ N/m, } c_s = 0.2 \text{ Ns/m, } c_c = 0 \) (Table 2).

### 4.1 SDOF model versus beam model

Comparison between the responses of the two models is made by constructing bifurcation diagrams of the attachment velocity, namely, the velocity at which the beam tip or the SDOF impacts the obstacle, by considering the driving frequency as a control parameter. Figure 10(a) shows the attachment velocity diagram of both the beam and the SDOF, when the latter is characterized by the pair of obstacle rigidity values \( k_0^B = 2.4 \times 10^5 \text{ N/m and } k_0^S = 2.05 \times 10^5 \text{ N/m.} \) As already mentioned, these values can be associated to a ‘medium’ level of rigidity; in fact, the global behaviour of the reduced order structural model resembles that of a non-impacting cantilever beam when the beam obstacle rigidity is less than \( k_0^B = 10^3 \text{ N/m, whereas the contact tends to become an impact against a rigid stop when it is larger than } k_0^B = 10^6 \text{ N/m.} \)

The frequency region \( 29–45 \text{ Hz} \) was chosen in order to test the reliability of the identification procedure also for values sufficiently away from the system pseudo-resonance frequency \( \omega_0 = 89.2 \text{ Hz, at } k_0^B = 2.4 \times 10^5 \text{ N/m, just used in the identification procedure. Moreover, the lower and upper bounds of the selected range approximately coincide with 1/3 and 1/2 of } \omega_0, \text{ thus corresponding to the likely occurrence of superharmonics of order three and two, respectively, in the response. This circumstance seems to be of special interest because it could be conveniently exploited to detect the existence of damage in the inverse problem, if the gap size were unknown [30]. The attachment velocity diagram of Fig. 10(a) was constructed by imposing natural initial conditions \((u(t), x(t), t) = 0, t \in [0, L] \text{ and } q(0) = 0, \text{ respectively)} \text{ to the beam and SDOF models.} \)

The results obtained by the equivalent SDOF are in good agreement with those of the beam in the whole investigated region. Different zones of periodic solutions, characterized by multiple cycles, are clearly recognizable; moreover, non-regular responses can occasionally and abruptly occur and coexisting attractors show their existence, owing to the circumstance that natural conditions may lead to quite different attractors when a control parameter is varied. However, the overall structure of the diagram is captured by both models and the robustness of the dominant periodic solutions is preserved. Of course, the simulation via SDOF model is much less time consuming with respect to the beam (MDOF) model.

### 4.2 Periodic solutions in the SDOF model

The investigation can now be deepened and widened at low computational cost by studying the behaviour of the sole equivalent SDOF; in fact, in the considered region of driving frequency, the same dynamic scenarios also hold for the beam. Applying the parametric continuation technique [31] from the natural initial conditions at \( \omega = 29.5 \text{ Hz} \) allowed us to identify ranges of driving frequency where the system motion exhibits different periodicity (calculated by measuring the time elapsed to move along an orbit), along with the transitions driving from one class of periodic response to another (Fig. 10(b)). Note that the data in Fig. 10(a) were obtained by starting for each frequency from initial natural conditions and hence,
jumping from an orbit to another can occasionally occur; whereas in Fig. 10(b), continuation technique guarantees the stability of the orbit for sufficiently small perturbations of the bifurcation parameter (frequency in this case). Basically, the two-cycle solution dominates the response in the whole considered frequency region. However, with increasing frequency, it is robustly replaced by two main higher cycle solutions in two intermediate ranges, where a three- and a six-cycle response settle down, respectively, as discussed in the sequel. In the initial narrow range (29.50–29.90 Hz), the solution is periodic with one cycle (blue line in Fig. 11(a)) and with the same period as that of the driving force. At about 29.85 Hz, an impending loop is observed in the corresponding blue cycle of Fig. 11(b), just before the onset of a two-cycle solution. Actually, in the relevant initial range of existence (29.90–30.095 Hz), the attachment velocity diagram reveals only the larger cycle (upper red line in Fig. 11(a)) of the new born two-cycle solution, because the smaller one (lower red line in Fig. 11(a)) remains within the obstacle and hence is (hardly) visible in the sole phase portrait (red cycle in Fig. 11(b)). Note that the period of the two-cycle solution is still equal to that of the driving force.

At 30.095 Hz, the smaller cycle of the two-cycle solution starts expanding while grazing the obstacle from the inside. In a first small range which extends up to 30.375 Hz, the system response exhibits a marked instability due to the coexistence of two competing attractors (Fig. 10(a)), one with two cycles and one with five cycles. Overall, the onset of the two-cycle solution is characterized by a substantially non-linear path of attachment velocity (Fig. 11(a)): a very steep ascend of the outer cycle branch between 29.900 and 30.095 Hz is followed by an almost abrupt increase of the slope of the inner one in between 30.095 and 30.375 Hz. Thereafter, the slopes of the two branches remain constant up to 33.50 Hz.

At 33.50 Hz, a discontinuous transition from two-cycle to three-cycle periodic solution occurs (Fig. 12(a)). The period of the new born solution is doubled with respect to the previous one, namely, it is twice the driving period (0.03 s → 0.06 s). The three-cycle solution exists up to about 34.10 Hz, where it passes back to the two-cycle solution (Fig. 12(b)) with half period (0.0529 s → 0.009 s) via another discontinuous bifurcation. In the range 34.20–35.68 Hz, the two-cycle solution is again stably periodic, with slightly increasing values of attachment velocity.

A similar sequence of forward/backward discontinuous transitions from/to the two-cycle solution is observed in Fig. 13(a). At about 35.685 Hz, the response triples its period (0.0281 s → 0.0843 s) and passes from two to six cycles (Fig. 13(b)), the latter remaining stable up to about 35.950 Hz. It is worth noting that grazing occurs in between 35.90 and 35.95 Hz. Then, another discontinuous transition drives the response back to the underlying two-cycle solution (Fig. 13(a)), with the period being reduced from 0.0834 s to approximately one-third (0.0282s). Again, the two-cycle solution remains
stably periodic in the relatively large range (36.0–41.5 Hz), with its amplitude non-linearly increasing along an ascending path (Fig. 14(a)).

Finally, a further discontinuous transition occurring at about 41.5 Hz marks the onset of a different two-cycle solution (red in Fig. 14(b)), whose period is approximately one-third (0.0282 s) of that (0.0834 s) of the dominant one. Note that in a somehow mirrored way to what previously observed at the first onset of the former two-cycle solution (Fig. 11(a)), in the last range (41.0–45.0 Hz) of the considered frequency region, the attachment velocity diagram does not reveal the inner cycle of the new born two-cycle solution, since it does not cross the obstacle and is generated in the flight region inside the outer cycle. An overall gently decreasing path of attachment velocity, with a progressively reduced slope of the solely visible outer cycle amplitude (blue), is observed in Fig. 14(a).

4.3 Sample responses in the beam model

The equivalent SDOF model has been identified in the range of medium–low values of obstacle rigidities \( (k_B^0 = 2.4 \times 10^5 \text{ N/m}, k_S^0 = 2.05 \times 10^5 \text{ N/m}) \) in Fig. 7, wherein the identification relation, equation (26), exhibits a minor sensitivity to the numerical approximation with which the pseudo-resonance frequencies were detected in a given range of driving frequency. Comparison of the ensuing non-linear dynamic response with that of the beam model has shown to be quite satisfactory. It is now worth to complement the analysis (1) by comparing the results obtainable with the two models for higher values of obstacle rigidities, where the identification relationship is affected by a larger numerical error and (2) by obtaining sample responses with the beam model for values of other problem parameters (typically, driving frequency and amplitude) away from the range wherein the identification was performed.

Figure 15 shows the diagram of attachment velocity as obtained with the beam and with the equivalent SDOF in the whole considered range of beam obstacle rigidity (note that the values on the abscissa axis can be referred to both \( k_B^0 \) and \( k_S^0 \)). The driving frequency and amplitude are equal to 40 Hz and 100 N,
respectively. It is worth reminding that the former value is already quite below the value $\omega_0 = 89.2 \text{ Hz}$ where the identification was actually performed, though belonging to the frequency range where the non-linear dynamic responses of the two models show a fairly good agreement. Such an agreement is herein confirmed in the range of soft/mid-soft obstacle rigidities as far as both the number of response cycles (basically two cycles in Fig. 15) and the values of attachment velocities are concerned. In contrast, moving to higher values of obstacle rigidity, meaningful differences may occur between the results of the two models, which is an expected outcome owing to the notably poorer approximation of the identification relationship. In particular, looking at the phase portraits (Fig. 16) obtained for the two different values $k_{Bo} = 10^6 \text{ N/m}$ and $k_{Bo} = 10^7 \text{ N/m}$, periodic responses with very low (very high) number of cycles are seen to occur with the beam (the SDOF) model in the former case (Fig. 16(a)), whereas in the latter case (Fig. 16(b)), a major qualitative difference occurs since the SDOF model exhibits a regular response and the beam model a non-regular one. This highlights how much care has to be paid to properly analysing the system with a more involved MDOF model, actually accounting for the effects of the higher modes, before assessing the capability of an alleged equivalent SDOF to simulate the actual behaviour of an impacting beam, frequently claimed in literature.

Using the beam model by way of example, Fig. 17 shows the diagram of attachment velocity obtained by considering the amplitude of the external force as control parameter, for an obstacle rigidity value $k_{Bo} = 10^7 \text{ N/m}$ and a driving frequency $\omega_0 = 67 \text{ Hz}$, which corresponds to the first resonance frequency of the beam without the obstacle. Different zones of apparently non-regular responses, characterized by either one or two bands, are recognizable in Fig. 17. Looking at, e.g. the sample phase portrait of the beam tip depicted in Fig. 18(a) for $F_0 = 160 \text{ N}$, the two bands...
are seen to correspond to the quasi-periodically varying values of attachment velocity in the sole relevant negative range, considered in Fig. 17, as highlighted by the resolved wiggle intersection with the velocity axis, visible in the phase portrait magnification in proximity of the obstacle reported in Fig. 18(b).

5 CONCLUSIONS

Non-smooth dynamics of a cantilever beam subjected to a transverse harmonic force and impacting onto a soft obstacle at its free boundary has been studied. Upon formulating the equations of motion of the impacting infinite-dimensional system, proper attention has been paid to identifying the mechanical properties of an equivalent SDOF piecewise linear impacting model, to be reliably used for getting hints on the single-mode dynamics of the actual infinite-dimensional system in the background via a return map approach. A MDOF of the impacting beam has been obtained via standard FEs.

Identification of a best fit relationship between the obstacle spring rigidity in the MDOF model and the corresponding one in the SDOF model has been pursued by comparing the relevant pseudo-resonance frequencies in a range of driving frequencies. An optimal identification curve has been obtained over a quite extended range of obstacle rigidity values. However, its approximation becomes weaker for higher values of the obstacle rigidity, due to some major disagreement of the pseudo-resonance frequencies of the two models in connection with the effect of the higher modes being neglected (accounted for) in the SDOF (beam) model. When moving to the non-linear dynamic responses, the identification is expected to work well basically within the considered ranges of system parameter values. However, comparing the bifurcation diagrams of attachment velocity as obtained with the two models for relatively low optimal values of obstacle rigidities, the identification is seen to work well also rather away from the underlying pseudo-resonance frequency. This favourable circumstance paved the way to reliably get hints on some main, mostly regular, features of non-linear dynamic response of the impacting beam by actually investigating in detail the behaviour of the sole equivalent impacting SDOF, with a definitely lower computational effort. Of course, as expected, when considering higher values of the optimal pair of obstacle rigidities, the two different reduced order models exhibit non-linear dynamic responses which can differ substantially with respect to each other, with also possible occurrence of non-regular instead of regular responses. Accordingly, a reliable and of course heavier investigation of the actual infinite-dimensional system has to be done via a proper reduced order model (MDOF) overcoming the limitations inherent to the minimal (SDOF) one.

Apart from specific results, the overall outcomes of the investigation allow for two major points useful to future analyses of the non-linear dynamics of impacting beams. (1) Upon suitably identifying a minimal order impacting model, it can be extensively referred to for getting hints on basic response features of the actual infinite-dimensional beam, by paying proper care to the range of system parameter values where it is reliably exploitable; on the other hand, there is substantial time saving compared to the computational model of the beam. (2) An actual MDOF model of the impacting beam has to be used in all other cases, though being computationally more onerous as well as more involved in terms of understanding and description of the non-linear dynamic response.

FUNDING

This work was supported by the Sapienza Università di Roma [grant number C26A098Y5S].

ACKNOWLEDGEMENTS

Dedicated to Hans Troger, a scientist and a man that the nonlinear dynamics community in mechanics is strongly missing.

© Authors 2011

REFERENCES


