Analysis of the Discrete-Time
GI/Geom(n)/1/N Queue

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Abstract—In this paper, we consider the discrete-time single-server finite-buffer late arrival with delayed access queue GI/Geom(n)/1/N. Whereas the interarrival times are independently identically distributed random variables with arbitrary probability mass function, the service times are geometrically distributed random variables with probability of a service completion during a slot dependent on the number of customers present in the system. Using the supplementary variable technique, we obtain probability distributions of numbers of customers in the system at arbitrary and prearrival epochs as well as an outside observer's distribution. In addition, we derive some important results which are used to develop relations between probabilities at prearrival and arbitrary epochs. Results obtained in this paper can be used in several areas such as performance evaluation of computer-communication systems.

Keywords—Discrete-time, Finite buffer, Geometric, Queue, State dependent.

1. INTRODUCTION

In recent years, there has been a growing interest in the analysis of discrete-time queues due to their applications in communication systems and other related areas. This can be judged by the fact that two books [1,2] have recently come out. Also, various special issues of some journals have been exclusively devoted to discrete-time queues and their applications [3]. In the past few years, not only have a large number of articles on various aspects of discrete-time queues already appeared in the literature, but they still continue to do so. One of the main reasons for the analysis of discrete-time queues is that continuous-time queuing models cannot accurately give the performance measures of discrete-time queuing models in which basic units are digital such as machine cycle time or bits and packets. Further, though discrete-time queues can be used to derive continuous-time results, the reverse is generally not true [2, p. v]. In fact, much of the importance of discrete-time queues stems from the fact that they can be used in the

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performance analysis of Broadband Integrated Services Digital Network (BISDN), Asynchronous Transfer Mode (ATM) and related computer communication technologies wherein the models for continuous-time queues do not fit.

In this paper, we consider a discrete-time analog of the continuous-time queue $GI/M(n)/1/N$ which has been discussed in [4,5]. It may, however, be mentioned that the discrete-time model differs from the corresponding continuous-time model in the sense that whereas in continuous-time models the probability of simultaneous arrivals and departures is zero, it is positive in the case of discrete-time models. We denote the discrete-time version of the model $GI/M(n)/1/N$ by $GI/\text{Geom}(n)/1/N$. In discrete-time queues, arrivals and departures occur around slot boundaries; further, either an arrival may have precedence over a departure or vice versa. The former case is referred to as arrival first (AF) policy and the latter as departure first (DF) policy [6]. Also, in AF policy, if the server is idle and a customer arrives, two cases arise:

(i) the customer enters service immediately, or
(ii) the customer's service starts in the following slot.

The first type of service rule is known as 'immediate access' whereas as the second type is known as 'delayed access' [7]. Late arrival and early arrival rules correspond to AF and DF, respectively. Here, we consider the $GI/\text{Geom}(n)/1/N$ queue with late arrival system and delayed access (LAS-DA) and carry out its analysis using the supplementary variable technique. The model is a generalization of the $GI/\text{Geom}/1/N$ LAS-DA queue discussed in [8] in the sense that here we consider a state-dependent service rate as opposed to a constant service rate. Further, the $GI/\text{Geom}(n)/1/N$ queue is more widely applicable than the simple $GI/\text{Geom}/1/N$ queue, as it can be used to reduce the loss probability by increasing the service rate. The analysis of the $GI/\text{Geom}(n)/1/N$ queue with late arrival system and immediate access (LAS-IA) can be carried out as has been done for the $GI/\text{Geom}/1/N$ LAS-IA queue [8]. The results have been presented in the form of recursive equations which can be programmed on any PC to obtain probabilities at prearrival and arbitrary epochs, as well as outside observer's distribution (for definitions, see later). We also obtain relations between probabilities at both prearrival and arbitrary epochs. It may be remarked that the method discussed here gives probabilities at both prearrival and arbitrary epochs, and hence, there is no apparent need to develop a relation between the two sets of probabilities. However, such a relation is derived for the sake of completeness and to show how our procedure can be used to develop relations besides providing state probabilities at various epochs.

The paper is organized as follows: assumptions and notations of the model are introduced in Section 2; basic equations of the model and its solution are discussed in Section 3; Sections 3.1 and 3.2 together provide probability distributions of numbers in system at prearrival, arbitrary and outside observer's epochs. Relations between prearrival and arbitrary epochs probabilities along with a few important results are discussed in Section 4. In Section 5, we present some important performance measures and finally numerical results are discussed in Section 6. An algorithm along with its complexity for computing state probabilities is given in the Appendix.

2. ASSUMPTIONS AND NOTATIONS

We assume that the arrivals occur at slot boundaries with late arrival and delayed access (LAS-DA); i.e., the late arriving customer is blocked from entering an empty service facility until the servicing interval terminates, [7, p. 193]. Departures also occur at slot boundaries. More specifically, time is slotted into intervals of equal length with the length of an interval taken as unity. Whereas arrivals occur in $(m-, m)$, departures take place in $(m, m+)$. We introduce the following notations for the $GI/\text{Geom}(n)/1/N$ LAS-DA queue:

\[ A \equiv \text{random variable (r.v.) denoting interarrival time} \]
\[ a_n \equiv P(A = n), \ n \geq 1, \ \text{probability mass function (pmf) of r.v. } A \]
\[ a \equiv \text{mean interarrival time} = \sum_{n=1}^{\infty} n a_n \]

\[ N = \text{system capacity} \]

\[ A(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{probability generating function (pgf) of interarrival-time distribution} \]

\[ a_0 = 0 \quad \text{and} \quad |z| \leq 1 \]

\[ S_i \equiv \text{r.v. denoting service time of a customer when } i \text{ customers are present in the system} \]

\[ \mu_i = \text{service completion rate when there are } i \text{ in the system} \]

\[ b(n, i) = P(S_i = n) = (1 - \mu_i)^{n-1} \mu_i, \quad 0 < \mu_i < 1, \quad n \geq 1 \]

\[ b_i = \text{mean service time when there are } i \text{ customers in the system} \]

\[ N_m = \text{number in system (before a potential arrival) at epoch } m- \]

\[ U_m = \text{remaining interarrival time at epoch } m- \]

\[ P_n(u) = \lim_{m \to \infty} P\{N_m = n, U_m = u\}, \quad u \geq 0 \]

\[ G_n(z) = \sum_{u=0}^{\infty} P_n(u)z^u, \quad |z| \leq 1, \quad \text{and} \quad G_n(1) = \sum_{u=0}^{\infty} P_n(u) = P_n = \text{arbitrary epoch distribution} \]

\[ P_n = \lim_{m \to \infty} P(N_m = n), \quad n = 0, 1, \ldots, N \]

\[ P' = \text{outside observer's distribution} \]

\[ P_n^- = \text{prearrival-epoch probabilities} \]

### 3. BASIC EQUATIONS AND SOLUTION OF THE MODEL

Relating the states of the system at two consecutive epochs \( m- \) and \( (m + 1)- \) and letting \( m \to \infty \), we have the following set of difference equations:

\[ P_0(u - 1) = P_0(u) + \mu_1 P_1(u), \quad (1) \]

\[ P_1(u - 1) = (1 - \mu_1) P_1(u) + a_u P_0(0) + \mu_2 P_2(u) + a_u P_1(0) \mu_1, \quad (2) \]

\[ P_n(u - 1) = (1 - \mu_n) P_n(u) + a_n (1 - \mu_{n-1}) P_{n-1}(0) + \mu_{n+1} P_{n+1}(u) + a_u P_n(0) \mu_n, \quad 2 \leq n \leq N - 2, \quad (3) \]

\[ P_{N-1}(u - 1) = (1 - \mu_{N-1}) P_{N-1}(u) + a_u (1 - \mu_{N-2}) P_{N-2}(0) + \mu_N P_N(u) + a_u \mu_N P_N(0), \quad (4) \]

\[ P_N(u - 1) = (1 - \mu_N) P_N(u) + (1 - \mu_N) a_u P_N(0) + (1 - \mu_{N-1}) a_u P_{N-1}(0). \quad (5) \]

Taking the pgf of (1)–(5), we get

\[ (z - 1) G_0(z) = \mu_1 G_1(z) - \mu_1 P_1(0) - P_0(0), \quad (6) \]

\[ [z - (1 - \mu_1)] G_1(z) = \mu_2 G_2(z) + A(z) P_0(0) + P_1(0) [\mu_1 A(z) + \mu_1 - 1] - \mu_2 P_2(0), \quad (7) \]

\[ [z - (1 - \mu_n)] G_n(z) = \mu_{n+1} G_{n+1}(z) + (1 - \mu_{n-1}) A(z) P_{n-1}(0) + P_n(0) [\mu_n A(z) + \mu_n - 1] - \mu_{n+1} P_{n+1}(0), \quad 2 \leq n \leq N - 2, \quad (8) \]

\[ [z - (1 - \mu_{N-1})] G_{N-1}(z) = \mu_N G_N(z) + (1 - \mu_{N-2}) A(z) P_{N-2}(0) + P_{N-1}(0) [\mu_{N-1} A(z) + \mu_{N-1} - 1] + \mu_N A(z) P_N(0) - \mu_N P_N(0), \quad (9) \]

\[ [z - (1 - \mu_N)] G_N(z) = (1 - \mu_{N-1}) A(z) P_{N-1}(0) + P_N(0) [(1 - \mu_N) A(z) - (1 - \mu_N)]. \quad (10) \]
Adding (6) to (10), we obtain
\[
\sum_{n=0}^{N} G_n(z) = \frac{A(z) - 1}{z - 1} \sum_{n=0}^{N} P_n(0). \tag{11}
\]
Letting \( z \to 1 \) in (11), we have
\[
\sum_{n=0}^{N} P_n(0) = \frac{1}{\alpha}. \tag{12}
\]

The left-hand side of (12) is the probability that an arrival is about to occur or it is arrival rate. Further, (12) is also used as a check over numerical results. To obtain prearrival \( P_{n-1}(0) \) and arbitrary \( P_n \) epochs' probabilities from (6) to (10), we first evaluate \( P_n(0) \) \( (0 \leq n \leq N) \) in the following manner. Setting \( z = 1 \) in (10), we get
\[
P_{N-1}(0) = \frac{\mu N}{1 - \mu_{N-1}} P_N. \tag{13}
\]
Again, setting \( z = 1 - \mu_N \) in (10) yields
\[
P_N(0) = \frac{(1 - \mu_{N-1})A(1 - \mu_N)}{(1 - \mu_N)[1 - A(1 - \mu_N)]} P_{N-1}(0). \tag{14}
\]
Now setting \( z = 1 - \mu_{N-1} \) in (9), we obtain
\[
P_{N-2}(0) = \frac{1}{(1 - \mu_{N-2})A(1 - \mu_{N-1})} \left[ (\mu_N - \mu_N A(1 - \mu_{N-1}))P_N(0) \right.
\]
\[+ (1 - \mu_{N-1} - \mu_{N-1} A(1 - \mu_{N-1})) P_{N-1}(0) - \mu_N G_N(1 - \mu_{N-1}) \right], \tag{15}
\]
where \( G_N(1 - \mu_{N-1}) \) can be obtained from (10) by setting \( z = 1 - \mu_i, \ i = N - 1, N - 2, \ldots, 2 \) and is given by
\[
G_N(1 - \mu_i) = \frac{1}{\mu_N - \mu_i} \left[ (1 - \mu_{N-1})A(1 - \mu_i)P_{N-1}(0) + P_N(0)(1 - \mu_N)(A(1 - \mu_i) - 1) \right], \tag{16}
\]
\[i = N - 1, N - 2, \ldots, 1.
\]

It may be remarked here that other values such as \( G_N(1 - \mu_{N-2}) \) which are being found above are needed in other equations; see, e.g., equation (18). Now setting \( z = 1 - \mu_n \) in (8) yields
\[
P_{n-1}(0) = \frac{1}{(1 - \mu_{n-1})A(1 - \mu_n)} \left[ \mu_{n+1} P_{n+1}(0) \right.
\]
\[+ \left( 1 - \mu_n - \mu_n A(1 - \mu_n) \right) P_n(0) - \mu_{n+1} G_{n+1}(1 - \mu_n) \right], \tag{17}
\]
\[n = N - 2, N - 3, \ldots, 2,
\]
where \( G_{n+1}(1 - \mu_n), \ n = N - 2, N - 3, \ldots, 2 \) can be obtained from (9) by setting \( z = 1 - \mu_j, \ j = N - 2, N - 3, \ldots, 2 \) in (9) and setting \( z = 1 - \mu_k, \ k = N - 3, N - 4, \ldots, 2 \) in (8). These substitutions lead to
\[
G_{N-1}(1 - \mu_j) = \frac{1}{\mu_{N-1} - \mu_j} \left[ \mu_N G_N(1 - \mu_j) + (1 - \mu_{N-2})A(1 - \mu_j)P_{N-2}(0) \right.
\]
\[+ P_{N-1}(0)\{\mu_{N-1}A(1 - \mu_j) + \mu_{N-1} - 1\} + \mu_N A(1 - \mu_j)P_N(0) - \mu_N P_N(0) \right], \tag{18}
\]
\[j = N - 2, N - 3, \ldots, 1,
\]
\[
G_n(1 - \mu_k) = \frac{1}{\mu_n - \mu_k} \left[ \mu_{n+1} G_{n+1}(1 - \mu_k) + (1 - \mu_{n-1})A(1 - \mu_k)P_{n-1}(0) \right.
\]
\[+ P_n(0)\{\mu_n A(1 - \mu_k) + \mu_n - 1\} - \mu_{n+1} P_{n+1}(0) \right], \tag{19}
\]
\[n = N - 2, N - 3, \ldots, 2, \quad k = n - 1, n - 2, \ldots, 1.
\]
Therefore, \( P_{n-1}(0), \, n = N - 2, N - 3, \ldots, 2 \) can be evaluated recursively using (17)--(19). Now setting \( z = 1 - \mu_1 \) in (7) yields
\[
P_0(0) = \frac{1}{A(1 - \mu_1)} \left[ \mu_2 P_2(0) + (1 - \mu_1 - \mu_1 A(1 - \mu_1)) P_1(0) - \mu_2 G_2(1 - \mu_1) \right],
\]
where \( G_2(1 - \mu_1) \) is known from (19).
Hence, \( P_n(0) \) \((n = 0, 1, \ldots, N)\) are completely known from (13)--(16) and (20) in terms of \( P_N \) which can be easily evaluated using the normalization condition. In the following subsection, we obtain distributions of numbers in the system at various epochs.

3.1. Probability Distribution of Numbers in the System at a Prearrival Epoch
Let \( P_n^- \) \((n = 0, 1, \ldots, N)\) be the probability that an arrival at a slot boundary finds \( n \) messages in the system. Then,
\[
P_n^- = \frac{P_n(0)}{\sum_{n=0}^{N} P_n(0)} = a P_n(0), \quad n = 0, 1, \ldots, N.
\]
It may be noted that to evaluate \( P_n^- \) from (21), we do not require the value of \( P_N \), since it cancels out in the numerator and denominator of the right-hand side of (21).

3.2. Probability Distribution of Numbers in the System at an Arbitrary Epoch
Arbitrary-epoch probabilities \( P_n \) \((n = 1, 2, \ldots, N - 1)\) can be obtained from (10) to (7) by setting \( z = 1 \) in terms of \( P_N \) and are given by
\[
P_{N-1} = \frac{1}{\mu_{N-1}} \left[ \mu_N P_N + P_{N-1}(0)(2\mu_{N-1} - 1) + (1 - \mu_{N-2})P_{N-2}(0) \right],
\]
\[
P_n = \frac{1}{\mu_n} \left[ \mu_{n+1} P_{n+1} + P_n(0)(2\mu_n - 1) + (1 - \mu_{n-1})P_{n-1}(0) - \mu_{n+1} P_{n+1}(0) \right],
\]
\[
2 \leq n \leq N - 2,
\]
\[
P_1 = \frac{1}{\mu_1} \left[ \mu_2 P_2 + P_1(0)(2\mu_1 - 1) + P_0(0) - \mu_2 P_2(0) \right].
\]
As \( P_n(0)'s \) \((n = 0, 1, \ldots, N)\) are known, \( P_{N-1}, P_{N-2}, \ldots, P_1 \) can be obtained recursively from (22)--(24) in terms of the unknown probability \( P_N \). Now, the only unknown probability is \( P_0 \) which can be obtained from (6) by differentiating it once w.r.t. \( z \) and setting \( z = 1 \). We obtain
\[
P_0 = \mu_1 G_1^{(1)}(1),
\]
where \( G_1^{(1)}(1) \) can be obtained recursively from (10) to (7) by differentiating them once w.r.t. \( z \) and setting \( z = 1 \). These are
\[
G_N^{(1)}(1) = \frac{1}{\mu_N} \left[ a (1 - \mu_{N-1}) P_{N-1}(0) + (1 - \mu_N) P_N(0) - P_N \right],
\]
\[
G_{N-1}^{(1)}(1) = \frac{1}{\mu_{N-1}} \left[ \mu_N G_N^{(1)}(1) + a (\mu_{N-1} P_{N-1}(0) \right.
\]
\[
+ (1 - \mu_{N-2}) P_{N-2}(0) + \mu_N P_N(0) - P_{N-1} \right],
\]
\[
G_n^{(1)}(1) = \frac{1}{\mu_n} \left[ \mu_{n+1} G_{n+1}^{(1)}(1) + a (\mu_n P_n(0) + (1 - \mu_{n-1}) P_{n-1}(0) - P_n \right], \quad N - 2 \leq n \leq 2
\]
and
\[
G_1^{(1)}(1) = \frac{1}{\mu_1} \left[ \mu_2 G_2^{(1)}(1) + a (\mu_1 P_1(0) + P_0(0) - P_1 \right].
\]
Now $P_n$ ($n = 0, 1, \ldots, N - 1$) is known from (25) to (22) in terms of $P_N$ which can be obtained using the normalizing condition

$$
\sum_{n=0}^{N} P_n = 1. \tag{30}
$$

The following remark concerning outside observer's distribution seems pertinent. In LASDA, since an outside observer's observation point falls in an interval somewhere between a potential departure and a potential arrival, the probability $P'_j$ ($0 \leq j \leq N$) that the outside observer sees $j$ in the system is the same as $P_j$ ($0 \leq j \leq N$).

4. RELATION BETWEEN PROBABILITIES AT PREARRIVAL AND ARBITRARY EPOCHS

In this section, we obtain a few important results which are used to derive relations between probabilities at prearrival and arbitrary epochs. Furthermore, these results have their own interpretations.

**Lemma 4.1.**

$$
\mu_N P_N = (1 - \mu_{N-1}) P_{N-1}(0), \tag{31}
$$

$$
\mu_n P_n = \mu_n P_n(0) + (1 - \mu_{n-1}) P_{n-1}(0), \quad N - 1 \leq n \leq 2, \tag{32}
$$

$$
\mu_1 P_1 = \mu_1 P_1(0) + P_0(0). \tag{33}
$$

**Proof.** The above results can be easily obtained from (13), (22)–(24) by repeated substitution of $\mu_i P_i$, $i = N, N - 1, \ldots, 1$.

**Lemma 4.2.**

$$
\sum_{i=1}^{N} \mu_i P_i = \sum_{i=0}^{N-1} P_i(0). \tag{34}
$$

Note that the left-hand side is the effective output rate.

**Proof.** Adding (31) to (33), we have (34).

**Lemma 4.3.**

$$
\frac{1 - P_N}{a} = \sum_{i=1}^{N} \mu_i P_i. \tag{35}
$$

This equation shows that the effective input rate is equal to the effective output rate, as it should be.

**Proof.** Using (12) and Lemma 4.2, we get (35).

**Lemma 4.4.** Relations between probabilities at arbitrary ($P_n$) and prearrival ($P_n^-$) epochs are given by

$$
P_N = \frac{(1 - \mu_{N-1})}{a \mu_N} P_{N-1}^-, \tag{36}
$$

$$
P_n = \frac{1}{a} P_n^- + \frac{(1 - \mu_{n-1})}{a \mu_n} P_{n-1}^-, \quad N - 1 \leq n \leq 2, \tag{37}
$$

$$
P_1 = \frac{1}{a} P_1^- + \frac{1}{a \mu_1} P_0^-, \tag{38}
$$

with

$$
P_0 = 1 - \sum_{i=1}^{N} P_i. \tag{39}
$$

**Proof.** Using (12) and (31)–(33), we easily get the above equations.
5. PERFORMANCE MEASURES

As performance measures are functions of state probabilities, they can easily be obtained from $P_n^{-}$ and $P_n$. The probability of blocking (PBL) and the probability that a customer has to wait ($\pi$) are given by $P_N^{-}$ and $\sum_{n=1}^{N-1} P_n^{-}$, respectively. Obviously, $P_0^{-} + \pi + \text{PBL} = 1$. The average number of customers in system ($L$) and in queue ($L_q$) are given by

$$L = \sum_{n=1}^{N} nP_n \quad \text{and} \quad L_q = \sum_{n=2}^{N} (n-1)P_n,$$

respectively.

It may be remarked here that the analysis of waiting time for the $GI/\text{Geom}(n)/1/N$ queue is more complicated. However, for the state independent service queue $GI/\text{Geom}/1/N$, it is carried out in [8]. For $GI/\text{Geom}(n)/1/N$, the mean waiting time, however, can be obtained from Little’s rule and is given by

$$W_q = \frac{aL_q}{(1 - P_N^{-})}.$$

6. NUMERICAL RESULTS

The procedure discussed in this paper has been tested for a variety of interarrival-time distributions including distributions with finite supports. The computations were performed on an IBM 386 PC. All the numerical results were checked to satisfy equation (12). For geometric interarrival-time distribution, state probabilities matched those obtained in [9] for the Geom(n)/Geom(n)/1/N queue by assuming state independent arrival probabilities. In Table 1, $P_n(C)$ indicates that the arbitrary epoch probabilities of Geom/Geom(n)/1/N were obtained using [9]. In this table, we also give results for some other interarrival-time distributions: deterministic and arbitrary. In addition, some performance measures are given at the bottom of Table 1.

<table>
<thead>
<tr>
<th>$GI = \text{Geom, } a = 5$</th>
<th>$GI = D, a = 4$</th>
<th>$GI = \text{arbitrary}^*$</th>
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<tr>
<td>$L_q$</td>
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<td>1.0659</td>
</tr>
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* $a_1 = 0.5, a_9 = 0.2, a_{10} = 0.1, a_{25} = 0.2$, and $a = 8.1$.

In Table 2, we provide numerical results for some large values of $N$ and geometric interarrival-time distribution.

It may be remarked here that the problem of instability arises for higher values of $N$. This instability is due to the closeness of $\mu_n$ and $\mu_{n-1}$ values when $N$ is large. Since the accuracy of computed probabilities deteriorates as $N$ decreases from large to small, the probabilities of loss, however, are correct. In some cases, the problem of inaccuracies may be resolved by reducing the size of the waiting space as right-tail probabilities become almost negligible. Similar remarks apply to the $D/\text{Geom}(n)/1/N$ queue, results of which for large $N$ are given in Table 3.
It is worth mentioning that computation of state probabilities with small magnitudes, i.e., both right and left tails especially when $N$ is large, is difficult. This is similar to what has already been noted in [10].

### 7. CONCLUSIONS

The procedure described in this paper can be used to analyze other queueing models such as the discrete-time multiserver $GI/Geom/c/N$ queue. It may be noted that the results for the $GI/Geom/c/N$ queue cannot be obtained from those of the $GI/Geom(n)/1/N$ queue due to the
Table 3. $D/Geom(n)/1/N$, $a = 4$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\mu_n$</th>
<th>$P_n$</th>
<th>$P_{n-}$</th>
<th>$P_n$</th>
<th>$P_{n-}$</th>
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The special nature of service pattern. However, in the continuous-time case, it is possible to derive the results for $GI/M/c/N$ from those of $GI/M(n)/1/N$ by taking

$$\mu_n = \begin{cases} n\mu, & 0 \leq n < c, \\ c\mu, & n \geq c. \end{cases}$$

The model $GI/Geom/c/N$ is under investigation and the results will be reported in the near future.
APPENDIX

Algorithm for computing state probabilities in GI/Geom(n)/1/N (LAS-DA).

Step 1: Set $P_N = 1$.

Step 2: Compute $P_{N-1}(0)$ and $P_N(0)$ from (13) and (14), respectively.

Step 3: For $i = N - 1, N - 2, \ldots, 1$, compute $G_N(1 - \mu_i)$ from (16).

Step 4: Compute $P_{N-2}(0)$ using (15).

Step 5: For $j = N - 2, \ldots, 1$, compute $G_{N-1}(1 - \mu_j)$ using (18).

Step 6: For $n = N - 2, \ldots, 2$, compute $P_{n-1}(0)$ from (17) and for $n = N - 2, N - 3, \ldots, 2$ and $k = n - 1, n - 2, \ldots, 1$, compute $G_n(1 - \mu_k)$ from (19).

Step 7: Compute $P_0(0)$ from (20).

Step 8: For $n = 0, 1, \ldots, N$, compute $P_n^-$ using (21).

Step 9: Compute $P_{N-1}$ using (22).

Step 10: For $n = N - 2, N - 3, \ldots, 2$, compute $P_n$ using (23).

Step 11: Compute $P_1$ using (24).

Step 12: Compute $G_N^{(1)}(1)$ and $G_{N-1}^{(1)}(1)$ using (26) and (27).

Step 13: For $n = N - 2, N - 3, \ldots, 2$, compute $G_n^{(1)}(1)$ using (28).

Step 14: Compute $G_1^{(1)}(1)$ using (29).

Step 15: Compute $P_0$ using (25).

Step 16: Sum $= \sum_{n=0}^{N} P_n$.

Step 17: For $n = 0, 1, \ldots, N$, compute $P_n = P_n / \text{Sum}$.

The overall complexity of the algorithm is $O(N^2)$.

REFERENCES