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Robust H_∞ Filtering of Stationary Discrete-Time Linear Systems with Stochastic Uncertainties*

E. Gershon† U. Shaked
Dept. Electrical Engineering-Systems
Tel-Aviv University
Tel-Aviv, Israel

I. Yaesh
Taas Israel Industries
Advanced Sys. Div.
P.O.B. 1044/77
Ramat Hasharon, Israel

Abstract

The problem of H_∞ filtering of stationary discrete-time linear systems with stochastic uncertainties in the state space matrices is addressed, where the uncertainties are modeled as white noise. The relevant cost function is the expected value of the standard H_∞ performance index with respect to the uncertain parameters. A previously developed stochastic bounded real lemma is applied which results in a modified Riccati inequality. This inequality is expressed in a linear matrix inequality form whose solution provides the filter parameters. The method proposed is applied also to the case where, in addition to the stochastic uncertainty, other deterministic parameters of the system are not perfectly known and are assumed to lie in a given polytope. The problem of mixed H_2/H_∞ filtering for the above system is also treated. The theory developed is demonstrated by a simple tracking example.

1 Introduction

The analysis and design of controllers and estimators for systems with stochastic uncertainties, which ensure a worst case performance bound in the H_∞ style, have received recently much attention (Boyd *et al.*, 1994; Costa and Kubrusly, 1996; Dragan and Morozan, 1997; Dragan and Stoica, 1998; El Ghaoui, 1995; Gershon *et al.*, 1998; Hinriechsen and Pritchard, 1998). An approach in which the parameter uncertainties are modeled as white noise processes in a linear system has been developed in (Boyd *et al.*, 1994; Dragan and Stoica, 1998; Gershon *et al.*, 1998) for the discrete-time state-feedback problem, and in (Dragan and Morozan, 1997; El Ghaoui, 1995) for continuous-time counterpart. The estimation problem of stochastic systems has been solved in (Dragan and Morozan, 1997; Hinriechsen and Pritchard, 1998) and (Dragan and Stoica, 1998; Gershon *et al.*, 1998) for the continuous-time and the discrete-time cases, respectively. Such models of uncertainty are encountered in many areas of applications (see Costa and Kubrusly, 1996 and the references therein).

Recently, the solution of the output-feedback problem for stochastic time-varying uncertain systems has been obtained for both the finite and infinite time cases (Gershon *et al.*, 1998). This solution is based on solving the filtering part using a Luenberger-type observer, and applying a filtering-type bounded real lemma (BRL). It results in a modified-Riccati recursion, which guarantees a given H_∞ estimation level, while minimizing an upper-bound on the covariance

*This work was supported by the C&M Maus Chair at Tel Aviv University.
†E-mail: *gershon@eng.tau.ac.il*

of the estimation error. The latter minimization was an indispensable part of the solution procedure there. The case of stationary filtering is also treated in (Gershon *et al.*, 1998).

An alternative and a more common approach considers the uncertainties to lie in a convex-bounded domain (polytopic type). This approach has been adopted by (Geromel and deOliviera, 1998; Geromel *et al.*, 1998; Palhares *et al.*, 1998). In (Palhares and Peres, 1998), applying the known H_∞ BRL (deSouza and Xie, 1992) on the uncertain system, a Riccati inequality has been obtained whose solution over the whole uncertainty polytope guarantees the existence of a solution to the problem by a single filter. This Riccati inequality has been expressed in a linear matrix inequality (LMI) form that is affine in the uncertain parameters. A single solution to the latter for all the vertices of the uncertainty polytope produced the required result (Gahinet, 1996). The mixed H_2/H_∞ problem has also been solved in (Palhares and Peres, 1998).

In the present paper we treat the general case where the stochastic uncertainty appears in all the system matrices, and where we allow for correlations between the uncertain parameters. This problem has been partially treated in (Dragan and Stoica, 1998), however, the system there did not allow for uncertainty in the measurement matrix and for correlations between the parameters.

We use the techniques of (Li and Fu, 1997) as applied in the solution of the deterministic polytopic problem (Geromel and deOliviera, 1998; Palhares and Peres, 1998). Necessary and sufficient conditions are derived for the existence of a solution in terms of LMIs. Our solution is based on a previously developed stochastic BRL (Gershon et al, 1998). The latter solution is extended to the case where the deterministic part of the system matrices lie in a convex bounded domain of a polytopic-type. Our theory is also applicable to the case where the covariance matrices of the stochastic parameters are not perfectly known and lie in a polytopic domain. We also solve the mixed H_2/H_∞ problem where, of all the filters that solve the stochastic H_∞ filtering problem, the one that minimizes an upper-bound on the estimation error variance is found. The applicability of our method is demonstrated in a gain-scheduled estimation example. We treat there a guidance motivated tracking problem and compare the results with those obtained by the Kalman-filter.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, (respectively, $P\geq 0$) for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (respectively, semi-definite). The space of square summable functions over $[0 \quad \infty]$ is denoted by $l_2[0 \quad \infty]$, and $||.||_2$ stands for the standard l_2-norm, $||u||_2=(\Sigma_{k=0}^{\infty}u_k^T u_k)^{1/2}$. We also denote by $\underset{v}{E}\{\cdot\}$ expectation with respect to v, by $\mathrm{Tr}\{\cdot\}$ the trace of a matrix and by δ_{ij} the Kronecker delta function.

2 Problem Formulation

We consider the following system:

$$\begin{array}{rl} x_{k+1} = (A+Dv_k)x_k+(B_1+Gr_k)w_k, & x_0=0 \\ y_k = (C+F\zeta_k)x_k+D_{21}w_k & \\ z_k = Lx_k & \end{array} \tag{1}$$

where $x_k\in\mathcal{R}^n$ is the system states, $y_k\in\mathcal{R}^r$ is the measurement, $w_k\in\mathcal{R}^q$ is the exogenous disturbance signal, $z_k\in\mathcal{R}^m$ is the state combination to be estimated and $\{v_k\}$, $\{r_k\}$ and $\{\zeta_k\}$ are standard random scalar sequences with zero mean that satisfy:

$$E\{v_kv_j\}=\delta_{kj},\quad E\{r_kr_j\}=\delta_{kj},\quad E\{\zeta_k\zeta_j\}=\delta_{kj},\quad E\{\zeta_kv_j\}=\alpha_k\delta_{kj}, |\alpha_k|<1,\quad \forall k,j \quad \geq 0.$$

and where $\{r_k\}$ is uncorrelated with $\{v_k\}$ and $\{\zeta_k\}$.

We consider the following filter for the estimation of z_k:

$$\begin{array}{l} \hat{x}_{k+1} = A_f \hat{x}_k + B_f y_k, \quad \hat{x}_0 = 0 \\ \hat{z}_k = C_f \hat{x}_k. \end{array} \tag{2}$$

Denoting

$$e_k = x_k - \hat{x}_k, \quad \xi_k^T = [\ x_k^T \quad \hat{x}_k^T\ ] \quad \text{and} \quad \tilde{z}_k = z_k - \hat{z}_k, \tag{3a-c}$$

we define, for a given scalar $\gamma > 0$, the following performance index

$$J_S \stackrel{\triangle}{=} \underset{v,\zeta,r}{E} \left\{ ||\tilde{z}||_2^2 - \gamma^2 ||w_k||_2^2 \right\}. \tag{4}$$

The problems addressed in this paper are :

i) Stochastic H_∞ filtering problem: Given $\gamma > 0$, and assume that w_k may depend only on the present and the past values of x_k, find an asymptotically stable linear filter of the form (2) that leads to an estimation error $\tilde{z}_k$ for which J_S of (4) is negative for all nonzero $\{w_k\}$ where $\{w_k\} \in l_2[0 \quad \infty)$.

ii) Stochastic mixed H_2/H_∞ filtering problem: Of all the asymptotically stable filters that solve problem (i), find the one that minimizes an upper-bound on the estimation error variance :

$$\lim_{k \to \infty} \underset{w,v,\zeta,r}{E} \left\{ \tilde{z}_k^T \tilde{z}_k \right\}.$$

3 Solutions

3.1 A BRL for systems with stochastic uncertainty

We bring first the lemma that was derived in (Gershon *et al.*, 1998) for the system

$$\begin{array}{l} x_{k+1} = (A + D_1 v_k + D_2 \zeta_k) x_k + (B + G r_k) w_k \\ z_k = L x_k \end{array} \tag{5}$$

where the scalar sequences $\{v_x\}, \{r_k\}$ and $\{\zeta_k\}$ are defined in section 2. The exogenous disturbance $\{w_k\}$ is assumed to be of finite energy and may depend only on current and past values of the state-vector. Considering the cost function

$$\hat{J} = \underset{v,r,\zeta}{E} \left\{ ||z_k||_2^2 - \gamma^2 ||w_k||_2^2 \right\}$$

and using the arguments of (Gershon *et al.*, 1998) the following holds:

Lemma 1 (Gershon *et al.*, 1998): *Consider the system of (5). Given $\gamma > 0$, a necessary and sufficient condition for $\hat{J}$ to be negative for all nonzero $\{w_k\}$ where $\{w_k\} \in l_2[0 \quad \infty)$ is that there exists a solution Q to*

$$-Q + A^T Q A + A^T Q B \Theta^{-1} B^T Q A + L^T L + D_1^T Q D_1 + D_2^T Q D_2 + \alpha [D_1^T Q D_2 + D_2^T Q D_1] < 0$$

which satisfies $\Theta > 0$, where

$$\Theta \stackrel{\triangle}{=} \gamma^2 I_q - B^T Q B - G^T Q G.$$

3.2 Stochastic H_∞-filtering

Problem (i) is solved by applying Lemma 1. Considering the system of (1) and the definitions of (3b) we obtain

$$\begin{aligned} \xi_{k+1} &= [\tilde{A} + \tilde{D}_1 v_k + \tilde{D}_2 \zeta_k]\xi_k + [\tilde{B} + \tilde{G} r_k] w_k \\ \tilde{z}_k &= \tilde{C}\xi_k \end{aligned} \tag{6}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ B_f C & A_f \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 \\ B_f D_{21} \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} G \\ 0 \end{bmatrix},$$

$$\tilde{D}_1 = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{D}_2 = \begin{bmatrix} 0 & 0 \\ B_f F & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} L & -C_f \end{bmatrix}. \tag{7a-f}$$

We arrive at the following result:

Theorem 1: *Consider the system of (6) and (4). Given $\gamma > 0$, a necessary and sufficient condition for J_S to be negative for all nonzero $\{w_k\}$, where $w_k \in l_2[0 \quad \infty)$, is that there exist*

$$R = R^T \in \mathcal{R}^{n\times n}, \quad W = W^T \in \mathcal{R}^{n\times n},, \quad Z \in \mathcal{R}^{n\times r}, \quad S \in \mathcal{R}^{n\times n} \quad \textit{and} \quad T \in \mathcal{R}^{m\times n},$$

such that

$$\Sigma(R, W, Z, S, T, \gamma^2) > 0 \tag{8}$$

where

$$\Sigma(R, W, Z, S, T, \gamma^2) \triangleq \left[\begin{array}{ccccccccccc}
R & 0 & 0 & 0 & 0 & 0 & 0 & -RA & 0 & -RB_1 & 0 \\
0 & W & 0 & 0 & 0 & 0 & 0 & -WA - ZC - S & S & -WB_1 - ZD_{21} & 0 \\
0 & 0 & R & 0 & 0 & 0 & 0 & -RD & 0 & 0 & 0 \\
0 & 0 & 0 & W & 0 & 0 & 0 & -WD - \alpha ZF & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & W & 0 & 0 & -\bar{\alpha} ZF & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & -RG & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & W & 0 & 0 & -WG & 0 \\
-A^T R & -A^T W - C^T Z^T - S^T & -D^T R & -D^T W - \alpha F^T Z^T & -\bar{\alpha} F^T Z^T & 0 & 0 & R & 0 & 0 & T^T - L^T \\
0 & S^T & 0 & 0 & 0 & 0 & 0 & 0 & W & 0 & -T^T \\
-B_1^T R & -B_1^T W - D_{21}^T Z^T & 0 & 0 & 0 & -G^T R & -G^T W & 0 & 0 & \gamma^2 I_q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & T - L & -T & 0 & I_m
\end{array}\right] \tag{9}$$

In the latter case, a filter in the form of (2) that achieves the negative J_S is given by:

$$A_f = -W^{-1}S, \quad B_f = -W^{-1}Z \quad and \quad C_f = T. \tag{10a-c}$$

Proof : Applying Lemma 1 on (6) and (7a-f) we obtain the following Riccati inequality

$$-Q + \tilde{A}^T Q\tilde{A} + \tilde{A}^T Q\tilde{B}\tilde{\Theta}^{-1}\tilde{B}^T Q\tilde{A} + \tilde{C}^T\tilde{C} + \tilde{D}_1^T Q\tilde{D}_1 + \tilde{D}_2^T Q\tilde{D}_2 + \alpha[\tilde{D}_2^T Q\tilde{D}_1 + \tilde{D}_1^T Q\tilde{D}_2] < 0$$

$$\tilde{\Theta} \triangleq \gamma^2 I_q - \tilde{B}^T Q\tilde{B} - \tilde{G}^T Q\tilde{G} \tag{11a,b}$$

Noticing that

$$\tilde{D}_1^T Q\tilde{D}_1 + \tilde{D}_2^T Q\tilde{D}_2 + \alpha\tilde{D}_2^T Q\tilde{D}_1 + \alpha\tilde{D}_1^T Q\tilde{D}_2 = (\tilde{D}_1 + \alpha\tilde{D}_2)^T Q(\tilde{D}_1 + \alpha\tilde{D}_2) + \bar{\alpha}^2\tilde{D}_2^T Q\tilde{D}_2$$

where $\bar{\alpha} \triangleq (1-\alpha^2)^{0.5}$, (11a) is written as

$$-Q + \tilde{A}^T Q\tilde{A} + \tilde{A}^T Q\tilde{B}\tilde{\Theta}^{-1}\tilde{B}^T Q\tilde{A} + \tilde{C}^T\tilde{C} + (\tilde{D}_1 + \alpha\tilde{D}_2)^T Q(\tilde{D}_1 + \alpha\tilde{D}_2) + \bar{\alpha}^2\tilde{D}_2^T Q\tilde{D}_2 < 0 \tag{12}$$

The latter can be readily put in the following LMI form

$$\hat{\Gamma}(Q) \triangleq \begin{bmatrix} -Q^{-1} & 0 & 0 & 0 & \tilde{A} & \tilde{B} & 0 \\ 0 & -Q^{-1} & 0 & 0 & (\tilde{D}_1 + \alpha\tilde{D}_2) & 0 & 0 \\ 0 & 0 & -Q^{-1} & 0 & \bar{\alpha}\tilde{D}_2 & 0 & 0 \\ 0 & 0 & 0 & -Q^{-1} & 0 & \tilde{G} & 0 \\ \tilde{A}^T & (\tilde{D}_1 + \alpha\tilde{D}_2)^T & \bar{\alpha}\tilde{D}_2^T & 0 & -Q & 0 & \tilde{C}^T \\ \tilde{B}^T & 0 & 0 & \tilde{G}^T & 0 & -\gamma^2 I_q & 0 \\ 0 & 0 & 0 & 0 & \tilde{C} & 0 & -I_m \end{bmatrix} < 0, \tag{13}$$

where we look for $Q > 0$ that satisfies the LMI.
Similarly to (Palhares and Peres, 1998), Q and Q^{-1} are partitioned as

$$Q \triangleq \begin{bmatrix} X & M \\ M^T & U \end{bmatrix} \text{and} \quad Q^{-1} \triangleq \begin{bmatrix} Y & N \\ N^T & V \end{bmatrix}.$$

Since (Palhares and Peres, 1998)

$$\begin{bmatrix} Y & I_n \\ I_n & X \end{bmatrix} > 0,$$

we therefore require below that

$$Y > 0, \quad \text{and} \quad X > Y^{-1}.$$

We also note that $I - XY = MN^T$ is of rank n. Defining :

$$J \triangleq \begin{bmatrix} Y & I_n \\ N^T & 0 \end{bmatrix} \quad \text{and} \quad \tilde{J} \triangleq diag\,[\ QJ, \quad QJ, \quad QJ, \quad QJ, \quad J, \quad I \quad I\ ]$$

we pre- and post-multiply (13) by $\tilde{J}^T$ and $\tilde{J}$, respectively. Using (7a-f) we obtain that (13) is equivalent to the requirement that

$$\left[\begin{array}{cccccccccccc}
-Y & -I_n & 0 & 0 & 0 & 0 & 0 & 0 & AY & A & B_1 & 0 \\
-I_n & -X & 0 & 0 & 0 & 0 & 0 & 0 & XAY+ZCY+\hat{Z} & XA+ZC & XB_1+ZD_{21} & 0 \\
0 & 0 & -Y & -I_n & 0 & 0 & 0 & 0 & DY & D & 0 & 0 \\
0 & 0 & -I_n & -X & 0 & 0 & 0 & 0 & XDY+\alpha ZFY & XD+\alpha ZF & 0 & 0 \\
0 & 0 & 0 & 0 & -Y & -I_n & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -I_n & -X & 0 & 0 & \bar{\alpha} ZFY & \bar{\alpha} ZF & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -Y & -I_n & 0 & 0 & G & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -I_n & -X & 0 & 0 & XG & 0 \\
YA^T & YA^TX+YC^TZ^T+\hat{Z}^T & YD^T & YD^TX+\alpha YF^TZ^T & 0 & \bar{\alpha} YF^TZ^T & 0 & 0 & -Y & -I_n & 0 & YL^T-\hat{Z}^T \\
A^T & A^TX+C^TZ^T & D^T & D^TX+\alpha F^TZ^T & 0 & \bar{\alpha} F^TZ^T & 0 & 0 & -I_n & -X & 0 & L^T \\
B_1^T & B_1^TX+D_{21}^TZ^T & 0 & 0 & 0 & 0 & G^T & G^TX & 0 & 0 & -\gamma^2 I_q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & LY-\hat{Z} & L & 0 & -I_m
\end{array}\right] < 0, \quad X > Y^{-1} > 0 \qquad (14)$$

where we define

$$Z \triangleq MB_f, \quad \tilde{Z} \triangleq C_f N^T \quad \text{and} \quad \hat{Z} \triangleq MA_f N^T. \qquad (15)$$

Pre- and post-multiplying (14) by Υ and Υ^T, respectively, where

$$\Upsilon \triangleq diag\{\begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, I_q, I_m\},$$

and where we denote

$$R \triangleq Y^{-1},$$

we obtain, defining

$$S \triangleq \hat{Z}R \quad \text{and} \quad T \triangleq \tilde{Z}R \qquad (16)$$

the requirement of (8), where we replace $X - R$ by W and multiply the resulting inequality by -1. If a solution to (8) exists it follows from (15) that

$$A_f = M^{-1}\hat{Z}N^{-T}, \quad B_f = M^{-1}Z \quad \text{and} \quad C_f = \tilde{Z}N^{-T}. \qquad (17)$$

Denoting the transfer function matrix of the filter of (2) by $H_{\hat{z}y}$ we find that :

$$H_{\hat{z}y}(\rho) = \tilde{Z}N^{-T}(\rho I - M^{-1}\hat{Z}N^{-T})^{-1}M^{-1}Z,$$

where ρ is the Z-transform variable. The latter equation is similar to:

$$H_{\hat{z}y}(\rho) = \tilde{Z}(\rho M N^T - \hat{Z})^{-1} Z = \tilde{Z}[\rho(I - XY) - \hat{Z}]^{-1} Z,$$

and (10) follows using (16). □□□

Due to the affinity of Σ of (8) in A, B_1, C and D_{21}, the result of Theorem 1 can be easily extended to the case where these matrices lie in convex bounded domain. In this case, it is required that (8) holds for all the vertices of the uncertain polytope for a single assembly of (R, S, Z, T, W).

Assuming that $A, B_1, C,$ and D_{21} lie in the following uncertainty polytope

$$\bar{\Omega} \stackrel{\triangle}{=} \{(A, B_1, C, D_{21}) | (A, B_1, C, D_{21}) = \sum_{i=1}^{l} \tau_i (A_i, B_{1i}, C_i, D_{21,i}); \tau_i \geq 0; \sum_{i=1}^{l} \tau_i = 1\}.$$

and denoting the set of the l vertices of this polytope by $\bar{\Psi}$ we obtain the following result:

Corollary 1: *Consider the system of (1) and (2). The performance index of (4) is negative for a given $\gamma > 0$, for any energy bounded nonzero $\{w_k\}$ and for any $(A, B_1, C, D_{21}) \in \bar{\Omega}$ if (8) is satisfied for all the vertices in $\bar{\Psi}$ by a single (R, Z, S, T, W). In the latter case the filter matrices are given by (10).*

3.3 Robust mixed Stochastic H_2/H_∞ filtering

The mixed stochastic H_2/H_∞ filter design is achieved by considering the filters that satisfy the H_∞ requirement and finding the one that minimizes an upper-bound on the estimation error variance. The latter is described by the following H_2 objective function :

$$J_2 = \lim_{k \to \infty} \underset{w,v,r,\zeta}{E} \left\{ \tilde{z}_k^T \tilde{z}_k \right\} = ||H_{\tilde{z}w}||_2^2,$$

where $H_{\tilde{z}w}$ is the transference in the system of (6), from w to $\tilde{z}$, and where we assume that the pair (A, C) of (1) is observable.

Denoting

$$\bar{P} \stackrel{\triangle}{=} \lim_{k \to \infty} \underset{w,v,r,\zeta}{E} \left\{ \xi_k \xi_k^T \right\},$$

we readily find that :

$$||H_{\tilde{z}w}||_2^2 = Tr\{\tilde{C} \bar{P} \tilde{C}^T\}$$

where $\bar{P} = \lim_{k \to \infty} P_k$ and

$$-P_{k+1} + \tilde{A} P_k \tilde{A}^T + \tilde{D}_1 P_k \tilde{D}_1^T + \tilde{D}_2 P_k \tilde{D}_2^T + \alpha(\tilde{D}_1 P_k \tilde{D}_2^T + \tilde{D}_2 P_k \tilde{D}_1^T) + \tilde{B} \tilde{B}^T = 0.$$

We are interested in deriving the corresponding observability-type result (Zhou et al., 1996) taking into account the stochastic nature of $\{v_k\}, \{\zeta_k\}, \{r_k\}$. Considering the following recursion

$$\tilde{Q}_k = \tilde{A}^T \tilde{Q}_{k+1} \tilde{A} + \tilde{D}_1^T \tilde{Q}_{k+1} \tilde{D}_1 + \tilde{D}_2^T \tilde{Q}_{k+1} \tilde{D}_2 + \alpha(\tilde{D}_1^T \tilde{Q}_{k+1} \tilde{D}_2 + \tilde{D}_2^T \tilde{Q}_{k+1} \tilde{D}_1) + \tilde{C}^T \tilde{C},$$

we obtain :

$$Tr\{P_{k+1} \tilde{Q}_{k+1} - P_k \tilde{Q}_k\} = Tr\{[\tilde{A} P_k \tilde{A}^T + \tilde{D}_1 P_k \tilde{D}_1^T + \tilde{D}_2 P_k \tilde{D}_2^T + \alpha(\tilde{D}_1 P_k \tilde{D}_2^T + \tilde{D}_2 P_k \tilde{D}_1^T) + \tilde{B} \tilde{B}^T] \tilde{Q}_{k+1}\}$$
$$-Tr\{P_k [\tilde{A}^T \tilde{Q}_{k+1} \tilde{A} + \tilde{D}_1^T \tilde{Q}_{k+1} \tilde{D}_1 + \tilde{D}_2^T \tilde{Q}_{k+1} \tilde{D}_2 + \alpha(\tilde{D}_1^T \tilde{Q}_{k+1} \tilde{D}_2 + \tilde{D}_2^T \tilde{Q}_{k+1} \tilde{D}_1) + \tilde{C}^T \tilde{C}]\}.$$

Since

$$\lim_{k\to\infty} Tr\{P_{k+1}\tilde{Q}_{k+1} - P_k\tilde{Q}_k\} = 0$$

and $Tr\{\alpha\beta\} = Tr\{\beta\alpha\}$ it follows that:

$$Tr\{\tilde{C}\bar{P}\tilde{C}^T\} = Tr\{\tilde{B}^T\tilde{Q}\tilde{B}\}$$

where

$$\tilde{Q} = \lim_{k\to\infty} \tilde{Q}_k.$$

Defining :

$$\Gamma(\tilde{Q}) = -\tilde{Q} + \tilde{A}^T\tilde{Q}\tilde{A} + \tilde{D}_1^T\tilde{Q}\tilde{D}_1 + \tilde{D}_2^T\tilde{Q}\tilde{D}_2 + \alpha(\tilde{D}_1^T\tilde{Q}\tilde{D}_2 + \tilde{D}_2^T\tilde{Q}\tilde{D}_1) + \tilde{C}^T\tilde{C},$$

and denoting the set

$$\Omega \triangleq \{\hat{Q}|\Gamma(\hat{Q}) \leq 0 \quad ; \quad \hat{Q} > 0\},$$

we obtain from the monotonicity property of the equation $\Gamma(\tilde{Q}) = 0$ that

$$J_B = Tr\{\tilde{B}^T\hat{Q}\tilde{B}\} \geq Tr\{\tilde{B}^T\tilde{Q}\tilde{B}\}, \quad \forall\hat{Q} \in \Omega. \tag{18}$$

To solve the stochastic mixed H_2/H_∞ problem we seek to minimize an upper-bound on J_B over Ω. Namely, assuming that there exists a solution to (12) we consider the following LMI:

$$\tilde{\Gamma}(\bar{Q}, H) \triangleq \begin{bmatrix} H & -\tilde{B}^T\bar{Q} \\ -\bar{Q}\tilde{B} & \bar{Q} \end{bmatrix} > 0, \quad \bar{Q} \in \Omega \tag{19}$$

where we want to find $\bar{Q}$ and H that minimize

$$J_\tau = Tr\{H\}. \tag{20}$$

It follow from (12) that

$$-Q + \tilde{A}^TQ\tilde{A} + \tilde{C}^T\tilde{C} + (\tilde{D}_1 + \alpha\tilde{D}_2)^TQ(\tilde{D}_1 + \alpha\tilde{D}_2) + \bar{\alpha}^2\tilde{D}_2^TQ\tilde{D}_2 < -\tilde{A}^TQ\tilde{B}\tilde{\Theta}^{-1}\tilde{B}^TQ\tilde{A}$$

Restricting, therefore, $\bar{Q}$ of (19) to the set of the solutions to (13), we clearly have that $\bar{Q} \in \Omega$. We are looking for Q and H that satisfy $\hat{\Gamma}(Q) < 0$ and $\tilde{\Gamma}(Q, H) > 0$ so that $Tr(H)$ is minimized.

Notice that the matrix built from the first and sixth column and row blocks in (13) resembles $\tilde{\Gamma}$. Hence Q of (12) satisfies also (19) for $H = \gamma^2 I_q$. This is in accordance with the well known fact that the solution to the H_∞ problem is an upper-bound to the solution of the corresponding H_2 problem (Bernstein and Haddad, 1989). We are clearly looking for a tighter bound on J_B.

The minimization of (20) can be put in LMI form that is affine in B_f, by pre- and post-multiplying (19) by $diag\{I, J^T\}$ and $diag\{I, J\}$, respectively, substituting for $\tilde{B}$ (using (7) and (17)) and pre- and post-multiplying the result by $\bar{\Lambda}$ and $\bar{\Lambda}^T$, respectively, where :

$$\bar{\Lambda} \triangleq diag\{I, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}\}.$$

We obtain the following result :

Theorem 2: *Consider the system of (6) and (4). Given $\gamma > 0$, a filter that yields $J_S < 0$ for all nonzero $\{w_k\} \in l_2[0 \quad \infty)$ and minimizes (18) is obtained if there exists a solution*

(R, S, Z, T, W, H) *to (8). The minimizing filter is obtained by simultaneously solving (8) and* $\bar{\Upsilon} > 0$ *and minimizing (20), where*

$$\bar{\Upsilon} \triangleq \begin{bmatrix} H & -B_1^T R & -B_1^T W - D_{21}^T Z^T \\ -RB_1 & R & 0 \\ -WB_1 - ZD_{21} & 0 & W \end{bmatrix}.$$

The filter matrices are given then by (10).

4 Example

We illustrate the use of the above theory in a guidance motivated tracking problem, where a scheduled estimation is obtained in spite of significant noise intensity that is encountered in the measurement of the scheduling parameter.

Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \omega$$

where x_1 is the relative separation between an interceptor and an evader, normal to a collision course, x_2 is its derivative, with respect to time, and ω represents the relative interceptor-evader maneuvers. The state x_2 has to be estimated via the following measurements :

$$\begin{aligned} y &= x_1/R + v_1 \\ R_m &= R(1 + v_2) \end{aligned} \qquad \text{(21a-b)}$$

where v_1 and v_2 are, additive and multiplicative white noise zero-mean signals in the bearing measurement and the measurement R_m of the range R, respectively. These noise signals stem from the characteristics of the measuring devices. Substituting (21b) in (21a) we have

$$y = x_1(1 + v_2)/R_m + v_1.$$

Given the variances of v_1 and v_2, it is desired to obtain an estimate that is scheduled by the measurement of R_m and achieves a given H_∞ estimation level. We solve the problem in discrete-time and we, therefore, apply sampling of period T. The resulting discrete-time system is the one described in (1) with $D = 0$, $G = 0$,

$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} T^2/2 & 0 \\ T & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & \bar{\rho} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The measurement of R_k is, then, given by

$$R_{m,k} = R_k(1 + \zeta_k).$$

The time-varying matrix C_k in (1) is

$$C_k = \begin{bmatrix} \frac{1}{Rm_k} & 0 \end{bmatrix},$$

and the time-varying version of F in (1) becomes $F_k = \begin{bmatrix} \frac{\sigma}{Rm_k} & 0 \end{bmatrix}$, where σ is the standard deviation error.

In our example we take $T = 0.025$sec, $\bar{\rho} = 0.001$, and $\sigma = 0.3$. The discretized version of the range R_k is

$$R_k = V_c(50 - k/40), \quad k \in [0, N], \quad V_c = 300\text{msec}$$

where $N = 1880$ is taken to match a time range of $[0, 47]$ sec. Since C_k varies significantly during the system operation in $k \in [0 \quad N]$, we consider it to be uncertain, varying in the interval described by the two vertices $[g_1 \quad 0]$ and $[g_2 \quad 0]$ where

$$g_1 \stackrel{\triangle}{=} 1/R_{m1} = 1/15,000, \quad \text{and} \quad g_2 \stackrel{\triangle}{=} 1/R_{mN} = 1/900.$$

The matrix F_k that corresponds to $k = 0$ and $k = N$ is similarly considered as an uncertain matrix lying in between $\begin{bmatrix} 0.3g_1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0.3g_2 & 0 \end{bmatrix}$.

Application of Theorem 1 directly, or the use of the corresponding polytopic results in (Palhares and Peres, 1998), would clearly cause an overdesign since it leads to a single H_∞ filter that satisfies the required estimation level over the whole intervals of uncertainty. Instead, since we measure $R_{m,k}$, we may use this noisy measurement of R_k to schedule the filter at time k. This scheduling is based on the fact that the LMI of (9) is affine in $\Phi = Z_k C_k$ and $\Psi = Z_k F_k$. Thus, if there is a solution (W, S, Z, T, R) to (9) with a zero last column in Φ, it will produce Z_k over the whole uncertainty polytope. The solution Φ (Ψ is redundant for this purpose) yields Z_1 and Z_N that correspond to R_{m1} and R_{mN}, respectively. Expressing R_{mk} as a convex combination of R_{m1} and R_{mN}, say $R_{mk} = \alpha_k R_{m1} + (1 - \alpha_k) R_{mN}$, $\alpha \in [0, 1]$, the matrix Z_k that satisfies $Z_k C_k = \Phi$ and $Z_k F_k = \Psi$ is obtained by $Z_k = \alpha_k Z_1 + (1 - \alpha_k) Z_N$. The corresponding B_{fi} in (2) is, then, given at any instant i by

$$B_{fi} = \alpha_i B_{f1} + (1 - \alpha_i) B_{fN}, \quad i \in [0, N],$$

where B_{f1} and B_{fN} are obtained from Z_1 and Z_N by (10b). The matrices A_f and C_f in (2) are constant. They are obtained by (10a,c).

We solved the problem for $\gamma = 30$ where we also added the requirement for a minimum upper-bound on the H_2-norm of the estimation error. We obtained $Z_1^T = \begin{bmatrix} -0.1142 & -20.0476 \end{bmatrix}$, $B_{f1}^T = \begin{bmatrix} 1.9035 & 0.0887 \end{bmatrix}$, $Z_N^T = \begin{bmatrix} -1.8874 & -333.0475 \end{bmatrix}$, and $B_{fN}^T = \begin{bmatrix} 31.8921 & 1.4825 \end{bmatrix}$. The pair (A_f, C_f) is

$$A_f = \begin{bmatrix} 0.9978 & 0.025 \\ -0.0001 & 0.9999 \end{bmatrix}, \quad C_f = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The resulting scheduled estimate is given by $\hat{z}_k = C_f \hat{x}_k$ where

$$\hat{x}_{k+1} = A_f \hat{x}_k + [\alpha_k B_{f1} + (1 - \alpha_k) B_{fN}] y_k, \quad \hat{x}_0 = 0.$$

Fig. 1 describes the resulting estimation error for a randomly selected seed in the routine that produced the white noise sequence $\{\zeta_k\}$. We compare it to the one obtained by applying the standard time-varying Kalman filter. It is shown that in spite of its stationary nature, and the time-varying property of the problem, our estimate $\hat{x}_k$ is considerably more robust at large values of k where the changes in C_k are accentuated.

5 Conclusions

In the present paper we solve the problem of stationary stochastic H_∞-filtering of discrete-time linear systems using LMI techniques. The problem was solved before in (Gershon *et al.*, 1998), by restricting the filter to be of the Luenberger type. Using a Riccati recursion the solution was obtained there only if in addition to the H_∞ requirement an upper-bound on the covariance of the estimation error is minimized in every instant. The solution of the present paper does not depend on the latter minimization and is not restricted to a specific structure of the filter.

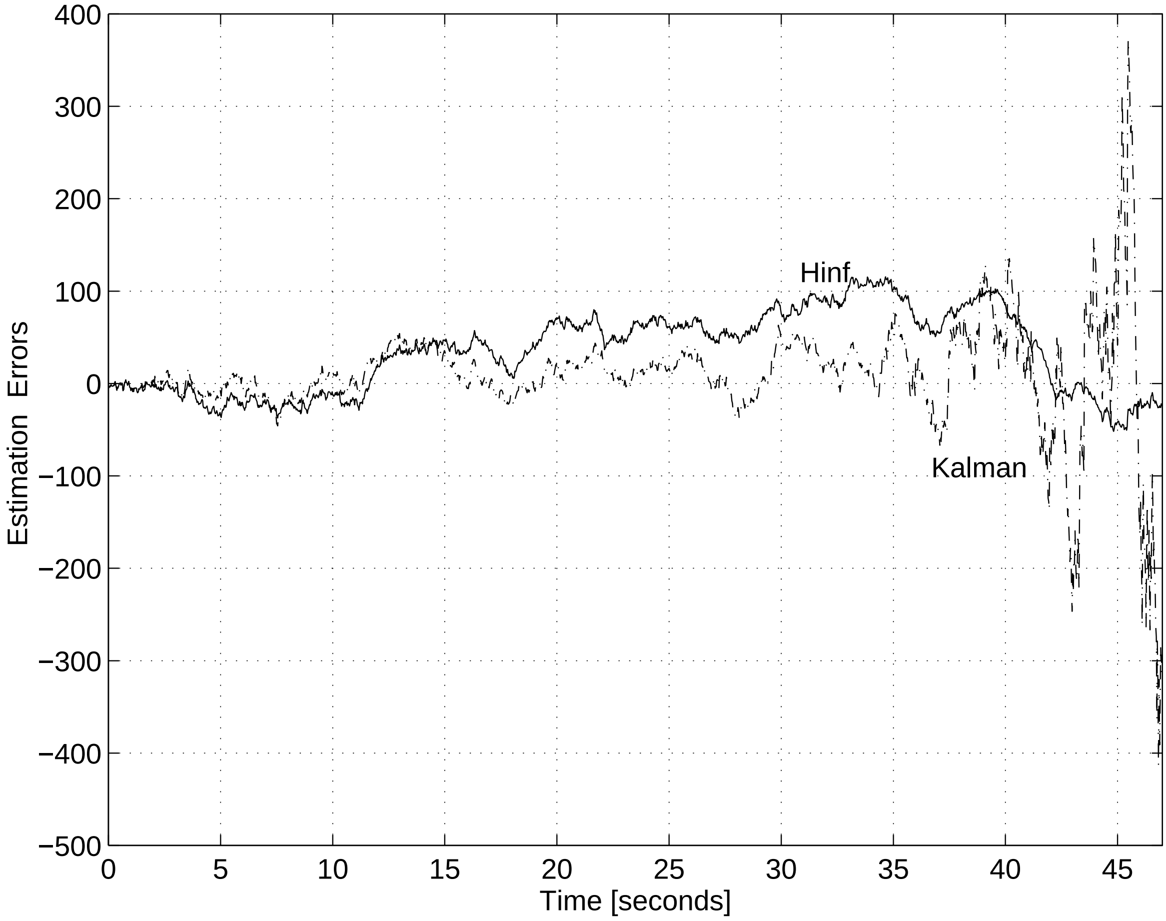

Figure 1: Comparison between the H_∞ and the Kalman Filters: Solid line – H_∞, dashed line – Kalman.

Using the LMI approach, the conditions for the existence of a solution to the problem were obtained in term of LMIs that are affine in the system and the filter parameters. This affinity allows also the consideration of deterministic uncertainty in the system, when the deterministic part of the system matrices lie in a given polytopic type domain. Our solution entails overdesign that stems from the quadratic stability nature of the solution. Under the requirement for this type of stability, the conditions we obtained for the existence of a solution to the problem are both necessary and sufficient.

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