LOCAL EXACT CONTROLLABILITY OF THE NAVIER–STOKES EQUATIONS WITH THE CONDITION ON THE PRESSURE ON PARTS OF THE BOUNDARY

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Abstract. In this paper we prove a local exact controllability of the Navier–Stokes equations with the condition on the pressure on parts of the boundary. The local exact controllability is proved by using a global Carleman inequality and an estimate of an equation adjoint to a linearized Navier–Stokes equation with the nonstandard boundary condition. We also obtain the null controllability of the linearized system by controls acting on an arbitrarily given subdomain.

Key words. Navier–Stokes equations, controllability, Carleman inequality, boundary condition on the pressure

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1. Introduction. Approximate controllability of the Navier–Stokes equations with homogeneous Dirichlet boundary conditions on a bounded domain by controls acting on an arbitrarily given subdomain or subboundary has been studied for many years. Some special cases have been solved (cf. [11], [12], [13], [22], [24], and [36]). However, for the general case, the problem is still open.

The approximate controllability of the Stokes equations was studied in many papers (cf. [14], [15], [20], and [34]; see the introduction in [16] and [22] and the references therein).

Global exact controllability by controls acting on a subdomain was obtained in the case of the Navier–Stokes equations on the 2-dimensional (2-D) manifold without boundary (cf. [13]) and the equations on 2-D or three-dimensional (3-D) tori (i.e., with periodic boundary conditions) (cf. [22]). Local exact controllability of the Navier–Stokes equations was studied in several papers as well. In [17], [18], [21], [29], and [30] equations with homogeneous Dirichlet boundary conditions and in [12], [26], and [27] equations with Navier slip boundary conditions, respectively, were studied. To study the exact controllability of the linearized equations, global Carleman inequalities were studied (cf. [16]). By the combination of the Carleman inequalities of a parabolic equation and an elliptic equation, the Carleman inequalities for the equations adjoint to the linearized Navier–Stokes equations with homogeneous Dirichlet boundary conditions were obtained (cf. [29] and [30]). In [17] and [18], using a new Carleman inequality for an elliptic equation in [31], the authors obtained new Carleman inequalities for the equations adjoint to the linearized Navier–Stokes equations with homogeneous Dirichlet boundary conditions, which result in reduction of the

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conditions on the given trajectories or the dimension of control. In [26], the authors obtained a Carleman inequality for the case of 3-D Stokes equations with a Navier slip boundary condition. Local exact controllability of the Navier–Stokes equations was obtained by the combination of global exact controllability of the linearized equation with a theorem of inverse operator (cf. [17], [18], [21], [22], [26], [27], [28], [29], and [30]).

In this paper we study local exact controllability of the Navier–Stokes equations with the condition on the pressure on parts of the boundary where there is flow. Due to the nonstandard boundary conditions and less regularity of solutions to the linearized problem, to obtain a Carleman inequality and apply a theorem of inverse operators we need to overcome more difficulties.

This paper consists of five sections. In the rest of section 1, we define notation and give linearization of the Navier–Stokes equations. In section 2, first, we obtain a formula for the boundary integral over the parts of the boundary where the pressure boundary condition is given (Lemma 2.4). Then, using this formula and following the method in [8] for a parabolic equation and the method in [30], we obtain a global Carleman inequality (Theorem 2.5) and an estimate called an observability inequality (Theorem 2.9) for the solutions to the equation adjoint to the linearized one. In section 3, we prove the null controllability of the linearized system with the nonstandard boundary condition (Theorem 3.2). In section 4, under Assumption 4.1 we prove a local exact controllability of the Navier–Stokes equations with the boundary condition (Theorems 4.7 and 4.8). In the appendix we will give proofs of some results used in sections 2–4.

Throughout this paper we will use the following notation. Let \( \Omega \) be a connected bounded open subset of \( \mathbb{R}^d \), \( l = 2, 3 \). \( \partial \Omega \in C^2 \), \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \), \( \Gamma_0 \cap \Gamma_1 = \emptyset \), and \( \Gamma_0 = \bigcup_j \Gamma_{ji} \), where \( \Gamma_{ji} \) are open subsets of \( \partial \Omega \). If \( \Gamma_0 = \emptyset \), then \( \Gamma_1 \) is connected. Let \( n(x) \) be the outward normal unit vector at \( x \) in \( \partial \Omega \), let \( Q = \Omega \times (0, T) \), and let \( 0 < T < \infty \).

Let \( \omega \) and \( \omega_0 \) be arbitrarily given subdomains such that \( \overline{\omega_0} \subset \omega \), \( Q_\omega = \omega \times (0, T) \), and \( Q_{\omega_0} = \omega_0 \times (0, T) \). Let \( W^s_0 \) be Sobolev spaces, let \( H^1(\Omega) = W^2_0(\Omega) \), let \( H^1(\Omega) = \{ H^1(\Omega) \} \), and let \( W^2_{2, \Gamma_1}(\Omega) = \{ v \in W^2_0(\Omega) : v|_{\Gamma_1} = 0 \} \). Let \( V = \{ u \in H^1(\Omega) : \text{div} u = 0, u|_{\Gamma_0} = 0, u \times n|_{\Gamma_1} = 0 \} \), and let \( H \) be the closure of \( V \) in \( L^2(\Omega) \) and \( D(\Omega) = \{ C^\infty(\Omega) : \text{div} u = 0 \} \). Define the spaces \( W \) and \( Z \) by \( W = \{ y \in L^2(0, T; V) : \frac{\partial y}{\partial \nu} \in L^2(0, T; V^*) \} \), where \( V^* \) is a dual space of \( V \), \( \| y \|_W = \| y \|_{L^2(0, T; V)} + \| \frac{\partial y}{\partial \nu} \|_{L^2(0, T; V^*)} \), and \( Z = \{ y \in L^2(0, T; D(\tilde{A})) : \frac{\partial y}{\partial \nu} \in L^2(0, T; H) \} \), \( \| y \|_Z = \| y \|_{L^2(0, T; D(\tilde{A}))} + \| \frac{\partial y}{\partial \nu} \|_{L^2(0, T; H)} \), where \( D(\tilde{A}) \) is defined by (2.7) in section 2. An inner product in the space \( L^2(\Omega) \) is denoted by \( \langle \cdot, \cdot \rangle \), and \( \langle \cdot, \cdot \rangle^* \) means the duality product between \( V \) and \( V^* \). Also, \( \langle \cdot, \cdot \rangle_{\Gamma_1} \) is an inner product in the \( L^2(\Gamma_1) \), and \( \langle \cdot, \cdot \rangle^*_\Gamma_1 \) means the duality product between \( H^1(\Gamma_1) \) and \( H^{-1}(\Gamma_1) \). The inner product and norms in \( \mathbb{R}^l \), respectively, are denoted by \( \langle \cdot, \cdot \rangle_{H^1} \) and \( | \cdot | \).

For convenience, in the case where \( l = 2 \), \( y = (y_1(x_1, x_2), y_2(x_1, x_2)) \) is identified with \( \tilde{y} = (y_1, y_2, 0) \), and so \( \text{rot} \ y = \text{rot} \tilde{y} \). Thus, for \( y = (y_1, y_2) \) and \( v = (v_1, v_2) \), \( \text{rot} \ y \times v \) is the 2-D vector consisting of the first two components of \( \text{rot} \tilde{y} \times \tilde{v} \).

Now we turn to linearization of the Navier–Stokes equations. To this end, let us consider the following problem:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (v, \nabla)v + \nabla p &= \hat{f} + \chi_p u, \\
\text{div} u &= 0, \\
v|_{\Gamma_0} &= a(x, t), \quad v \times n|_{\Gamma_1} = b(x, t) \times n, \quad (p + \frac{1}{2} | v |^2)|_{\Gamma_1} = \varphi_1(x, t), \\
v(0) &= v_0,
\end{aligned}
\]
where \( \chi_\omega \) is the characteristic function of the set \( \omega \).

Let \((\hat{v}, \hat{p})\) be a solution to the following problem (which was considered in [7]):

\[
\begin{align*}
\frac{\partial \hat{v}}{\partial t} - \nu \Delta \hat{v} + (\hat{v}, \nabla) \hat{v} + \nabla \hat{p} &= \hat{f}, \\
\text{div} \hat{v} &= 0, \\
\hat{v}|_{\Gamma_0} &= a(x, t), \\
\hat{v} \times n|_{\Gamma_1} &= b(x, t) \times n, \\
\hat{v}(0) &= \hat{v}_0.
\end{align*}
\]  

(1.2)

Assume that

\[
\hat{v} \in W^1_{\infty}(0, T; W^2_\alpha(\Omega)), \quad \alpha > l.
\]  

(1.3)

Setting \( v = \hat{v} + y, \ p = \hat{p} + p_1 \) in (1.1) and using the facts that

\[-\Delta v = \text{rot} \text{rot} v - \text{grad} (\text{div} v), \quad (v, \nabla)v = \text{rot} v \times v + \frac{1}{2} \text{grad} |v|^2,
\]

we have the following equation for \( y \):

\[
\begin{align*}
\frac{\partial y}{\partial t} + \nu \text{rot} \text{rot} y + \text{rot} \hat{v} \times y + \text{rot} y \times \hat{v} + \text{rot} y \times y + \nabla q &= \chi_\omega u, \\
\text{div} y &= 0, \\
y|_{\Gamma_0} &= 0, \\
y \times n|_{\Gamma_1} &= 0, \\
q|_{\Gamma_1} &= 0, \\
y(0) &= y_0.
\end{align*}
\]  

(1.4)

where \( q = p_1 + \hat{v} \cdot y + \frac{1}{2}|y|^2, \ y_0 = v_0 - \hat{v}_0 \).

Neglecting \( \text{rot} y \times y \) in (1.4), we get the following initial boundary value problem of the linearized Navier–Stokes equation:

\[
\begin{align*}
\frac{\partial y}{\partial t} + \nu \text{rot} \text{rot} y + \text{rot} \hat{v} \times y + \text{rot} y \times \hat{v} + \nabla q &= \chi_\omega u, \\
\text{div} y &= 0, \\
y|_{\Gamma_0} &= 0, \\
y \times n|_{\Gamma_1} &= 0, \\
q|_{\Gamma_1} &= 0, \\
y(0) &= y_0.
\end{align*}
\]  

(1.5)

Since

\[
(\text{rot} \hat{v} \times w) = - (\text{rot} \hat{v} \times w, v) \quad \forall v, w \in H^1(\Omega),
\]  

(1.6)

\[
(\text{rot} u \times \hat{v}, w) = (\text{rot} \hat{v} \times w) = (u, \text{rot} \hat{v} \times w) \quad \forall u, w \in V,
\]

we see that the equation adjoint to the first equation of (1.5) is

\[
-\frac{\partial w}{\partial t} + \nu \text{rot} \text{rot} w - \text{rot} \hat{v} \times w + \text{rot} (\hat{v} \times w) + \nabla p = 0.
\]  

(1.7)

We end this section and leave the study of (1.7) until the next section.

2. A global Carleman inequality and an estimate. We start this section by studying (1.7) put forward in section 1. However, instead of the backward problem (1.7), we consider the following initial boundary value problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \nu \text{rot} \text{rot} y - \text{rot} \hat{v} \times y + \text{rot} (\hat{v} \times y) + \nabla p &= f, \\
\text{div} y &= 0, \\
y|_{\Gamma_0} &= 0, \\
y \times n|_{\Gamma_1} &= 0, \\
p|_{\Gamma_1} &= \varphi_2(x, t) \in L_2(0, T; H^{-\frac{l}{2}}(\Gamma_1)), \\
y(0) &= y_0 \in H.
\end{align*}
\]  

(2.1)
By orthogonal decomposition of space

\[ \text{L}_2(\Omega) = H \oplus G_{0, \Gamma_1}(\Omega), \]

\[ G_{0, \Gamma_1}(\Omega) = \{ u = \text{grad} p : p \in H^1(\Omega), p|_{\Gamma_1} = 0 \text{ (or constant)} \}, \]

without any influence on the boundary condition on the pressure on \( \Gamma \) that \( \dot{u} \in L(2; L_2(\Omega)) \) refers to the case \( f \in L_2(0, T; H) \). Thus, we can assume that \( f \in L_2(0; T; H) \). For \( u, v \in V \bigcap D(\Omega) \), we therefore have

\[ \nu(\text{rot} \text{ rot} \ u, v)_\Omega = \nu(\text{rot} \ u, \text{rot} \ v) - \nu(\text{rot} \ u \times n, v)|_{\Gamma_1} \]
\[ = \nu(\text{rot} \ u, \text{rot} \ v) - \nu(\text{rot} \ u, n \times v)|_{\Gamma_1} = \nu(\text{rot} \ u, \text{rot} \ v). \]

In view of (2.2) and (2.3) we can give the following definition.

**Definition 2.1.** A function \( y \in L_2(0, T; V) \bigcap C([0, T]; H) \) is called a solution to (2.1) if \( \frac{\partial y}{\partial t} \in L_2(0, T; V^*) \), \( y(0) = y_0 \), and

\[ (\frac{\partial y}{\partial t}, v) + \nu(\text{rot} \ y, \text{rot} \ v) + (-\text{rot} \ \dot{v} \times y + \text{rot}(\dot{v} \times y), v) \]
\[ = (f, v) - \langle \varphi_2(x, t), v \cdot n \rangle_{\Gamma_1} \quad \forall v \in L_2(0, T; V). \]

Define an operator \( A \in (V \rightarrow V^*) \) by

\[ \langle Au, v \rangle = \nu(\text{rot} \ u, \text{rot} \ v) \quad \forall u, v \in V. \]

Then,

\[ \langle Au, v \rangle \leq K \| u \|_V \| v \|_V. \]

In the case of a 3-D domain with \( \partial \Omega \in C^{1,1} \) (cf. Proposition 1.1 in [1] or Lemma 1.5 in [5] if \( \Gamma_0 \neq \emptyset \) and p. 286 in [10] if \( \Gamma_0 = \emptyset \)) we have

\[ \langle Au, u \rangle \geq a \| u \|_V^2, \quad a > 0. \]

2-D domains for which (2.6) is valid will be considered in section 5 (see Lemma 5.3).

Therefore, in what follows we will assume that (2.6) holds and will discuss the cases of 2-D and 3-D domains together.

Let

\[ \tilde{A} : D(\tilde{A}) = \{ u \in V : Au \in \text{L}_2(\Omega) \}, \quad Au = \tilde{A} u \quad \forall u \in D(\tilde{A}), \]

\[ \| u \|_{D(\tilde{A})} = \| u \|_V + \| Au \|_{\text{L}_2(\Omega)}. \]

Then, the operator \( \tilde{A} \) is self-adjoint.

Define an operator \( B(t) \in (V \rightarrow H) \) by

\[ (Bw, v)_{\text{L}_2} = (-\text{rot} \ \dot{v} \times w + \text{rot}(\dot{v} \times w), v)_{\text{L}_2} \quad \forall v \in H. \]

For \( w \in D(\tilde{A}) \), by (1.3),

\[ \| Bu \|_{\text{L}_2(\Omega)} \leq c \| \dot{v} \|_{\text{H}^1(\Omega)} \cdot \| w \|_V \leq c \| \dot{v} \|_{\text{H}^1(\Omega)} \| w \|_{V}^{1/2} \cdot \| w \|_{D(\tilde{A})}^{1/2}, \]

where \( c \) is independent of \( t \).
The existence and uniqueness of a solution to (2.1) are equivalent to the property of a solution \( u \in W \) to the problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (A + B)u &= \bar{f}, \\
u(0) &= y_0,
\end{aligned}
\]

where \( \bar{f} \in L_2(0, T; V^*) \) is defined by

\[
\langle \bar{f}, v \rangle = \langle f, v \rangle - \langle \varphi_2(x, t), v \cdot n \rangle_{\Gamma_1}, \quad \forall v \in L_2(0, T; V).
\]

In view of (2.5), (2.6), (2.9), and (2.10) it is easy to verify the existence and uniqueness of a solution to (2.11) and an estimate

\[
\|u\|_W \leq c(\|y_0\|_H + \|\bar{f}\|_{L_2(0, T; V^*)})
\]

(cf. Chapter 6 in [23]).

**Remark 2.2.** Since for \( v \in V \)

\[
\langle \varphi_2(x, t), v \cdot n \rangle_{\Gamma_1} = \langle \varphi_2(x, t) + c(t), v \cdot n \rangle_{\Gamma_1},
\]

\( p \) is determined up to a constant with respect to \( x \). Thus, to fix it when we consider \( p \), we assume that \( p \) satisfies \( \int_Q p(x, t) \, dx = 0 \), namely, \( \int_Q p \, dx \, dt = 0 \).

**Lemma 2.3.** Suppose that \( y_0 \in V, f \in L^2(0, T; H) \), and \( \varphi_2 \in L_2(0, T; H^{1/2}(\Gamma_1)) \). Then, there exists a solution \( u \in Z \) to (2.11) such that

\[
\|u\|_Z \leq c \left( \|y_0\|_V + \|f\|_{L_2(0, T; H)} + \|\varphi_2\|_{L_2(0, T; H^{1/2}(\Gamma_1))} \right).
\]

**Proof.** It is easy to see that \( \widetilde{A}^{-1} = A \) is self-adjoint and compact in \( H \). Therefore, \( A \) possesses an orthogonal sequence of eigenfunctions \( w_i \) such that

\[
Aw_i = \lambda_i w_i, \quad \lambda_i > 0, \quad w_i \in V,
\]

\[
\lambda_i \to \infty \quad \text{as} \quad i \to \infty.
\]

By (2.4)–(2.6), the norm given by the inner product \( \langle Av, v \rangle \) in \( V \) is equivalent to the original one in \( V \). We use the eigenfunctions as bases in \( H \). If \( w \in D(A) \), then \( Aw \in L_2(\Omega) \) and \( \text{div} \, Aw = 0 \). Thus, \( (Aw) \cdot n \big|_{\Gamma_1} \in H^{-1/2}(\Gamma_1) \). Then, when \( \varphi_2(x, t) \in L_2(0, T; H^{1/2}(\Gamma_1)) \),

\[
\left| \langle \varphi_2(x, t), Aw \cdot n \rangle_{\Gamma_1} \right| \leq \| \varphi_2(x, t) \|_{H^{1/2}(\Gamma_1)} \cdot \| Aw \cdot n \|_{H^{-1/2}(\Gamma_1)} \leq c \| \varphi_2(x, t) \|_{H^{1/2}(\Gamma_1)} \cdot \| Aw \|_{L_2(\Omega)}
\]

(cf. Theorem 1.2, Chapter 1 in [37]). Taking (2.14) into account and following the method in Theorem 3.10 in [37] we come to the conclusion and complete the proof of Lemma 2.3. \( \square \)

**Lemma 2.4.** Suppose that \( y, z \in \{ u \in H^2(\Omega) : u \times n \big|_{\Gamma_1} = 0 \} \). Then,

\[
\left\langle \frac{\partial y}{\partial n}, z \right\rangle_{\Gamma_1} + (k(x)y, z)_{\Gamma_1} = (\text{div} \, y \cdot z, n)_{\Gamma_1},
\]

where \( k(x) = \text{div} \, n(x) \in C^1(\Gamma_1) \).
Proof. By the trace theorem, it is enough to prove the assertion for \( y, z \in C^2(\bar{\Omega}) \).

By Lemma 7 in [32]

\[
\left( \frac{\partial y}{\partial n} \right)_{R^1} + (k(x,y,n))_{R^1} + \text{div}_{\Gamma_1} y_t = \text{div} y \quad \text{on } \Gamma_1,
\]

where \( y_t \) is the component of \( y \) tangent to \( \Gamma_1 \) and \( \text{div}_{\Gamma_1} y_t \) is the divergence of \( y_t \) in the tangent coordinate system on \( \Gamma_1 \). Since \((y \times n)|_{\Gamma_1} = 0\), from this we get

\[
(2.15) \quad \left( \frac{\partial y}{\partial n} \right)_{R^1} + (k(x,y,n))_{R^1} = \text{div} y \quad \text{on } \Gamma_1
\]

(cf. Lemma 5.1 in the appendix).

Multiplying (2.15) by \(|z|\), we have

\[
\left( \frac{\partial y}{\partial n} \right)_{R^1} + (k(x,y,z))_{R^1} = \text{div} y \cdot (z,n)_{R^1},
\]

which shows the assertion. \( \Box \)

It is known that there exists a function \( \psi \in C^2(\bar{\Omega}) \) (cf. Lemma 1.1, Chapter 1 in [21]) such that

\[
\psi(x) > 0 \quad \forall x \in \Omega, \quad \psi|_{\partial \Omega} = 0, \quad \text{and} \quad |\nabla \psi| > 0 \quad \forall x \in \Omega \setminus \partial \Omega.
\]

Then we can define

\[
\begin{align*}
\varphi(x,t) &= e^{\lambda \psi(x)} / [t(T-t)]^8, \\
\tilde{\varphi}(x,t) &= e^{-\lambda \psi(x)} / [t(T-t)]^8, \\
\alpha(x,t) &= \left( e^{\lambda \psi} - e^{\lambda^2 \|\psi\|_{C^0(\Omega)}} \right) / [t(T-t)]^8, \\
\tilde{\alpha}(x,t) &= \left( e^{-\lambda \psi} - e^{-\lambda^2 \|\psi\|_{C^0(\Omega)}} \right) / [t(T-t)]^8, \\
\tilde{\alpha}(x,t) &= \alpha(x,t) |_{\partial \Omega} = 1 - e^{\lambda^2 \|\psi\|_{C^0(\Omega)}} / [t(T-t)]^8, \\
\tilde{\phi}(t) &= 1 / [t(T-t)]^8,
\end{align*}
\]

where \( \lambda \) is a positive parameter which will be chosen later.

Next we will obtain a global Carleman inequality which will be used later.

THEOREM 2.5 (Carleman inequality). Suppose that \( y_0 \in V, f \in L_2(0,T;H) \), and \( \varphi_2(x,t) = 0 \). Then, there exists a number \( \lambda > 1 \) such that for every \( \lambda > \lambda \) there exists a \( s_0(\lambda) \) such that for every \( s \geq s_0(\lambda) \) the solutions to (2.1) satisfy

\[
I(s) \equiv \int_Q \left[ \frac{1}{s^2} \frac{\partial \psi}{\partial n} + s \varphi |\nabla y|^2 + s^3 \tilde{\phi}^3 |y|^2 \right] e^{2s\alpha} dx dt
\]

\[
\leq C \left\{ \int_Q \left[ \tilde{f}^2 \left( e^{2s\alpha} + (s\tilde{\phi})^2 e^{2s\tilde{\alpha}} \right) \right] dx dt
\]

\[
+ \int_{Q_{\tilde{\alpha}}} \left[ s^3 \tilde{\phi}^3 |y|^2 + (s\tilde{\phi}) \frac{\partial}{\partial n} p \right] e^{2s\alpha} dx dt \right\} \quad \forall s \geq s_0(\lambda),
\]

where \( C \) depends on \( \lambda \).

To prove the above theorem, let us first consider the following problem:

\[
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} - \nu \Delta u + \nabla p = \tilde{f}, \\
\text{div} u = 0, \\
|u|_{\Gamma_0} = 0, \quad u \times n |_{\Gamma_1} = 0, \quad p |_{\Gamma_1} = 0, \\
u(0) = u_0,
\end{array} \right.
\]

\[
(2.17)
\]
where \( u_0 \in V \).

Now, let \( \tilde{f} \in L_2(0, T; H) \). Since the space \( C^1([0, T]; D(\hat{A})) \) is dense in the space \( Z \) (cf. Theorem 2.1 in [33]), the space \( C^1([0, T]; D(\hat{A}) \cap H^2) \) is also dense in \( Z \). For the solution \( u \) to (2.17), let \( \{u_m\} \) be a sequence such that \( u_m \rightarrow u \) in \( Z \) and \( u_m \in C^1([0, T]; D(\hat{A}) \cap H^2) \). Taking \( f_m \equiv 2A_{m} + Au_m \), we know that \( \tilde{f}_m \rightarrow \tilde{f} \) in \( L_2(0, T; H) \). Thus, taking (2.2) into account we have that for a \( p_m \in L_2(0, T; H^1(\Omega)) \)

\[
\begin{cases}
\frac{\partial u_m}{\partial t} - \nu \Delta u_m + \nabla p_m = f_m, \\
\text{div } u_m = 0, \\
u u_m |_{\Gamma_0} = 0, \
x \times n |_{\Gamma_1} = 0, \\
u m(0) = u_0
\end{cases}
\]

(2.18)

and

\[
\frac{\partial u_m}{\partial t} \rightarrow \frac{\partial u}{\partial t} \quad \text{in} \quad L_2(0, T; H), \]

\[
u u_m \rightarrow u \quad \text{in} \quad L_2(0, T; V).
\]

From (2.17)–(2.19) we have

\[
\nabla p_m \rightarrow \nabla p \quad \text{in} \quad L_2(0, T; H^{-1}),
\]

which implies \( p_m \rightarrow p \) in \( L_2(0, T; L_2(\Omega)) \) (cf. Proposition 1.2, Chapter 1 in [37]). Since the functions \( \varphi e^{2s \alpha}, \varphi e^{2s \alpha} \), \( \varphi \alpha e^{2s \alpha}, \) and \( \varphi \alpha e^{2s \alpha} \) are continuous on \([0, T] \times \Omega\), inequality (2.16) for \( u \) is true by passing to the limit, provided the inequality holds for \( u_m \). Thus, we consider (2.18) for \( u_m \). For convenience we will omit indices \( m \).

**Lemma 2.6.** There exists a number \( \lambda > 1 \) such that for an arbitrary \( \lambda > \hat{\lambda} \) there is a \( s_0(\lambda) \) with the property that for every \( s \geq s_0(\lambda) \) the solutions to (2.18) satisfy

\[
\int_Q \left[ 2s \varphi |\nabla u|^2 + s^3 \varphi^3 |u|^2 \right] (e^{2s \alpha} + e^{2s \alpha})dxdt 
\]

\[
\leq C \left[ \int_Q \|\tilde{f} - \nabla p\|^2 (e^{2s \alpha} + e^{2s \alpha})dxdt + \int_{Q_-} s^3 \varphi^3 |u|^2 (e^{2s \alpha} + e^{2s \alpha})dxdt \right] ,
\]

where \( C > 0 \) depends on \( \lambda \).

**Proof.** Let \( y = e^{k_1 \psi(x)} u \). Then \( y \) satisfies

\[
\begin{cases}
\frac{\partial y}{\partial t} - \nu \Delta y = g, \\
\text{div } y = k_1 (\nabla \psi(x), y)|_{\Gamma_1}, \\
y |_{\Gamma_0} = 0, \
x \times n |_{\Gamma_1} = 0, \\
y(0) = e^{k_1 \psi(x)} u_0,
\end{cases}
\]

(2.21)

where \( k_1 \) is a constant such that

\[
k_2(x) \equiv k(x) - k_1 \frac{\partial \psi(x)}{\partial n} \geq 0 \quad \text{on} \quad \Gamma_1,
\]

and

\[
g \equiv \frac{e^{k_1 \psi(x)}(\tilde{f} - \nabla p) - 2k_1 \nu \sum_{i=1}^{n} \left( \frac{\partial \psi}{\partial x_i}, \frac{\partial y}{\partial x_i} \right)_{\Gamma_1}}{\nu k_1 \Delta \psi - \nu k_1^2 |\nabla \psi|^2} y.
\]

(2.22)

It is enough to prove (2.20) with \( y \) and \( g \) instead of \( u \) and \( \tilde{f} - \nabla p \), because by \( y = e^{k_1 \psi(x)} u \) from the inequality for \( y \) the one for \( u \) can be obtained.
We will proceed as in [8]. Let
\begin{align}
\hat{L}y &= \frac{\partial y}{\partial t} - \nu \Delta y
\end{align}
and
\begin{align}
w(x,t) = e^{s\alpha}y(x,t), \quad \hat{w}(x,t) = e^{s\alpha}y(x,t).
\end{align}
Define the operators $P$ and $\hat{P}$, respectively, by
\begin{align}
Pw &= e^{s\alpha}L e^{-s\alpha}w, \quad \hat{P}w = e^{s\alpha}\hat{L} e^{-s\alpha}w.
\end{align}
By (2.25) and (2.26), we get
\begin{align}
Pw &= e^{s\alpha}\left[\frac{\partial}{\partial t} - \nu \Delta\right] e^{-s\alpha}w = e^{s\alpha}g,
\end{align}
\begin{align}
\hat{P}\hat{w} &= e^{s\alpha}\hat{L} e^{-s\alpha}\hat{w} = e^{s\alpha}g.
\end{align}
Rewrite the left-hand side of (2.27) as follows:
\begin{align}
Pw &= \frac{\partial w}{\partial t} - \nu \Delta w + 2s\lambda\varphi\nu(\nabla\psi, \nabla w)_{R^1} + s\lambda^2\varphi^2|\nabla \psi|^2w + s\lambda\varphi|\Delta\psi|w
\end{align}
\begin{align}
- s^2\lambda^2\varphi^2|\nabla \psi|^2w - s\frac{\partial \alpha}{\partial t}w.
\end{align}
Define the operators $L_1$ and $L_2$, respectively, by
\begin{align}
L_1w &= -\nu \Delta w - \lambda^2 s^2\varphi^2\nu|\nabla \psi|^2w - s\frac{\partial \alpha}{\partial t}w
\end{align}
and
\begin{align}
L_2w &= \frac{\partial w}{\partial t} + 2s\lambda\varphi\nu(\nabla\psi, \nabla w)_{R^1} + 2s\lambda^2\varphi^2|\nabla \psi|^2w.
\end{align}
Then, from (2.29) we get
\begin{align}
L_1w + L_2w = f_s,
\end{align}
where
\begin{align}
f_s(x,t) &= ge^{s\alpha} - s\lambda\varphi\nu w\Delta \psi + s\lambda^2\varphi^2|\nabla \psi|^2w.
\end{align}
Thus,
\begin{align}
\|f_s\|^2_{L_2(Q)} &= \|L_1w\|^2_{L_2(Q)} + \|L_2w\|^2_{L_2(Q)} + 2(L_1w, L_2w)_{L_2(Q)}.
\end{align}
By (2.30) and (2.31), we have
\begin{align}
(L_1w, L_2w)_{L_2(Q)} &= A_1 + A_2 + A_3,
\end{align}
where
\begin{align}
A_1 &= (-\nu \Delta w - \lambda^2 s^2\varphi^2\nu|\nabla \psi|^2w - s\frac{\partial \alpha}{\partial t}w, \frac{\partial w}{\partial t} + 2s\lambda^2\varphi^2|\nabla \psi|^2w)_{L_2(Q)},
\end{align}
\begin{align}
A_2 &= -(2s^3\lambda^3\varphi^3\nu|\nabla \psi|^2w + 2s^2\lambda\varphi \frac{\partial \alpha}{\partial t}w, \nu(\nabla \psi, \nabla w)_{R^1})_{L_2(Q)},
\end{align}
\begin{align}
A_3 &= -(\nu \Delta w, 2s\lambda\varphi(\nabla\psi, \nabla w)_{R^1})_{L_2(Q)}.
\end{align}
On the other hand, since

\[ \text{div } w = s \lambda \phi \frac{\partial \psi}{\partial n}(w, n)_{R^i} + k_1 \frac{\partial \psi}{\partial n}(w, n)_{R^i} \quad \text{on } \Gamma_1, \]

by Lemma 2.4 we have

\[
\int_{\Sigma_1} \left( \frac{\partial w}{\partial n}, [\frac{\partial w}{\partial t} + 2s \lambda^2 \phi \nu \nabla \psi \nabla w] \right)_{R^i} \, d\sigma
\]

(2.37)

\[
= -\int_{\Sigma_1} 2k_2(x) s \lambda^2 \phi \nu |\nabla \psi|^2 |\nabla w|^2 \, d\sigma
\]

\[
+ \int_{\Sigma_1} s \lambda \phi \nabla \psi, w \right)_{R^i} ([\frac{\partial w}{\partial t} + 2s \lambda^2 \phi \nu \nabla \psi \nabla w], n)_{R^i} \, d\sigma,
\]

where \(|w|^2(0) = |w|^2(T) = 0\) was used and \(k_2(x)\) is as in (2.22).

Using (2.37) and integrating by parts, we have

(2.38)

\[
A_1 = \int_Q \left\{ \sum_i \nu \left( \frac{\partial^2 w}{\partial x_i \partial t}, \frac{\partial w}{\partial x_i} \right)_{R^i} - \frac{1}{2} s \lambda^2 \phi \nu |\nabla \psi|^2 \frac{\partial w}{\partial t} - \frac{1}{2} \frac{\partial \phi}{\partial t} \frac{\partial w}{\partial t} + 2s \lambda^2 \phi \nu |\nabla \psi|^2 |\nabla w|^2
\]

\[
+ 2s \lambda^2 \nu \sum_i \left( \frac{\partial w}{\partial x_i}, \frac{\partial (\phi \nu |\nabla \psi|^2)}{\partial x_i} \right)_{R^i} - 2s \lambda^4 \phi^3 \nu^2 |\nabla \psi|^4 w^2
\]

\[
- 2s \lambda^2 \phi \frac{\partial \phi}{\partial t} \nu |\nabla \psi|^2 w^2 \right\} \, dx \, dt
\]

\[
+ \int_{\Sigma_1} 2k_2(x) s \lambda^2 \phi \nu |\nabla \psi|^2 w^2 \, d\sigma
\]

\[
- \int_{\Sigma_1} s \lambda \phi \nabla \psi, w \right)_{R^i} ([\frac{\partial w}{\partial t} + 2s \lambda^2 \phi \nu \nabla \psi \nabla w], n)_{R^i} \, d\sigma.
\]

Also integrating by parts, we get

(2.39)

\[
A_2 = -\int_Q \left[ s^3 \lambda^3 \phi^3 \nu^2 |\nabla \psi|^2 \nabla \psi, \nabla w^2 \right]_{R^i} + s^2 \lambda \phi \nu \frac{\partial \phi}{\partial t} (\nabla \psi, \nabla w^2)_{R^i} \right] \, dx \, dt
\]

\[
= \int_Q \left[ 3s^3 \lambda^4 \phi^3 \nu^2 |\nabla \psi|^4 w^2 + s^3 \lambda^3 \phi^3 \nu^2 |\nabla \psi|^2 \sum_i \frac{\partial}{\partial x_i} (|\nabla \psi|^2 \frac{\partial \phi}{\partial x_i}) \right] \, dx \, dt
\]

\[
+ \int_Q \sum_i \frac{\partial}{\partial x_i} (s^2 \lambda \phi \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x_i}) w^2 \, dx \, dt
\]

\[
- \int_{\Sigma_1} \left[ s^3 \lambda^3 \phi^3 \nu^2 |\nabla \psi|^2 + s^2 \lambda \phi \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial \sigma} \right] \, w^2 \, d\sigma
\]
and

\[
A_3 = \int_Q \left[ 2s\lambda^2\varphi^2|\nabla w, \nabla \psi|_{R^1}^2 + 2s\lambda\varphi\nu^2\sum_i \left( w_{x_i} \sum_j \psi_{x_jx_j} w_{x_j} \right)_{R^1} \right] + 2s\lambda\varphi\nu^2\sum_i \left( w_{x_i} \sum_j \psi_{x_j} w_{x_jx_j} \right)_{R^1} \right) \right] dxdt - \int_\Sigma 2s\lambda\varphi\nu^2 \frac{\partial \psi}{\partial n} |\frac{\partial w}{\partial n}|^2 d\sigma \\
(2.40) = \int_Q \left[ 2s\lambda^2\varphi^2|\nabla w, \nabla \psi|_{R^1}^2 + 2s\lambda\varphi\nu^2\sum_i \left( w_{x_i} \sum_j \psi_{x_jx_j} w_{x_j} \right)_{R^1} \right] + 2s\lambda\varphi\nu^2\sum_i |w_{x_i}|^2 \right] dxdt + \int_\Sigma \left[ -2s\lambda\varphi\nu^2 \frac{\partial \psi}{\partial n} |\frac{\partial w}{\partial n}|^2 + 2s\lambda\varphi\nu^2 \sum_i |w_{x_i}|^2 \right] d\sigma, \\
\]

where \( \psi_{x_i} = \frac{\partial \psi}{\partial n} \cdot n_i \) was used. Let us transform the integral in (2.40). In view of

\[
(2.41) \quad \frac{\partial w}{\partial x_i} = \lambda s\varphi \psi_{x_i} w + e^{2\alpha} y_{x_i},
\]

we have

\[
(2.42) \quad -\int_{\Sigma_0} 2s\lambda\varphi\nu^2 \frac{\partial \psi}{\partial n} |\frac{\partial w}{\partial n}|^2 d\sigma = -\int_{\Sigma_0} 2s\lambda\varphi\nu^2 \frac{\partial \psi}{\partial n} e^{2\alpha} |\frac{\partial y}{\partial n}|^2 d\sigma.
\]

Using

\[
(2.43) \quad |\frac{\partial w}{\partial n}|^2 = s^2\lambda^2\varphi^2 |\frac{\partial \psi}{\partial n}|^2 |w|^2 + e^{2\alpha} \frac{\partial \psi}{\partial n} |\frac{\partial y}{\partial n}|^2 + 2s\lambda\varphi \frac{\partial \psi}{\partial n} e^{2\alpha} (w, \frac{\partial y}{\partial n})_{R^1}
\]

and Lemma 2.4, we have

\[
(2.44) \quad -\int_{\Sigma_1} 2s\lambda\nu^2 \frac{\partial \psi}{\partial n} |\frac{\partial w}{\partial n}|^2 d\sigma = -\int_{\Sigma_1} 2s\lambda\nu^2 \frac{\partial \psi}{\partial n} \left[ s^2\lambda^2\varphi^2 |\frac{\partial \psi}{\partial n}|^2 |w|^2 + e^{2\alpha} |\frac{\partial y}{\partial n}|^2 \right] d\sigma + \int_{\Sigma_1} 4\nu^2 s^2\lambda^2\varphi^2 k_2(x) |\frac{\partial \psi}{\partial n}|^2 |w|^2 d\sigma.
\]

Using Lemma 2.4 and

\[
\sum_i |w_{x_i}|^2 = \sum_i s^2\lambda^2\varphi^2 |\psi_{x_i}|^2 |w|^2 + \sum_i e^{2\alpha} |\frac{\partial y}{\partial x_i}|^2 + 2s\lambda\varphi \frac{\partial \psi}{\partial n} e^{2\alpha} (w, \frac{\partial y}{\partial n})_{R^1},
\]

we obtain

\[
(2.45) \quad \int_\Sigma s\lambda\varphi\nu^2 \frac{\partial \psi}{\partial n} \sum_i |w_{x_i}|^2 d\sigma = \int_\Sigma s\lambda\varphi\nu^2 \frac{\partial \psi}{\partial n} \left[ s^2\lambda^2\varphi^2 |\nabla \psi|^2 |w|^2 + \sum_i e^{2\alpha} |\frac{\partial y}{\partial x_i}|^2 \right] d\sigma \\
- \int_\Sigma 2s\lambda\nu^2 \frac{\partial \psi}{\partial n} (k_2(x) |w|^2 d\sigma.
\]
By virtue of (2.35), (2.38)–(2.40), (2.42), (2.44), and (2.45), we have
(2.46)

\[
(L_1 w, L_2 w)_{L_2(Q)}
\]

\[
= \int_Q \left[ s^3 \lambda^4 \varphi^2 |\nabla \psi|^2 \right] |w|^2 + s \lambda^2 \varphi \nu^2 |\nabla \psi|^2 |\nabla w|^2 + 2s \lambda^2 \varphi \nu^2 |(\nabla \psi, \nabla w)|_R^2 \, dx \, dt
\]

\[
+ X_1 + \int_\Sigma s \lambda \varphi \nu^2 \frac{\partial \psi}{\partial n} \left[ s^3 \lambda^2 \varphi^2 |\nabla \psi|^2 w^2 + \sum_i s^2 \alpha_i |\frac{\partial \psi}{\partial x_i}|^2 \right] \, d\sigma
\]

\[- \int_{\Sigma_1} s \lambda \varphi (\nabla \psi, w)_R \left( \frac{\partial w}{\partial \nu} + 2s \lambda^2 \varphi \nu \nabla \psi \cdot w \right) \, d\sigma
\]

\[- \int_{\Sigma_0} \sum_i s \lambda \varphi \nu^2 |s^2 \lambda^2 \varphi^2 |\nabla \psi|^2 |w|^2 + e^{2s \alpha} |\frac{\partial \psi}{\partial n}|^2 \, d\sigma
\]

\[- \int_{\Sigma_1} 2s \lambda \varphi \nu^2 \frac{\partial \psi}{\partial n} \left[ s^2 \lambda^2 \varphi^2 |\frac{\partial \psi}{\partial n}|^2 |w|^2 + e^{2s \alpha} |\frac{\partial \psi}{\partial n}|^2 \right] \, d\sigma
\]

\[+ \int_{\Sigma_1} 2k_2(x)s \lambda^2 \varphi \nu |\nabla \psi|^2 |w|^2 \, d\sigma + \int_{\Sigma_1} 2s^2 \lambda^2 \varphi \nu k_2(x)w(\frac{\partial \psi}{\partial n})^2 |w|^2 \, d\sigma,
\]

where
(2.47)

\[X_1 = \int_Q \left[ 2s \lambda^2 \varphi^2 \left( w, \sum_i \frac{\partial w}{\partial x_i}, \frac{\partial \varphi}{\partial x_i} |\nabla \psi|^2 \right) \right] \left( |T - 2t| \right) + \frac{1}{2} \frac{\partial \psi}{\partial n} \left( \lambda^2 s^2 \varphi^2 |\nabla \psi|^2 \right) |w|^2
\]

\[+ \frac{1}{2} \sum_i \frac{\partial^2 \varphi}{\partial x_i^2} |w|^2 + 2s \lambda \varphi \nu^2 \sum_i \left( \frac{\partial w}{\partial x_i} \left( \sum_j \frac{\partial^2 \varphi}{\partial x_j \partial x_i} \frac{\partial w}{\partial x_i} \right) \right) \left( |T - 2t| \right) - 2\lambda^2 s^2 \varphi \nu |\nabla \psi|^2 |w|^2
\]

\[- s \lambda \varphi \nu^2 \left| \nabla w \right|^2 \sum_i \frac{\partial^2 \varphi}{\partial x_i^2} + |w|^2 \varphi^3 \lambda^3 s^3 \sum_i \frac{\partial \varphi}{\partial x_i} \left( \frac{\partial w}{\partial x_i} |\nabla \psi|^2 \right)
\]

\[+ \sum_i \frac{\partial \varphi}{\partial x_i} \left( s^3 \lambda^2 \varphi^3 \right) \frac{\partial w}{\partial x_i} \frac{\partial \psi}{\partial x_i} |w|^2 \right] \, dx \, dt.
\]

On the other hand, by differentiation and estimation of function \( \alpha \) we obtain
(2.48)

\[\left| \frac{\partial \alpha}{\partial t} \right| \leq \frac{|T - 2t|}{e^{2s \psi}} \left| e^{\lambda \psi} - e^{\lambda \psi \left| \theta \right| \gamma \theta} \right| \varphi^2 \leq C_1(\lambda) \varphi^2.
\]

Similarly,
(2.49)

\[\left| \frac{\partial^2 \alpha}{\partial t^2} \right| \leq C^2(\lambda) \varphi^3.
\]

Taking \( s_0(\lambda) = \max\{C_1(\lambda), C_2(\lambda)\} \) and using (2.47)–(2.49), for all \( s > s_0(\lambda), \lambda \geq 1 \), we have
(2.50)

\[|X_1| \leq C \int_Q \left[ \left( s^3 \lambda^3 \varphi^3 + s^2 \lambda^4 \varphi^3 \right) |w|^2 + s \lambda \varphi |\nabla \psi|^2 \right] \, dx \, dt.
\]

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Starting from (2.28), in a way similar to (2.32) we have

\begin{equation}
\hat{L}_1 \hat{w} + \hat{L}_2 \hat{w} = \hat{f}_s,
\end{equation}

where

\begin{equation}
\hat{L}_1 \hat{w} = -\nu \Delta \hat{w} - \lambda^2 s^2 \phi^2 |\nabla \psi|^2 \hat{w} - s \frac{\partial \alpha}{\partial t} \hat{w},
\end{equation}

\begin{equation}
\hat{L}_2 \hat{w} = \frac{\partial \hat{w}}{\partial t} - 2s \lambda \phi \nu (\nabla \psi, \nabla \hat{w})_R + 2s \lambda^2 \phi \nu |\nabla \psi|^2 \hat{w},
\end{equation}

and

\begin{equation}
\hat{f}_s = g e^{s\alpha} + s \lambda \phi \nu w \Delta \psi + s \lambda^2 \phi \nu |\nabla \psi|^2 \hat{w}.
\end{equation}

Thus,

\begin{equation}
\|\hat{f}_s\|_{L^2(Q)}^2 = \|\hat{L}_1 \hat{w}\|_{L^2(Q)}^2 + \|\hat{L}_2 \hat{w}\|_{L^2(Q)}^2 + 2(\hat{L}_1 \hat{w}, \hat{L}_2 \hat{w})_{L^2(Q)}.
\end{equation}

Taking \(\hat{w}|_{\Sigma} = w|_{\Sigma}, \phi|_{\Sigma} = \phi|_{\Sigma}, \hat{\alpha}|_{\Sigma} = \alpha|_{\Sigma}\) into account, similar to (2.46), from (2.52), (2.53) we have

\begin{equation}
(\hat{L}_1 \hat{w}, \hat{L}_2 \hat{w})_{L^2(Q)}
\end{equation}

\begin{align*}
&= \int_Q \left[ s^3 \lambda^4 \phi^2 \nu^2 |\nabla \psi|^4 |\hat{w}|^2 + s \lambda^2 \phi \nu^2 |\nabla \psi|^2 |\nabla \hat{w}|^2 + 2s \lambda^2 \phi \nu^2 (\nabla \psi, \nabla \hat{w})^2 \right] dx dt \\
&+ X_2 + \int_{\Sigma} s \lambda \phi \nu^2 \frac{\partial \psi}{\partial \nu} \left[ s^2 \lambda^2 \phi^2 |\nabla \psi|^2 |w|^2 + \sum_i \nu \left[ 2s \lambda^2 \phi \nu \left| \frac{\partial \psi}{\partial n} \right|^2 \right] d\sigma \\
&\phantom{=} + \int_{\Sigma} s \lambda \phi (\nabla \psi, w)_R ((\frac{\partial \psi}{\partial n} + 2s \lambda^2 \phi \nu \nabla \psi^2 w), n)_R d\sigma \\
&\phantom{=} + \int_{\Sigma} \left[ s^3 \lambda^3 \phi^3 \nu^2 |\nabla \psi|^2 + s^2 \lambda \nu^2 \phi \frac{\partial \nu}{\partial n} \frac{\partial \psi}{\partial n} w^2 d\sigma - \int_{\Sigma} 2s \lambda^2 \phi \nu^2 \left( \frac{\partial \nu}{\partial n} \right)^2 |w|^2 d\sigma \\
&\phantom{=} + \int_{\Sigma} 2s \lambda^2 \nu^2 \frac{\partial \psi}{\partial n} \left[ s^2 \lambda^2 \phi^2 |\frac{\partial \psi}{\partial n}|^2 |w|^2 + \nu \left( \frac{\partial \psi}{\partial n} \right)^2 \right] d\sigma \\
&\phantom{=} + \int_{\Sigma} 2k_2(x)s \lambda \phi \nu |\nabla \psi|^2 |w|^2 d\sigma + \int_{\Sigma} 2s^2 \lambda^2 \phi^2 k_2(x) \left( \frac{\partial \psi}{\partial n} \right)^2 |w|^2 d\sigma,
\end{align*}

where \(X_2\) is expressed by the right-hand side of (2.47) with \(w, \phi, \frac{\partial \phi}{\partial n}\), and powers of \(\lambda\) replaced by \(\hat{w}, \phi, \frac{\partial \phi}{\partial n}\), and powers of \((-\lambda)\), respectively.

Similar to (2.50) for \(X_1\) we have for all \(s > s_0^2(\lambda), \lambda \geq 1\),

\begin{equation}
|X_2| \leq C \int_Q [(s^3 \lambda^3 \phi^3 + s^2 \lambda^4 \phi^3) |\hat{w}|^2 + s \lambda \phi |\nabla \hat{w}|^2] dx dt.
\end{equation}
By virtue of (2.34), (2.46), (2.55), and (2.56), we have
(2.58)
\[ \|f_s\|_{L^2(Q)}^2 + \|	ilde{f}_s\|_{L^2(Q)}^2 = \|	ilde{L}_1\tilde{w}\|_{L^2(Q)}^2 + \|L_1w\|_{L^2(Q)}^2 + \|	ilde{L}_2\tilde{w}\|_{L^2(Q)}^2 + \|L_2w\|_{L^2(Q)}^2 + 2 \int_Q \left[ s^3 \lambda^4 \phi^3 \frac{\partial^2}{\partial t^2} |\nabla \psi| |\tilde{w}| + s^3 \lambda^4 \phi^3 |\nabla |\tilde{w}| \right] \right] \left( L_2^2 + \|L_2\|_{L^2(Q)}^2 \right)
+ 2 \int_Q \left[ s^3 \lambda^4 \phi^3 \frac{\partial^2}{\partial t^2} |\nabla \psi| |\tilde{w}| + s^3 \lambda^4 \phi^3 |\nabla \tilde{w}| \right] \right] \left( L_2^2 + \|L_2\|_{L^2(Q)}^2 \right) + \int_{\Sigma_1} 4k_2(x) s^2 \phi |\nabla \psi| |w|^2 \, d\sigma + \int_{\Sigma_1} 4k_2(x) s^2 \phi |\nabla \psi| |\tilde{w}|^2 \, d\sigma.

On the other hand, there exists a \( \beta > 0 \) such that
\[ |\nabla \psi(x)| > \beta > 0 \quad \forall x \in \Omega \setminus \omega_0. \]

Using this fact and (2.22), (2.50), and (2.57), choosing \( \tilde{\lambda} \) large enough, from (2.33), (2.54), and (2.58) we have
(2.59)
\[ \left[ \int_{Q_{s_0}} \left( \lambda \phi^3 |\nabla w|^2 + \lambda \phi^3 |\nabla \tilde{w}|^2 + s \phi |\nabla w|^2 + s \phi |\nabla \tilde{w}|^2 \right) dxdt 
+ \|ge^{s\alpha}\|_{L^2(Q)}^2 + \|ge^{s\alpha}\|_{L^2(Q)}^2 \right] \quad \forall s \geq s_0(\lambda), \forall \lambda \geq \tilde{\lambda}. \]

On the other hand, from (2.30), (2.31), and (2.48) we get
(2.60)
\[ \int_Q \frac{1}{s^2} |\nu \Delta w|^2 \, dxdt \leq \int_Q \frac{1}{s^2} |L_1w|^2 \, dxdt + C \int_Q \lambda s \phi^3 |w|^2 \, dxdt + C \int_Q \frac{1}{s} |L_1w|^2 \, dxdt + C \int_Q \frac{1}{s} |L_2w|^2 \, dxdt. \]

(2.61)
\[ \int_Q \frac{1}{s^2} \left| \frac{\partial w}{\partial t} \right|^2 \, dxdt \leq \int_Q \frac{1}{s^2} |L_2w|^2 \, dxdt + C \int_Q \left[ \lambda s \phi |\nabla w|^2 + \lambda s \phi |w|^2 \right] \, dxdt. \]
Also, from (2.52) and (2.53) we get

\[
\int_{Q} \frac{1}{s\varphi} |\nu \Delta \tilde{w}|^2 \, dxdt \leq \int_{Q} \frac{1}{s\varphi} |\bar{L}_1 \tilde{w}|^2 \, dxdt + C \int_{Q} \lambda^4 s^3 \tilde{\varphi} |\tilde{w}|^2 \, dxdt \\
+ C \int_{Q} s\tilde{\varphi}^3 |\tilde{w}|^2 \, dxdt,
\]

(2.62)

\[
\int_{Q} \frac{1}{s\varphi} \left| \frac{\partial \tilde{w}}{\partial t} \right|^2 \, dxdt \leq \int_{Q} \frac{1}{s\varphi} |\bar{L}_2 \tilde{w}|^2 \, dxdt + C \int_{Q} |\lambda^2 s\tilde{\varphi} |\nabla \tilde{w}|^2 \\
+ \lambda^4 s\tilde{\varphi} |\tilde{w}|^2 |dxdt.
\]

(2.63)

If necessary, taking \( s_0(\lambda) \) large enough for corresponding \( \lambda \), from (2.59)–(2.63) we have

\[
\int_{Q} \left[ \frac{1}{s\varphi} \frac{\partial w}{\partial t} \right]^2 + \frac{1}{s\varphi} |\nu \Delta w|^2 + \frac{1}{s\varphi} |\Delta \tilde{w}|^2 \right] \, dxdt \\
\leq C \left[ \int_{Q_{s_0}} \left( s^3 \lambda^4 \varphi^3 |w|^2 + s^3 \lambda^4 \varphi^3 |\tilde{w}|^2 + s^3 \lambda^4 \varphi^3 |\nabla w|^2 + s^3 \lambda^4 \varphi^3 |\nabla \tilde{w}|^2 \right) \right] \\
+ C \left[ \|g e^{\alpha}\|^2_{L_2(Q)} + \|g e^{\tilde{\alpha}}\|^2_{L_2(Q)} \right] \quad \forall s \geq s_0(\lambda),
\]

(2.64)

By virtue of (2.25), from (2.64) we get

\[
\int_{Q} \left[ \left( \frac{1}{s\varphi} \frac{\partial y}{\partial t} \right)^2 + \frac{1}{s\varphi} |\nu \Delta y|^2 + s^2 \lambda^2 \varphi |\nabla y|^2 + s^3 \lambda^4 \varphi^3 |y|^2 \right) e^{2s\alpha} \\
+ \left( \frac{1}{s\varphi} \frac{\partial \tilde{y}}{\partial t} \right)^2 + \frac{1}{s\varphi} |\nu \Delta \tilde{y}|^2 + s^2 \lambda^2 \tilde{\varphi} |\nabla \tilde{y}|^2 + s^3 \lambda^4 \tilde{\varphi}^3 |\tilde{y}|^2 \right) e^{2s\tilde{\alpha}} \right] \, dxdt \\
\leq C(\lambda) \left[ \int_{Q_{s_0}} \left( s^3 \lambda^4 \varphi^3 |y|^2 e^{2s\alpha} + s^3 \lambda^4 \varphi^3 |y|^2 e^{2s\tilde{\alpha}} + s^3 \lambda^4 \varphi^3 |\nabla y|^2 \right. \\
+ s^2 \lambda^2 \varphi e^{2s\alpha} |\nabla y|^2 \right) \\
+ \|g e^{\alpha}\|^2_{L_2(Q)} + \|g e^{\tilde{\alpha}}\|^2_{L_2(Q)} \right] \quad \forall s \geq s_0(\lambda), \forall \lambda \leq \lambda.
\]

(2.65)

Now, taking a function \( \rho(x) \in C_0^\infty(\omega) \) such that \( \rho(x) \equiv 1 \) on \( \omega_0 \) and \( \rho(x) > 0 \) and arguing as in [8], from (2.65) we have

\[
\int_{Q} \left[ \frac{1}{s\varphi} \left| \frac{\partial y}{\partial t} \right|^2 + \frac{1}{s\varphi} |\nu \Delta y|^2 + s\varphi |\nabla y|^2 + s^3 \varphi^3 |y|^2 \right) e^{2s\alpha} + e^{2s\tilde{\alpha}} \right] \, dxdt \\
\leq C \left[ \int_{Q} |g|^2 (e^{2s\alpha} + e^{2s\tilde{\alpha}}) \, dxdt + \int_{Q_{s_0}} s^3 \varphi^3 |y|^2 (e^{2s\alpha} + e^{2s\tilde{\alpha}}) \, dxdt \right].
\]

From this inequality we derive our conclusion.
LEMMA 2.7. Assume that \( \tilde{f} \in L_2(0,T;L_2(\Omega)) \), \( \text{div} \tilde{f} \in L_2(0,T;L_2(\Omega)) \). Then, there exists a number \( \lambda > 1 \) such that for every \( \lambda > \lambda \) there is a \( s_0(\lambda) \) with the property that for every \( s \geq s_0(\lambda) \) the solutions to (2.17) satisfy

\[
I(s) = \int_{\Omega} \left[ \frac{1}{\rho^2} \left| \frac{\partial y}{\partial \tau} \right|^2 + s \overline{\varphi} |\nabla y|^2 + s^3 \overline{\varphi}^3 |y|^2 \right] e^{2s\theta} dx dt 
\]

(2.66)

\[
\leq C \left\{ \int_{\Omega} |\tilde{f}|^2 e^{2s\theta} + (s \overline{\varphi})^2 e^{2s\theta} \right\} dx dt + \int_{\Omega} |\text{div} \tilde{f}|^2 (s \overline{\varphi})^\lambda e^{2s\theta} dx dt 
+ \int_{Q_{\omega_0}} s^3 \overline{\varphi}^3 |y|^2 + (s \overline{\varphi})^\lambda \right\} e^{2s\theta} dx dt \right\} \forall s \geq s_0(\lambda),
\]

where \( C \) depends on \( \lambda \).

Proof. Applying Lemma 2.6 for \( \Pi \tilde{f} \) instead of \( \tilde{f} \) and arranging the result, we have that there exists a number \( \lambda > 1 \) such that for every \( \lambda > \lambda \) there is a \( s_0(\lambda) \) such that for every \( s \geq s_0(\lambda) \) the solutions to (2.18) satisfy

\[
\int_{\Omega} \left[ \frac{1}{\rho^2} \left| \frac{\partial y}{\partial \tau} \right|^2 + s \overline{\varphi} |\nabla y|^2 + s^3 \overline{\varphi}^3 |y|^2 \right] (e^{2s\theta} + e^{2s\tilde{\theta}}) dx dt 
\]

\[
\leq C \left[ \int_{\Omega} |\tilde{f}|^2 |\nabla p|^2 (e^{2s\theta} + e^{2s\tilde{\theta}}) dx dt + \int_{Q_{\omega_0}} s^3 \overline{\varphi}^3 |y|^2 (e^{2s\theta} + e^{2s\tilde{\theta}}) dx dt \right].
\]

Taking \( s_0(\lambda) \) large enough, from this inequality we have that for all \( s \geq s_0(\lambda) \), for all \( \lambda \geq \tilde{\lambda} \)

\[
I(s) = \int_{\Omega} \left[ \frac{1}{\rho^2} \left| \frac{\partial y}{\partial \tau} \right|^2 + s \overline{\varphi} |\nabla y|^2 + s^3 \overline{\varphi}^3 |y|^2 \right] e^{2s\theta} dx dt 
\]

(2.67)

\[
\leq C \left[ \int_{\Omega} |\tilde{f}|^2 e^{2s\theta} dx dt + \int_{\Omega} |\nabla p|^2 e^{2s\theta} dx dt + \int_{\omega_0} s^3 \overline{\varphi}^3 |y|^2 e^{2s\theta} dx dt \right].
\]

On the other hand, from (2.1) and (2.2)

(2.68)

\[
\Delta p = \text{div} \left[ \text{rot} \hat{v} \times y - \text{rot} (\hat{v} \times y) \right] + \text{div} \tilde{f}.
\]

Considering

\[
\text{div} \left[ \text{rot} \hat{v} \times y - \text{rot} (\hat{v} \times y) \right] = \text{div}(\text{rot} \hat{v} \times y) = y \cdot \Delta \hat{v} - \text{rot} \hat{v} \cdot \text{rot} y
\]

and the assumption (1.3) on \( \hat{v} \), we have

(2.69)

\[
\| \text{div} \left[ \text{rot} \hat{v} \times y - \text{rot} (\hat{v} \times y) \right] \|_{L_2(\Omega)} \leq C(\|y\|_{L_2(\Omega)} + \|\nabla y\|_{L_2(\Omega)}),
\]

where \( C \) is independent of \( t \).

By virtue of Theorem 1.2 in [30], from (2.68) and (2.69) we have

\[
\int_{\Omega} s^{-\frac{1}{2}} |\nabla p(\cdot,t)|^2 e^{2s\varphi} dx \leq c \left( \|p(\cdot,t)\|_{H^s(\partial\Omega)} e^{2s} 
+ \int_{\Omega} s^{-\frac{1}{2}} (|\nabla y|^2 + |y|^2 + |\text{div} \tilde{f}|^2) e^{2s\varphi} dx 
+ \int_{\Omega} s^2 p^2(\cdot,t) e^{2s\varphi} dx \right).
\]
where \( \tilde{\varphi} = e^{\lambda \hat{\varphi}} \). Here and in what follows we do not distinguish newly taken \( s_0(\lambda) \).

From above, by the same argument as (2.12) in [30], we have

\[
\int_{\Omega} |\nabla p(\cdot, t)|^2 e^{2s\alpha} \, dx \leq c \left( \int_{Q_{s_0}} \|p(\cdot, t)\|^2_{H^\alpha(\partial\Omega)} (s\tilde{\varphi}(t))^\frac{3}{2} e^{2s\alpha(t)} \right)
+ \int_{\Omega} (s\tilde{\varphi})^\frac{3}{2} (|\nabla \tilde{y}|^2 + |\tilde{y}|^2 + |\nabla \tilde{f}|^2) e^{2s\alpha} \, dx
+ \int_{\omega_0} (s\tilde{\varphi})^\frac{1}{2} p^2(\cdot, t) e^{2s\alpha} \, dx.
\]

(2.70)

From (2.67) and (2.70) we get

\[
I(s) \leq c \left( \int_{Q_{s_0}} [s^3 \tilde{\varphi}^3 |\tilde{y}|^2 + (s\tilde{\varphi})^\frac{3}{2} p^2] e^{2s\alpha} \, dx \, dt \right)
+ \int_{0}^{T} (s\tilde{\varphi}(t))^\frac{3}{2} ||p||^2_{H^\alpha(\partial\Omega)} e^{2s\alpha} \, dt
+ \int_{Q} |\tilde{f}|^2 e^{2s\alpha} \, dx \, dt
\quad \forall s \geq s_0(\lambda).
\]

(2.71)

Now, by the same argument as (2.16) in [30], we have

\[
\int_{0}^{T} (s\tilde{\varphi}(t))^\frac{3}{2} ||p(\cdot, t)||^2_{H^\alpha(\partial\Omega)} e^{2s\alpha} \, dt \leq c \left( \int_{0}^{T} (s\tilde{\varphi}(t))^\frac{3}{2} ||p(\cdot, t)||^2_{L^2(\omega_0)} e^{2s\alpha} \, dt \right)
+ \int_{Q} (s\tilde{\varphi})^\frac{3}{2} |\tilde{y}|^2 e^{2s\alpha} + |\tilde{f}|^2 (s\tilde{\varphi})^\frac{3}{2} e^{2s\alpha} \, dx \, dt.
\]

(2.72)

By (2.71), (2.72), and the fact that \( \hat{\alpha}(t) \leq \alpha(x, t) \), we can obtain (2.66) for the solutions to (2.18), from which we come to the conclusion by passing to the limits.

Now, we are in the position to prove Theorem 2.5.

Proof of Theorem 2.5. Put \( F = \text{rot } \hat{\varphi} \times y - \text{rot } (\hat{\varphi} \times y) + f \). By Lemma 2.3 and (1.3), \( F \in L_2(0, T; L_2(\Omega)) \). Since \( \text{div } F = \text{div } (\text{rot } \hat{\varphi} \times y) \in L_2(0, T; L_2(\Omega)) \), applying Lemma 2.7 to (2.17) with \( F \) instead of \( \tilde{f} \) and moving the term \( \int_{Q} |\text{div } (\text{rot } \hat{\varphi} \times y)|^2 (s\tilde{\varphi})^\frac{3}{2} e^{2s\alpha} \, dx \, dt \) to the left-hand side, we can obtain the conclusion and complete the proof of Theorem 2.5.

Now we begin to obtain an observability inequality which will be used in the next section.

Let us consider the problem

\[
\begin{cases}
-\frac{\partial z}{\partial t} + \nu \text{rot rot } z - \text{rot } \hat{\varphi} \times z + \text{rot } (\hat{\varphi} \times z) + \nabla p = f, \\
\text{div } z = 0, \\
z|_{\Gamma_0} = 0, \quad z \times n|_{\Gamma_1} = 0, \quad p|_{\Gamma_1} = 0, \\
z(T) = z_T(x).
\end{cases}
\]

(2.73)

Lemma 2.8. Suppose that \( f \in L_2(0, T; H) \) and \( z_T(x) \in V \). Then, solutions to (2.73) satisfy

\[
\| \frac{\partial z}{\partial t} \|_{L_2(0, T; H)} + \| z \|_{L_2(0, T; D(\hat{A}))} \leq c (\| z \|_{L_2(0, T; H)} + \| f \|_{L_2(0, T; H)}).
\]
Then,
\[
\begin{align*}
\frac{\partial y}{\partial t} + \|y\|_{L_2(\Omega)} + \|y\|_{L_2(T;\mathcal{D}(\mathcal{A}))} &
\leq C \left( \|y\|_{L_2(\Omega)} + \|y\|_{L_2(T;\mathcal{D}(\mathcal{A}))} \right)
\end{align*}
\]
in problem (2.1) when \(y_0 \in V\). Take a function \(\rho\) such that
\[
\rho \in C^\infty([0, T]), \quad \rho(t) = 1 \quad \forall t \in [T/2, T], \quad \rho(t) = 0 \quad \forall t \in (0, T/4)
\]
and put \(Y = \rho y\). Then, we have the problem
\[
\begin{align*}
\frac{\partial Y}{\partial t} + \nu \text{rot} Y - \text{rot} \ (\hat{v} \times Y) + \nabla \rho p = Y \frac{\partial \rho}{\partial t} + \rho f,
\end{align*}
\]
(2.74)
\[
\begin{align*}
Y|_{\Gamma_0} = 0, \quad Y \times n|_{\Gamma_1} = 0, \quad \rho p|_{\Gamma_1} = 0, \\
Y(T) = 0.
\end{align*}
\]

Applying Lemma 2.3 to (2.74) on the interval \([T/4, T]\) and using the fact that \(Y = \rho y\) on \([T/2, T]\), we get the conclusion. \(\blacksquare\)

Take a function \(\ell \in C^\infty([0, T])\) such that \(\ell(t) = t\) for all \(t \in [T/2, T]\), \(\ell(t) > 0\) for all \(t \in [0, T]\).

Let
\[
k(x, t) = \frac{\hat{\lambda}^\psi - \hat{\lambda}_0^2 \|\psi\|_{C(\Omega)}}{[\ell(t)(T-t)]^s},
\]
(2.75)
where \(\hat{\lambda} > \hat{\lambda}_0\) is taken such that
\[
\max\{k(x, t) \mid x \in \bar{\Omega}\} < \frac{95}{100} \min\{k(x, t) \mid x \in \bar{\Omega}\} \quad \forall t \in [0, T]
\]
and \(\hat{\lambda}\) is the number in Theorem 2.5.

Let
\[
k(t) = \min\{k(x, t) \mid x \in \bar{\Omega}\}, \quad \tilde{k}(t) = \max\{k(x, t) \mid x \in \bar{\Omega}\}.
\]
(2.76)

Then,
\[
k(x, t) = \alpha_{\hat{\lambda}}(x, t) = \frac{\hat{\lambda}^\psi(x) - \hat{\lambda}_0^2 \|\psi\|_{C(\Omega)}}{[\ell(t)(T-t)]^s} \quad \forall (x, t) \in \Omega \times \left(\frac{3}{4}T, T\right).
\]

**Theorem 2.9** (observability inequality). Suppose that \(z\) is the solution to (2.73); then there exists \(\hat{s} > 0\) such that
\[
\|z(0)\|_V^2 + \int_Q (T-t)^{\hat{s}} |z|^2 e^{\hat{s}k} dx dt \leq c \left( \int_Q |f|^2 e^{\hat{s}k} dx dt + \int_{Q_\omega} |z|^2 e^{\hat{s}k} dx dt \right),
\]
where \(c\) is independent of \(f\) and \(z_T(x)\).

**Proof.** We adapt the method in [30]. Define
\[
r(x, t) = \int_{\frac{T}{2}}^t z(x, \tau) d\tau, \quad g(x, t) = \int_{\frac{T}{2}}^t p(x, \tau) d\tau, \quad \tilde{f}(x, t) = \int_{\frac{T}{2}}^t f(x, \tau) d\tau.
\]
Then,
\[
\int_{\frac{T}{2}}^{t} (\text{rot} \hat{v} \times z)(\tau) \, d\tau = \text{rot} \hat{v}(t) \times \int_{\frac{T}{2}}^{t} z(\tau) \, d\tau - \int_{\frac{T}{2}}^{t} \left[ \text{rot} \frac{\partial \hat{v}}{\partial \tau}(\tau) \times \int_{\frac{T}{2}}^{\tau} z(\rho) \, d\rho \right] \, d\tau,
\]
\[
\int_{\frac{T}{2}}^{t} \text{rot} (\hat{v} \times z)(\tau) \, d\tau = \text{rot} \left[ \hat{v}(t) \times \int_{\frac{T}{2}}^{t} z(\tau) \, d\tau \right] - \int_{\frac{T}{2}}^{t} \text{rot} \left[ \frac{\partial \hat{v}}{\partial \tau}(\tau) \times \int_{\frac{T}{2}}^{\tau} z(\rho) \, d\rho \right] \, d\tau,
\]
\[
\int_{\frac{T}{2}}^{t} \frac{\partial z}{\partial t} \, dt = z(t) - z \left( \frac{T}{2} \right) = \frac{\partial r}{\partial t} - z \left( \frac{T}{2} \right).
\]
Using this, we have
\[
\begin{cases}
-\frac{\partial r}{\partial t} + \nu \text{rot } r - \text{rot } \hat{v} \times r + \text{rot} (\hat{v} \times r) + \nabla g \\
= \tilde{f} - z \left( \frac{T}{2} \right) - \int_{\frac{T}{2}}^{t} \left( \text{rot} \frac{\partial \hat{v}}{\partial \tau} \times r \right)(\xi) \, d\xi + \int_{\frac{T}{2}}^{t} \text{rot} \left[ \frac{\partial \hat{v}}{\partial \tau} \times r \right](\xi) \, d\xi,
\end{cases}
(2.77)
\]
\[
\begin{align*}
\text{div } r &= 0, \\
r \big|_{\Gamma_0} &= 0, \quad r \times n \big|_{\Gamma_1} = 0, \quad g \big|_{\Gamma_1} = 0, \\
r(T) &= \int_{\frac{T}{2}}^{T} z(t) \, dt \in V.
\end{align*}
\]
From (2.77), by the same argument as (2.28) of [30], we can get
\[
\| g(t) \|_{L^2(\omega)} \leq c \| z(\cdot, T) \|_{L^2(\omega)} + \| r(\cdot, t) \|_{L^2(\omega)} + \| z \|_{L^2((\frac{T}{2}, t) \times \omega)} + \| \tilde{f}(\cdot, t) \|_{L^2(\omega)},
(2.78)
\]
where (1.3) was used.

Considering (1.3), we can prove
\[
\int_{\Omega} \left| \int_{\frac{T}{2}}^{t} \left( \text{rot} \frac{\partial \hat{v}}{\partial \tau} \times r \right)(\xi) \, d\xi \right|^2 \, e^{2\alpha t} \, dx \, dt 
\leq c(\lambda) \int_{\Omega} \left( |\nabla r|^2 + |r|^2 \right) e^{2\alpha t} \, dx \, dt,
(2.79)
\]
\[
\int_{\Omega} \left| - \int_{\frac{T}{2}}^{t} \left( \text{rot} \frac{\partial \hat{v}}{\partial \tau} \times r \right)(\xi) \, d\xi \right|^2 \left( s \tilde{\phi} \right)^{\frac{3}{2}} e^{2\alpha t} \, dx \, dt 
\leq c(\lambda) \int_{\Omega} \left( |\nabla r|^2 + |r|^2 \right) \left( s \tilde{\phi} \right)^{\frac{3}{2}} e^{2\alpha t} \, dx \, dt.
(2.80)
\]
Also, we get estimate
\[
\int_{\Omega} \left| \text{div} \left( - \int_{\frac{T}{2}}^{t} \left( \text{rot} \frac{\partial \hat{v}}{\partial \tau} \times r \right)(\xi) \, d\xi \right) + \int_{\frac{T}{2}}^{t} \left( \text{rot} \frac{\partial \hat{v}}{\partial \tau} \times r \right)(\xi) \, d\xi \right|^2 \left( s \tilde{\phi} \right)^{\frac{1}{2}} e^{2\alpha t} \, dx \, dt 
\leq c(\lambda) \int_{\Omega} \left( \left( s \tilde{\phi} \right)^{\frac{1}{2}} |\nabla r|^2 + (s \tilde{\phi})^{\frac{3}{2}} |r|^2 \right) e^{2\alpha t} \, dx \, dt.
(2.81)\]
Using (2.79), (2.80), and (2.81) and applying Theorem 2.5, from (2.77) we have

\[
\int_Q \left[ \frac{1}{2} |z|^2 + s \dot{\varphi} |\nabla r|^2 + s^3 \dot{\varphi}^3 |r|^2 \right] e^{2 \alpha s} dx dt
\]

(2.82)

\[
\leq c(\lambda) \left\{ \int_Q \left( s \dot{\varphi} \left( |f|^2 + |z(x, \frac{T}{2})|^2 \right) e^{2 \alpha s} dx dt \right.
\right.
\]

\[
+ \int_{Q_{\infty}} \left[ (s \dot{\varphi}) \frac{4}{3} g^2 e^{2 \alpha s} + s^3 \dot{\varphi}^3 |r|^2 e^{2 \alpha s} \right] dx dt \right\} \quad \forall s \geq s_0(\lambda).
\]

Let us estimate

\[
I \equiv \int_Q (T - t)^{3/2} |z|^2 e^{sk} dx dt.
\]

Taking \( \lambda = \tilde{\lambda} \) and putting \( \hat{s} = 2s_0(\tilde{\lambda}) \), by the definition of \( k(x, t) \), (2.75), and (2.82), we get

\[
I_1 \equiv \int_{\Omega \times \left( \frac{T}{2}, T \right)} (T - t)^{3/2} |z|^2 e^{sk} dx dt \leq c(s) \left( \int_{Q_{\infty}} \frac{1}{(T - t)^{3/2}} g^2 e^{sk} dx dt \right.
\]

(2.83)

\[
+ \int_{Q_{\infty}} \frac{1}{(T - t)^{3/2}} |r|^2 e^{sk} dx dt + \int_{Q_{\infty}} \frac{|f|^2}{(T - t)^{1/2}} e^{sk} dx dt + \| z(\cdot, \frac{T}{2}) \|^2_{H} \right) \equiv I'_1 \quad \forall s \geq \hat{s},
\]

where the fact that \( s \dot{\varphi} \frac{4}{3} e^{2 \alpha s} \) is continuous on \( \bar{Q} \) was used. Using the estimate of the solution (cf. (2.13)), we have

\[
I_2 \equiv \int_{\Omega \times \left( 0, \frac{T}{2} \right)} (T - t)^{3/2} |z|^2 e^{sk} dx dt \leq c(s) \| z \|^2_{L_2(0, \frac{T}{2}; L_2(\Omega))}
\]

(2.84)

\[
\leq c(s) \left( \| z(\frac{T}{2}) \|^2_{H} + \| f \|^2_{L_2(0, \frac{T}{2}; V')} \right).
\]

Now, (2.83) and (2.84) imply

\[
(2.85) \quad I \leq c(s) \left( I'_1 + \| z \left( \frac{T}{2} \right) \|^2_{H} + \| f \|^2_{L_2(0, \frac{T}{2}; H)} \right).
\]

By Lemma 2.8, Proposition 2.1 of Chapter 1 in [33], and Theorem 3.2 of Chapter 1 in [33], we get

\[
\left\| z \left( \frac{T}{2} \right) \right\|_{V'} \leq \left( \frac{\partial z}{\partial t} \right)_{L_2(0, \frac{T}{2}; H)} + \| z \|^2_{L_2(0, \frac{T}{2}; L_2(D(\tilde{\lambda}))}
\]

(2.86)

\[
\leq c \left( \| z \|^2_{L_2(0, \frac{T}{2}; H)} + \| f \|^2_{L_2(0, \frac{T}{2}; H)} \right).
\]

By the fact that

\[
\| z \|^2_{L_2(0, \frac{T}{2}; H)} \leq c(s) I_1,
\]
we have
\[ \| z \|_{L^2(0, T; H)}^2 \leq c(s) \left( \left\| z \left( \frac{T}{2} \right) \right\|_H^2 + \| f \|_{L^2(0, T; H)}^2 + I_1 \right). \]

Thus, from (2.86) we have
\[ \left\| z \left( \frac{T}{2} \right) \right\|_V^2 \leq c(s) \left( \left\| z \left( \frac{T}{2} \right) \right\|_H^2 + \| f \|_{L^2(0, T; H)}^2 + I_1 \right). \]

Hence, (2.85) and (2.87) imply
\[
\int_Q (T-t)^k |z|^2 e^{\lambda k} dx dt + \| z \left( \frac{T}{2} \right) \|_V^2 \\
\leq c(s) \left( I_1 + \left\| z \left( \frac{T}{2} \right) \right\|_H^2 + \| f \|_{L^2(0, T; H)}^2 \right).
\]

Fixing \( s = \hat{s} = 2\hat{s}_0(\hat{\lambda}) \), by (2.78), (2.83), and (2.88), we have
\[
\int_Q (T-t)^k |z|^2 e^{\lambda k} dx dt + \| z \left( \frac{T}{2} \right) \|_V^2 \\
\leq c \left( \int_{Q_{\omega}} (T-t)^{-24} |r|^2 e^{\lambda k} dx dt \\
+ \int_{Q_{\omega}} \left\| \frac{|\hat{f}|^2 + |\hat{z}|^2 + |e^{\lambda k}|}{(T-t)^{24}} \right\|_V^2 e^{\lambda k} dx dt + \int_0^T \| z \left( \frac{T}{2} \right) \|_H^2 + \| f \|_{L^2(0, T; H)}^2 \right).
\]

Next, we will use (2.75) and (2.76) to prove
\[
\int_{Q_{\omega}} \frac{|\hat{f}|^2 + |\hat{z}|^2 + |e^{\lambda k}|}{(T-t)^{24}} e^{\lambda k} dx dt \\
\leq c \left( \int_Q \left| f \right|^2 e^{\frac{\hat{\lambda}}{2}\hat{k}} dx dt + \int_{Q_{\omega}} \left| z \right|^2 e^{\frac{\hat{\lambda}}{2}\hat{k}} dx dt \right).
\]

First, we have
\[
I' = \int_{Q_{\omega}} \frac{|\hat{f}|^2}{(T-t)^{24}} e^{\lambda k} dx dt \\
\leq c \int_{Q_{\omega}} \left| \int_0^t |f(r)|^2 dr \right|^2 e^{\frac{\hat{\lambda}}{2}\hat{k}(t)} e^{\frac{\hat{\lambda}}{2}\hat{k}(t)} (T-t)^{24} dx dt.
\]

Since \( e^{\frac{\hat{\lambda}}{2}\hat{k}(t)} \) is bounded on \( Q_{\omega} \), we have
\[ I' \leq c(\hat{s}) \int_{Q_{\omega}} \int_0^t |f(r)|^2 dr e^{\frac{\hat{\lambda}}{2}\hat{k}(r)} e^{\frac{\hat{\lambda}}{2}\hat{k}(r)} dx dt. \]

It is easy to verify that \( e^{\frac{\hat{\lambda}}{2}\hat{k}(t)-\hat{k}(r)} \) is bounded when \( \frac{T}{4} \leq r \leq t \leq \frac{3}{4}T \) or \( 0 \leq t \leq r \leq \frac{T}{2} \). Also, \( e^{\frac{\hat{\lambda}}{2}\hat{k}(r)} \) is decreasing on \( \frac{T}{4}T \leq r \leq t \leq T \). Thus,
\[
I' \leq c(\hat{s}) \left( \int_{Q_{\omega}} \int_0^t e^{\frac{\hat{\lambda}}{2}\hat{k}(r)} e^{\frac{\hat{\lambda}}{2}\hat{k}(r)} |f(r)|^2 dx dr dt \\
+ \int_{Q_{\omega}} \int_0^t e^{\frac{\hat{\lambda}}{2}\hat{k}(r)} |f(r)|^2 dx dr dt \right) \\
\leq c(\hat{s}) \int_{Q_{\omega}} |f(r)|^2 e^{\frac{\hat{\lambda}}{2}\hat{k}(r)} dx dt.
\]
In this way we can prove (2.90).

Similarly, we have

\begin{equation}
\int_{Q_{\omega}} (T-t)^{-24} |r|^2 e^{i\mathbf{k} \cdot x} dxdt \leq c \int_{Q_{\omega}} |z|^2 \cdot e^{\frac{\mathbf{\bar{n}}}{n} i\mathbf{k} \cdot x} dxdt,
\end{equation}

(2.91)

\begin{equation}
\int_{0}^{T} \frac{\|z\|_{L^2((\frac{T}{2}, t) \times \omega)}}{(T-t)^{24}} e^{i\mathbf{k} \cdot x} dt \leq c \int_{Q_{\omega}} |z|^2 \cdot e^{\frac{\mathbf{\bar{n}}}{n} i\mathbf{k} \cdot x} dxdt,
\end{equation}

(2.92)

\begin{equation}
\int_{Q} \frac{\hat{f}^2 e^{i\mathbf{k} \cdot x} dxdt}{(T-t)^{24}} + \|f\|_{L^2(0, T; L^2)}^2 \leq c \int_{Q} |f|^2 \cdot e^{\frac{\mathbf{\bar{n}}}{n} i\mathbf{k} \cdot x} dxdt.
\end{equation}

(2.93)

Therefore, (2.89)–(2.93) imply

\begin{equation}
\int_{Q} (T-t)^{8} |z|^2 e^{i\mathbf{k} \cdot x} dxdt + \|z(\frac{T}{2})\|^2
\end{equation}

\begin{equation}
\leq c \left( \int_{Q_{\omega}} |z|^2 \cdot e^{\frac{\mathbf{\bar{n}}}{n} i\mathbf{k} \cdot x} dxdt + \int_{Q} |f|^2 \cdot e^{\frac{\mathbf{\bar{n}}}{n} i\mathbf{k} \cdot x} dxdt + \|z(\frac{T}{2})\|^2 \right).
\end{equation}

(2.94)

To prove the conclusion it suffices to prove

\begin{equation}
\int_{Q} (T-t)^{8} |z|^2 e^{i\mathbf{k} \cdot x} dxdt + \|z(\frac{T}{2})\|^2
\end{equation}

\begin{equation}
\leq c \left( \int_{Q_{\omega}} |z|^2 \cdot e^{\frac{\mathbf{\bar{n}}}{n} i\mathbf{k} \cdot x} dxdt + \int_{Q} |f|^2 \cdot e^{\frac{\mathbf{\bar{n}}}{n} i\mathbf{k} \cdot x} dxdt \right).
\end{equation}

(2.95)

Supposing that (2.95) is not valid, we can choose a sequence \{(z_n, p_n, f_n)\} such that

\begin{equation}
\int_{Q} (T-t)^{8} |z_n|^2 e^{i\mathbf{k} \cdot x} dxdt + \|z_n(\frac{T}{2})\|^2
\end{equation}

\begin{equation}
> n \left( \int_{Q_{\omega}} |z_n|^2 \cdot e^{\frac{\mathbf{\bar{n}}}{n} i\mathbf{k} \cdot x} dxdt + \int_{Q} |f_n|^2 \cdot e^{\frac{\mathbf{\bar{n}}}{n} i\mathbf{k} \cdot x} dxdt \right).
\end{equation}

(2.96)

Here and in what follows, subsequences are renumbered with the same indices. Put

\begin{equation}
Z_n = \frac{z_n}{\|z_n(\frac{T}{2})\|_H}, \quad F_n = \frac{f_n}{\|z_n(\frac{T}{2})\|_H}, \quad \text{and} \quad Q_n = \frac{p_n}{\|z_n(\frac{T}{2})\|_H}.
\end{equation}

Then, \((Z_n, Q_n)\) is a solution to (2.73) with \(f_n\) instead of \(f\) and

\begin{equation}
\left\| Z_n \left( \frac{T}{2} \right) \right\|_H = 1.
\end{equation}

(2.97)

By (2.94) and (2.96), we have \(N\) such that for all \(n \geq N\)

\begin{equation}
\frac{n}{2} \left[ \int_{Q_{\omega}} |Z_n|^2 e^{\frac{\mathbf{\bar{n}}}{n} i\mathbf{k} \cdot x} dxdt + \int_{Q} |F_n|^2 e^{\frac{\mathbf{\bar{n}}}{n} i\mathbf{k} \cdot x} dxdt \right] \leq c,
\end{equation}

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(2.98) \[ \left\| Z_n \left( \frac{T}{2} \right) \right\|_{V} \leq c. \]

Thus, we have

\[
\begin{align*}
\int_{Q_\omega} \left| Z_n \right|^2 e^{\frac{\eta}{n}} \hat{s} \hat{k} dx dt & \to 0, \\
\int_{Q} \left| F_n \right|^2 e^{\frac{\eta}{n}} \hat{s} \hat{k} dx dt & \to 0.
\end{align*}
\] (2.99)

Using Lemma 2.3 and (2.97)–(2.99), we can prove the existence of a solution \( Z \in W \) to (2.73) such that

\[
\int_{\omega \times (0,T-\epsilon)} \left| Z(x,t) \right|^2 dx dt = 0,
\] (2.100)

\[
\left\| Z \left( \frac{T}{2} \right) \right\|_{H} = 1.
\]

Since \( \epsilon > 0 \) is arbitrary, by Theorem 2.5, \( Z \equiv 0 \) on \( Q \). This is a contradiction to (2.100), which proves our assertion. \( \square \)

3. Null controllability of the linearized problem. In this section, we establish a result which is very important in the proof of the main result of this paper in the next section.

Let us consider the problem

\[
\begin{align*}
Ly & \equiv \frac{\partial y}{\partial t} + \nu \text{rot rot } y + \text{rot } \hat{v} \times y + \text{rot } y \times \hat{v} = -\nabla p + f + u, \\
\text{div } y & = 0, \\
y \big|_{\Gamma_0} & = 0, \quad y \times n \big|_{\Gamma_1} = 0, \\
p \big|_{\Gamma_1} & = 0, \\
y(0) & = y_0 \in V.
\end{align*}
\] (3.1)

Remark 3.1. A solution to (3.1) is defined as in Definition 2.1. Also, similar to Lemma 2.3, existence and estimates of a solution to (3.1) in the space \( Z \) can be obtained.

Let

\[
\eta(x,t) = -\hat{s}k(x,t), \quad \theta(x,t) = (1 - \chi_\omega) \frac{e^{\eta}}{(T-t)^{8}} + \chi_\omega,
\]

and let \( L_2(Q, \theta) \) be a weighted \( L_2 \)-space with weight \( \theta(x,t) \).

Let

\[
F(Q, \theta) = \{ f \in L_2(Q) : \exists f_1 \in L_2(Q, \theta), \exists f_2 \in L_2(0,T; W_{2,1}^1(\Omega)), f = f_1 + \nabla f_2 \},
\]

\[
\| f \|_{F(Q, \theta)} = \inf \left\{ \left( \| f_1 \|_{L_2(Q, \theta)} + \| \nabla f_2 \|_{L_2(\Omega)} \right)^{\frac{1}{2}} : f_1 + \nabla f_2 \in F \right\},
\]

\[
Y(Q) = \left\{ y \in Z : Ly \in F(Q, \theta), e^{-\hat{s}k} y \in Z \right\}.
\]
\[ \|y\|_Y = \|Ly\|_{F(Q, \theta)} + \|e^{-\frac{i}{s} k} y\|_Z. \]

**Theorem 3.2.** Suppose that \( y_0 \in V, f \in F(Q, \theta) \). Then, there exists a solution \((y, p, u)\) to (3.1) such that
\[
\begin{align*}
y(T) &= 0, \\
(y, u) &\in Y \times L_2(Q_\omega), \\
\|\langle y, u \rangle\|_{Y \times L_2(Q_\omega)} &\leq c(\|y_0\|_V + \|f\|_{F(Q, \theta)}).
\end{align*}
\]

**Proof.** Define an operator \( B^*(t) \in (V \rightarrow H \leftarrow V^*) \) by
\[
\langle B^*(t)y, v \rangle = \langle \text{rot} \times y + \text{rot} y \times \hat{v}, v \rangle \quad \forall y, v \in V.
\]

The existence of a solution \((y, p)\) in \( Z \times L_2(0, T; H^1) \) to (3.1) is equivalent to the existence of a solution \( y \in Z \) to the following problem:
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial y}{\partial t} + (A + B^*)y = \Pi(f + u), \\
y(0) = y_0 \in V,
\end{array} \right.
\end{align*}
\]
where \( \Pi \) is the projector from \( L_2 \) onto \( H \) by (2.2). Problem (3.2) means
\[
\begin{align*}
\left\{ \begin{array}{l}
\Pi Ly = \Pi(f + u), \\
y(0) = y_0 \in V.
\end{array} \right.
\end{align*}
\]

First, suppose that \( f \in L_2(Q, \theta) \) and \( f|_{Q_\omega} = 0 \).

Let us consider the extremal problem of (3.2) with the set of controls \( U = L_2(0, T; L_2(\Omega)) \) and a cost functional
\[
J_k(y, u) = \frac{1}{2} \int_Q \rho_k |y|^2 dxdt + \frac{1}{2} \int_Q m_k |u|^2 dxdt \rightarrow \inf,
\]
where
\[
\rho_k = e^{-\frac{\alpha k(x)(T - s)^3}{10(T - s + 1/2)^8}}, \quad m_k(x, t) = \left\{ \begin{array}{ll}
e^{-\frac{\alpha k(x)(T - s)^3}{10(T - s + 1/2)^8}}, & x \in \bar{\omega}, \\
k, & x \in \Omega \setminus \bar{\omega}.
\end{array} \right.
\]

It is easy to verify the existence of a unique solution to the problem, which is denoted by
\[
(\hat{y}_k, \hat{u}_k) \in Z \times L_2(Q).
\]

Let \( y_0 \in D(\hat{A}) \). By the change of \( y \) with \( y(t) = \frac{x - t}{s} \cdot y_0 \), we have problem (3.1) with \( y(0) = 0, f \in L_2(0, T; L_2(\Omega)) \). Then by the Lagrange principle (cf. Theorem 1.5, Chapter 2 in [19]) there exists a \( z_k \in L_2(0, T; H) \) such that
\[
\int_Q \rho \hat{y}_k h dx dt - \left\langle \frac{\partial h}{\partial t} + (A + B^*)h, z_k \right\rangle = 0 \quad \forall h \in \{ v \in Z : v(0) = 0 \},
\]
\[
\int_Q (m_k \hat{u}_k \cdot u + z_k \cdot u) dx dt \geq 0 \quad \forall u \in L_2(0, T; L_2(\Omega)).
\]
From (3.4) we get
\[ -\frac{\partial z_k}{\partial t} + (A + B) z_k = \rho_k \hat{y}_k, \]
(3.6)
\[ z_k(T) = 0. \]
(3.7)
From (3.5) we have
\[ m_k \hat{u}_k + z_k = 0. \]
(3.8)
If \( y_0 \in V \), then there exists an arbitrarily small \( \varepsilon > 0 \) such that \( y(\varepsilon) \in D(\hat{A}) \). Thus, as above we obtain (3.6)–(3.8) on \( (\varepsilon, T) \). Since \( \varepsilon \) is arbitrarily small, we again get (3.6)–(3.8) on \((0, T)\).

By Theorem 2.9, from (3.6) and (3.7) we have
\[ \|z_k(0)\|_V^2 + \int_Q (T-t)\|z_k\|^2 e^{sk} \, dx \, dt \]
(3.9)
\[ \leq c \int_Q e^{\frac{\rho_k}{(T-t)}} \|y_k\|^2 \, dx \, dt + c \int_Q e^{\frac{\rho_k}{2}} \|z_k\|^2 \, dx \, dt. \]

It is easy to prove that
\[ |\sigma_k(t) e^{\frac{\rho_k}{2}}| < 1 \text{ on } Q, \quad |m_k e^{\frac{\rho_k}{2}}| < 1 \text{ on } Q_\omega. \]
(3.10)
By (3.8)–(3.10),
\[ \|z_k(0)\|_V^2 + \int_Q (T-t)\|z_k\|^2 e^{sk} \, dx \, dt \]
(3.11)
\[ \leq c \left( \int_Q \rho_k \|\hat{y}_k\|^2 \, dx \, dt + \int_Q m_k \|\hat{u}_k\|^2 \, dx \, dt \right). \]

Making an inner product with \( \hat{y}_k \) in (3.6) and transforming it, we have
\[ 0 = \left( z_k, \frac{\partial \hat{y}_k}{\partial t} + A \hat{y}_k + B^* \hat{y}_k \right)_H - (z_k(0), \hat{y}_k(0))_H - \int_Q \rho_k \|\hat{y}_k\|^2 \, dx \, dt, \]
(3.12)
\[ 0 = (z_k, f + \hat{u}_k)_H - (z_k(0), \hat{y}_k(0))_H - \int_Q \rho_k \|\hat{y}_k\|^2 \, dx \, dt. \]

By (3.8) and (3.12), we get
\[ \int_Q \rho_k \|\hat{y}_k\|^2 \, dx \, dt + \int_Q m_k \|\hat{u}_k\|^2 \, dx \, dt = \int_Q f \cdot z_k \, dx \, dt - (z_k(0), \hat{y}_k(0))_H. \]
(3.13)

Thus, by (3.11) and (3.13), we can get
\[ J_k(\hat{y}_k, \hat{u}_k) \leq c \left( \|y_0\|_H^2 + \|f\|_{L^2(Q, \theta)}^2 \right). \]
(3.14)
By (3.14), we can choose a subsequence \( \{ (\hat{y}_k, \hat{u}_k) \} \) such that
\[ (\hat{y}_k, \hat{u}_k) \rightharpoonup (y, u) \text{ weakly in } Z \times L^2(Q), \]
(3.15)
(3.17) \[ \int_Q \rho_k \hat{y}_k^2 \, dx \, dt + \int_Q m_k |\hat{u}_k|^2 \, dx \, dt \leq c \left( \|y_0\|_{L^2}^2 + \|f\|_{F(Q, \theta)}^2 \right). \]

Passing to the limit in (3.2), we can see that \((y, u)\) satisfies (3.1). Taking (3.16) into account and using Remark 3.1, we have

(3.18) \[ \| (y, u) \|_{L^2(Q)} \leq c \left( \|y_0\|_{L^2} + \|f\|_{F(Q, \theta)} \right). \]

To get the estimate in the conclusion we need only to get

(3.19) \[ \| Ly \|_{F(Q, \theta)} + \| ye^{-\hat{\delta} k} \|_{Z}^2 \leq c \left( \|y_0\|_{V}^2 + \|f\|_{F(Q, \theta)}^2 \right). \]

From (3.18) we have

(3.20) \[ \| Ly \|_{F(Q, \theta)} \leq c \left( \|y_0\|_{V} + \|f\|_{F(Q, \theta)} \right). \]

Now, let us prove

(3.21) \[ \| ye^{-\hat{\delta} k} \|_{Z} \leq c \left( \|y_0\|_{V} + \|f\|_{F(Q, \theta)} \right). \]

For \( \varepsilon \in (0, T) \) there exists \( K(\varepsilon) \) independent of \( k \) such that \( |\rho_k(x,t)| \leq K(\varepsilon) \). Thus, we have

(3.22) \[ \| \rho_k \hat{y}_k \|_{L^2(0,T-\varepsilon; L^2(\Omega))} \leq K_1(\varepsilon). \]

By (3.17) and (3.11), \( \{ \|z_k(0)\|_{V} \} \) is bounded. Thus, by Lemma 2.3, (3.22), and (3.6), we have

\[ \|z_k\|_{Z(0,T-\varepsilon)} \leq K_2(\varepsilon), \]

where \( Z(0,T-\varepsilon) \) is the space \( Z \) with \( T-\varepsilon \) instead of \( T \). By (3.8), \( \|m_k \hat{u}_k\|_{Z(0,T-\varepsilon)} \leq K_2(\varepsilon) \). Since \( \varepsilon \) is arbitrarily small, without loss of generality, there exists a sequence \( \hat{u}_k(x,t) \) such that

(3.23) \[ \hat{u}_k(x,t) \rightarrow u(x,t) \quad \text{almost everywhere in } Q_\omega. \]

By (3.17), (3.23), and Fatou’s theorem,

(3.24) \[ \| (y, u) \|_{L^2_1(Q, e^{-\hat{\delta} k})} \leq c. \]

Putting \( \tilde{y} = ye^{-\hat{\delta} k}, \tilde{p} = pe^{-\hat{\delta} k}, g = (f - \frac{2}{5} \hat{\delta} \hat{u}_k)e^{-\hat{\delta} k}, \tilde{u} = ue^{-\hat{\delta} k}, \) and \( \hat{y}_0 = y_0 e^{-\hat{\delta} k} \), we have

\[ \begin{cases}
L \tilde{y} = \nabla \tilde{p} + g + \chi_\omega \tilde{u}, \\
\text{div} \tilde{y} = 0, \\
\tilde{y}|_{\Gamma_0} = 0, \\
\tilde{y} \times n|_{\Gamma_1} = 0, \\
\tilde{y}(0) = \hat{y}_0.
\end{cases} \]

By (3.18) and (3.24), we can prove

(3.25) \[ \| g \|_{L^2_2(Q)} + \| \tilde{u} \|_{L^2_2(Q)} \leq c(\|y_0\|_{V} + \|f\|_{F(Q, \theta)}). \]

From (3.25) and (3.26), we get (3.21). Combining (3.20) and (3.21), we have (3.19).

The fact that \( y \in Z \) implies \( y(T) = 0 \).

Let us consider the case where \( f \in F(Q, \theta), f = f_1 + \nabla f_2, f_1 \in L^2_2(Q, \theta), \) and \( f_2 \in L^2_2(0,T; W^{1,1}_1(\Omega)) \). Let \((y, p, u)\) be the solution when \( f = (1 - \chi_\omega)f_1 \). Then, \((y, p + f_2, u + \chi_\omega f_1)\) is a solution corresponding to \( f \).
4. Local exact controllability of the nonlinear problem. In what follows we consider the case such that the following assumption is true.

**Assumption 4.1.** \(Au \in H\) implies \(\Delta u \in L^2\).

**Remark 4.2.** \(Au = f \in H\) means that \(u\) is a unique solution to the problem

\[
\Delta u + \nabla p = f, \quad \text{div } u = 0, \quad u|_{\Gamma_0} = 0, \quad u \times n|_{\Gamma_1} = 0, \quad p|_{\Gamma_1} = 0.
\]

Under Hypothesis 2.1 in [6] we know that \(u \in H^2\) (cf. Theorem 1.2 in [6]), and so Assumption 4.1 is satisfied. But, it is not obvious yet whether Assumption 4.1 is true or not when \(\Gamma_0 \cap \Gamma_1 \neq \emptyset\).

**Lemma 4.3.** Suppose that \(y \in D(\bar{A})\). Then, under Assumption 4.1 \(\text{rot } y \in H^1(\Omega)\) and \(\|\text{rot } y\|_{H^1(\Omega)} \leq K\|y\|_{D(\bar{A})}\).

**Proof.** It is enough to prove the conclusion for \(y \in V \cap H^2\). Let \(l = 3\). In a neighborhood of a point \(\bar{x} \in \partial \Omega\) let us introduce local curvilinear coordinates \((u,v,w)\) such that the lines \((u),(v)\), and \((w)\) are orthogonal at all points to each other, the linear \((u)\) is the outward normal \(n\) to \(\partial \Omega\), and the surface \(u = 0\) coincides with \(\partial \Omega\). Let the original orthogonal coordinates \((x_1,x_2,x_3)\) be expressed by \(x_1(u,v,w), x_2(u,v,w), x_3(u,v,w)\), and \(ds_u, ds_v,\) and \(ds_w\) are, respectively, the differentials of lengths of the lines \((u),(v)\), and \((w)\). Then,

\[
ds_u = L_u du, \quad ds_v = L_v dv, \quad ds_w = L_w dw,
\]

where \(L_u, L_v, \) and \(L_w\) are Lamé’s coefficients defined by

\[
\begin{align*}
L_u &= \sqrt{\left(\frac{\partial u}{\partial u}\right)^2 + \left(\frac{\partial u}{\partial v}\right)^2 + \left(\frac{\partial u}{\partial w}\right)^2}, \\
L_v &= \sqrt{\left(\frac{\partial v}{\partial u}\right)^2 + \left(\frac{\partial v}{\partial v}\right)^2 + \left(\frac{\partial v}{\partial w}\right)^2}, \\
L_w &= \sqrt{\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 + \left(\frac{\partial w}{\partial w}\right)^2},
\end{align*}
\]

(cf. equation (2.13.2) in [25]). In the curvilinear coordinates \((u,v,w)\) we have (cf. equation (2.13.8) in [25])

\[
\text{rot } y = \frac{1}{L_u L_v L_w} \left[ \frac{\partial}{\partial w} (L_w y_{eu}) - \frac{\partial}{\partial u} (L_u y_{ew}) \right] e_u + \frac{1}{L_u L_v L_w} \left[ \frac{\partial}{\partial v} (L_v y_{eu}) - \frac{\partial}{\partial v} (L_v y_{ev}) \right] e_v + \frac{1}{L_u L_v L_w} \left[ \frac{\partial}{\partial v} (L_w y_{eu}) - \frac{\partial}{\partial v} (L_w y_{ew}) \right] e_w.
\]

Let \(y_{eu}, y_{ev}, \) and \(y_{ew}\) be, respectively, projections of \(y\) on the unit vectors \(e_u, e_v,\) and \(e_w\) tangent to the lines \((u),(v),\) and \((w)\). When \(y \in D(\bar{A})\), \(y_{eu} = 0\) and \(y_{ev} = 0\) on \(\partial \Omega\). Thus, by (4.3) the component of \(\text{rot } y\) in the direction of \(e_u\) on \(\partial \Omega\) vanishes, that is, \((\text{rot } y, n)_{H^1(\partial \Omega)} = 0\).

Putting \(w = \text{rot } y\), we have that \(\text{div } w = 0\), \(\text{rot } w = -\Delta y \in L^2(\Omega)\). Thus, by Theorem 2.9 in [3], we have that \(w = \text{rot } y \in H^1(\Omega)\) and \(\|\text{rot } y\|_{H^1(\Omega)} \leq K(\|y\|_V + \|\Delta y\|_{L^2(\Omega)})\).

Let \(l = 2, \ y = (y_1, y_2),\) and \(w = \text{rot } y\). Then,

\[
\text{rot } w = \text{rot } \text{rot } y = \text{rot } \text{rot } y = -\Delta \bar{y} = -(\Delta y_1, \Delta y_2, 0) \in L^2(\Omega).
\]
On the other hand,

\[
\begin{align*}
\text{(4.5)} & \quad w = (w_1, w_2, w_3) = (0, 0, w_3), \quad \text{rot } w = \left( \frac{\partial w_3}{\partial x_2}, \frac{\partial w_3}{\partial x_1}, 0 \right).
\end{align*}
\]

Thus, (4.4) and (4.5) imply

\[
- \frac{\partial w_3}{\partial x_2} = \Delta y_1 \in L_2(\Omega), \quad \frac{\partial w_3}{\partial x_1} = \Delta y_2 \in L_2(\Omega),
\]

which shows that \( ||\text{rot } y||_{H^1(\Omega)} \leq K(||y||_V + ||\Delta y||_{L_2(\Omega)}) \) in the case that \( l = 2 \).

Now, let us introduce a new norm in \( D(A) \) by \( ||u||_\Delta = ||u||_V + ||\Delta u||_{L_2} \). Then, the operator \( A \in (D(\tilde{A}) \rightarrow H) \) is continuous, injective, and surjective. Thus, by the Banach theorem there exists a continuous inverse operator \( A^{-1} \), and so \( ||u||_\Delta \leq K||Au||_H \leq K||u||_{D(\tilde{A})} \), where \( ||\cdot||_{D(\tilde{A})} \) is given by (2.8). Taking this into account, we come to the conclusion.

**LEMMA 4.4.** Under Assumption 4.1 the operator \( (y_1, y_2) \mapsto \text{rot } y_1 \times y_2 \) is bilinear continuous from \( Y(Q) \times Y(Q) \) to \( F(Q, \theta) \).

**Proof.** Clearly, the operator is bilinear. Let us prove the continuity. Applying \( \eta = -\tilde{s}k < -\tilde{s}k \), we have

\[
||\text{rot } y_1 \times y_2||_{L_2(Q, \theta)}^2 \leq c \left[ \int_{Q \setminus Q_\omega} |\text{rot } y_1|^2 \cdot |y_2|^2 \cdot e^{\eta (T - t)}^{-\tilde{s}} \, dx \, dt + \int_{Q_\omega} |\text{rot } y_1|^2 \cdot |y_2|^2 \, dx \, dt \right] \leq c \int_Q |\text{rot } y_1|^2 \cdot |y_2|^2 \cdot e^{-\tilde{s}k (T - t)}^{-\tilde{s}} \, dx \, dt \leq c \int_Q |\text{rot } y_1|^2 \cdot |y_2|^2 \cdot e^{-\tilde{s}k} \cdot e^{\tilde{s}k} (T - t) \, dx \, dt \leq c \int_Q e^{-\tilde{s}k} |\text{rot } y_1|^2 \cdot |y_2|^2 \, dx \, dt.
\]

Since \( e^{-\tilde{s}k} y_1 \in L_2(0, T; D(\tilde{A})) \), by Lemma 4.3

\[
||e^{-\tilde{s}k} \text{rot } y_1||_{L_2(0, T; L_2(\Omega))} \leq c ||e^{-\tilde{s}k} \text{rot } y_1||_{L_2(0, T; H^1(\Omega))} \leq c ||e^{-\tilde{s}k} y_1||_{L_2(0, T; D(\tilde{A}))}.
\]

Also, from

\[
\frac{\partial}{\partial t}(e^{-\tilde{s}k} y_2) \in L_2(0, T; D(\tilde{A})), \quad \frac{\partial}{\partial t}(e^{-\tilde{s}k} y_2) \in L_2(0, T; H)
\]

we have (cf. Proposition 2.1 and Theorem 3.2 of Chapter 1 in [33])

\[
\text{(4.8)} \quad e^{-\tilde{s}k} y_2 \in L_\infty(0, T; V), \quad ||e^{-\tilde{s}k} y_2||_{L_\infty(0, T; V)} \leq K ||e^{-\tilde{s}k} y_2||_{L_2(0, T; V)}. \]

By (4.6)–(4.8), we have

\[
\text{(4.9)} \quad ||\text{rot } y_1 \times y_2||_{L_2(Q, \theta)}^2 \leq K ||y_1||_V^2 \cdot ||y_2||_V^2.
\]
This ends the proof of Lemma 4.4. □

To prove the local exact controllability of the Navier–Stokes equation we use the following result.

**Proposition 4.5** (cf. [30, Theorem 4.1]). Suppose that $X, Z$ are Banach spaces and $A : X → Z$ is a continuously differentiable map. Assume that for $x_0 \in X$, $z_0 \in Z$ the equality $A(x_0) = z_0$ holds and the derivative $A'(x_0) : X → Z$ of the map at $x_0$ is an epimorphism. Then there exists $\varepsilon > 0$ such that for any $z \in Z$ which satisfies the condition $\|z - z_0\|_Z < \varepsilon$ there exists a solution $x \in X$ of the equation $A(x) = z$.

Let $\hat{f} \in L_2(0, T; V^*)$ and $\varphi_1(x, t) \in H^{\frac{1}{2}}(\Gamma_1)$ in (1.1).

**Definition 4.6.** A function $v \in L_2(0, T; V)$ is called a solution to (1.1) if

$$\frac{\partial v}{\partial t} + \nu(\nabla v, \nabla v) + \nabla p = \hat{f} + \chi_\omega u, \quad \text{div } v = 0,$$

$$v|_{\Gamma_0} = a(x, t), \quad v \times n|_{\Gamma_1} = b(x, t) \times n, \quad (p + \frac{1}{2} \|v\|^2)|_{\Gamma_1} = \varphi_1(x, t),$$

$$v(0) = y_0 + \hat{v}(0), \quad v(T) = \hat{v}(T).$$

**Proof.** The conclusion is equivalent to local null controllability of (1.4). The existence of a solution $y \in Z$ to (1.4) (cf. Remark 3.1) is equivalent to the existence of a solution $y \in Z$ to the following problem:

$$\begin{cases}
\frac{\partial y}{\partial t} + (A + B^*) y + \Pi(\nabla y \times y) = \Pi \chi_\omega u, \\
y(0) = y_0 \in V.
\end{cases} \tag{4.10}$$

Let $X = Y(Q) \times L_2(Q_\omega)$, $Z = \Pi F(Q, \theta) \times V$, and $A(y, u) = (\Pi N(y, u), y(0))$, where

$$N(y, u) = \frac{\partial y}{\partial t} + \nu \text{rot } y + \hat{v} \times y + \text{rot } \hat{y} \times y + \text{rot } y \times \hat{y} - \chi_\omega u$$

and $\Pi$ is one in (3.2). Putting $x_0 = (0, 0)$ and $z_0 = (0, 0)$, we have $A(x_0) = z_0$. By Lemma 4.4 and the definition of space $F(Q, \theta)$, the operator $A \in (X → Z)$ and is continuously differentiable. It is easy to verify

$$A'(x_0)(y, u) = \{\Pi L(y, u), y(0)\}$$

$$\equiv \{\Pi \left(\frac{\partial y}{\partial t} - \nu \Delta y + \text{rot } \hat{v} \times y + \text{rot } \hat{y} \times y - \chi_\omega u\right), y(0)\}.$$ 

By Theorem 3.2, there exists a solution to (3.1) such that $(y, u) \in X$ for $f \in F(Q, \theta)$ and $y_0 \in V$. This means

$$\begin{cases}
\frac{\partial y}{\partial t} + (A + B^*) y - \Pi \chi_\omega u = \Pi f, \\
y(0) = y_0
\end{cases}$$
(cf. (3.3)). Thus, $A'(y_0) \in (X \rightarrow Z)$ is an epimorphism.

Therefore, by Proposition 4.5 when $\|y_0\|_V \leq \delta$ for sufficiently small $\delta > 0$, there exists a solution $(y, u) \in X$ to the problem $HN(y, u) = 0$, which means that $(y, u)$ satisfies (4.10).

By the definition of $Y(Q)$, we see that $y(T) = 0$, which shows our assertion. \qed

Let $\Gamma_c$ be an open subset of $\Gamma_0$ such that $\tilde{\Gamma}_c \subset \Gamma_0$. Let us consider the following boundary control problem:

$$
\begin{aligned}
\frac{\partial v}{\partial t} - \nu \Delta v + (v, \nabla)v + \nabla p &= 0, \\
\text{div} \ v &= 0, \\
v|_{\Gamma_0 \setminus \Gamma_c} &= 0, \\
v|_{\Gamma_c} &= u, \\
v \times n|_{\Gamma_1} &= 0, \\
(p + \frac{1}{2}|v|^2)|_{\Gamma_1} &= 0 (\text{or } c(t)), \\
v(0) &= v_0,
\end{aligned}
$$

(4.11)

where $c(t)$ is independent of $x$.

**Theorem 4.8** (local exact controllability by boundary controls). Under Assumption 4.1 there exists $\varepsilon > 0$ such that for $v_0 \in V$, $\|v_0\|_V \leq \varepsilon$ there exists a control $u \in L_2(0, T; H^2(\Gamma_c))$ to problem (4.11) by which $v(T) = 0$.

Proof. Putting $\hat{v}(x, t) \equiv 0$, $\hat{f} \equiv 0$, and $\varphi_1(x, t) \equiv 0$ (or $c(t)$), we have problem (4.11) with $u = 0$. Let us consider a connected domain $\tilde{\Omega} \supset \Omega$ such that $\tilde{\Omega} \setminus \partial \tilde{\Omega} \subset \Gamma_c$ and $\partial \tilde{\Omega} \subset C^2$. Let $\tilde{v}_0$ be the extension of $v_0$ such that $\tilde{v}_0 = 0$ on $\tilde{\Omega} \setminus \Omega$. Set $\omega = \tilde{\Omega} \setminus \Omega$. Let us consider the following control problem on $\Omega \times (0, T)$:

$$
\begin{aligned}
\frac{\partial \tilde{v}}{\partial t} - \nu \Delta \tilde{v} + (\tilde{v}, \nabla)\tilde{v} + \nabla p &= \chi_\omega \tilde{u}, \\
\text{div} \ \tilde{v} &= 0, \\
\tilde{v}|_{\partial \tilde{\Omega} \times [0, T]} &= 0, \\
\tilde{v} \times n|_{\Gamma_1} &= 0, \\
(p + \frac{1}{2}|\tilde{v}|^2)|_{\Gamma_1} &= 0 (\text{or } c(t)), \\
\tilde{v}(0) &= \tilde{v}_0, \\
\tilde{v}(T) &= 0.
\end{aligned}
$$

By Theorem 4.7, there exists a control $\tilde{u}$ to the above problem. Now $u = v|_{\Gamma_c \times (0, T)} \in L_2(0, T; H^2(\Gamma_c))$ is the control in the assertion. \qed

**5. Appendix.** In this section we consider 2-D domains for which two kinds of norms are equivalent (see Lemma 5.3). For the convenience of readers we first give a proof of a special case of Lemma 7 in [32] which was used in the proof of Lemma 2.4.

**Lemma 5.1.** Suppose that $y \in H^2$ and $y \times n|_{\Gamma_1} = 0$; then there exists $k(x) \in C^1(\Gamma_1)$ such that

$$
\left( \frac{\partial y}{\partial n}, n \right)_{\Gamma_1} + (k(x)y, n)_{\Gamma_1} = \text{div} \ y|_{\Gamma_1},
$$

where $k(x)$ depends only on the form of the surface $\Gamma_1$.

Proof. By the trace theorem (cf. Theorem 1.2, Chapter 1 in [37]), it suffices to prove the conclusion for $y \in C^2(\Omega)$. In a neighborhood of a point $\tilde{x} \in \Gamma_1$ let us consider the local curvilinear coordinates $(u, v, w)$ (if $l = 2$, then $(u, v)$) as in the proof of Lemma 4.3. In the new coordinates $(u, v, w)$ we have (cf. equation (2.13.6) in [25])

$$
\text{div} \ y = \frac{1}{L_u L_w} \left[ \frac{\partial}{\partial u} \left( L_u L_w y_{w_u} \right) + \frac{\partial}{\partial v} \left( L_u L_w y_{v_u} \right) + \frac{\partial}{\partial w} \left( L_u L_w y_{w_v} \right) \right].
$$

(5.1)
From $y \times n|_{\Gamma_1} = 0$ we have $y_{e_v} = 0$ and $y_{e_w} = 0$. Using $L_u = 1$, from (5.1) we have
\[
\frac{\partial y_v}{\partial n} + \left(\frac{1}{L_v} \frac{\partial L_v}{\partial n} + \frac{1}{L_w} \frac{\partial L_w}{\partial n}\right)y_{e_v} = \text{div} y \text{ on } \Gamma_1.
\]

Now, considering that $y_{e_u} = (y, n)_{R^l}$, from this we have
\[
(5.2) \quad \left(\frac{\partial y}{\partial n}, n\right)_{R^l} + (k(\bar{x})y, n)_{R^l} = \text{div} y \text{ on } \Gamma_1, \quad k(\bar{x}) = \frac{1}{L_v} \frac{\partial L_v}{\partial u} + \frac{1}{L_w} \frac{\partial L_w}{\partial u}.
\]

On the other hand,
\[
(5.3) \quad \frac{\partial x}{\partial s_u} = e_u = n, \quad \frac{\partial x}{\partial s_v} = e_v, \quad \frac{\partial x}{\partial s_w} = e_w \text{ on } \Gamma_1.
\]

Since $L_u = 1$, by (4.2), (5.2), and (5.3),
\[
k(\bar{x}) = \frac{1}{L_v} \left(\frac{\partial x_v}{\partial v} \cdot \frac{\partial x_v}{\partial u} + \frac{\partial x_v}{\partial v} \cdot \frac{\partial x_v}{\partial u} + \frac{\partial x_v}{\partial v} \cdot \frac{\partial x_v}{\partial u}\right) + \frac{1}{L_w} \left(\frac{\partial x_v}{\partial w} \cdot \frac{\partial x_v}{\partial u} + \frac{\partial x_v}{\partial w} \cdot \frac{\partial x_v}{\partial u} + \frac{\partial x_v}{\partial w} \cdot \frac{\partial x_v}{\partial u}\right)
\]
\[
= \left(\frac{\partial x_v}{\partial s_u}, \frac{\partial x_v}{\partial s_w}\right)_{R^3} + \left(\frac{\partial x_v}{\partial s_u}, \frac{\partial x_v}{\partial s_w}\right)_{R^3} = (e_v, \frac{\partial n}{\partial s_u}) + (e_w, \frac{\partial n}{\partial s_w}).
\]

Let us prove that $k(\bar{x})$ is independent of the choice of coordinate system on the surface $\Gamma_1$. By differentiation of $(n, n) = 1$, we have that $(\frac{\partial n}{\partial s_u}, n) = 0$, $(\frac{\partial n}{\partial s_w}, n) = 0$. This shows that $\frac{\partial n}{\partial s_u}, \frac{\partial n}{\partial s_w} \in T_2(\Gamma_1)$, where $T_2(\Gamma_1)$ is a tangent space to the manifold $\Gamma_1$ at $\bar{x}$. Therefore, since $\{e_v, e_w\}$ is an orthogonal base in $T_2(\Gamma_1)$, there exists a matrix $A$ such that
\[
(5.4) \quad \left(\begin{array}{c}
\frac{\partial n}{\partial s_u} \\
\frac{\partial n}{\partial s_w}
\end{array}\right) = A \left(\begin{array}{c}
e_v \\
e_w
\end{array}\right), \quad A = \left(\begin{array}{cc}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right).
\]

From this we have $k(\bar{x}) = \text{Tr}(A) \equiv a_{11} + a_{22}$. Let $\{\tilde{v}, \tilde{w}\}$ be a new coordinate on $\Gamma_1$ and $A$ be the matrix as in (5.4) associated with the new coordinate. Define a matrix $C$ by
\[
\left(\begin{array}{cc}
\frac{\partial x_v}{\partial s_v} & \frac{\partial x_v}{\partial s_w} \\
\frac{\partial x_w}{\partial s_v} & \frac{\partial x_w}{\partial s_w}
\end{array}\right).
\]

Then,
\[
(5.5) \quad \left(\begin{array}{c}
e_v \\
e_w
\end{array}\right) = C \left(\begin{array}{c}
e_v \\
e_w
\end{array}\right) \quad \text{and} \quad \left(\begin{array}{c}
\frac{\partial n}{\partial s_u} \\
\frac{\partial n}{\partial s_w}
\end{array}\right) = C \left(\begin{array}{c}
\frac{\partial n}{\partial s_u} \\
\frac{\partial n}{\partial s_w}
\end{array}\right).
\]

By (5.4) and (5.5), we know that $A = CAC^{-1}$, which shows that the polynomial $P(\lambda) = |A - \lambda I|$ is invariant under transformation of the coordinate system. This implies that $k(\bar{x}) = \text{Tr}(A) \equiv a_{11} + a_{22}$ also is invariant. From the fact that the boundary is compact and $\partial \Omega \in C^2$, we have $k(\bar{x}) \in C^1(\bar{\Gamma}_1)$.

Unlike the Navier–Stokes equations with the standard boundary conditions, the
equation with the nonstandard boundary condition like ours mainly was studied in the case where \( l = 3 \) (cf. [1], [4], [5], [10], and [35]). Some authors asserted that results of 2-D equations are special ones of 3-D equations. But, equivalence of \( \| y \|_{H^1} \) and \( \| y \|_{L^2} + \| \text{rot } y \|_{L^2} \) for \( y \in V \) which guarantees the formula (2.6) was proved for the connected bounded domains of \( R^3 \) belonging to \( C^{1,1} \) and the convex polyhedrons (cf. Proposition 1.1 in [1], Lemma 1.5 in [5], and Lemma 1.4 in [10]). ([2] dealt with convex polygons. For weaker solutions to the problem on nonconvex polyhedrons and polygons we refer readers to [9]). Owing to the conditions for the boundary (cf. equations (A.1), (A.2), and (A.3) in [10]), the equivalence for 3-D domains does not imply one for 2-D smooth domains. Thus, we consider the equivalence of the norms for smooth 2-D bounded domains.

**Definition 5.2.** In a bounded domain of \( R^2 \) putting outward normal vectors on the right-hand side, let us travel along its boundary. If the outward normal unit vector at a point of the boundary rotates clockwise, then the boundary at the point is said to be concave.

**Lemma 5.3.** Suppose that \( \Omega \) is a 2-D bounded domain with a boundary in \( C^{1,1} \) and \( \Gamma_1 \) is not concave. Then \( \| y \|_{H^1} \) and \( \| \text{rot } y \|_{L^2} \) are equivalent for \( y \in V \).

**Proof.** It suffices to prove for \( V \cap H^2(\Omega) \) that

\[
\| \nabla y \|_{L^2(\Omega)} = \int_{\partial \Omega} (y, \frac{\partial y}{\partial n})_R^2 \, dx + (\text{rot } y, y)_{L^2(\Omega)}
\]

\[= (y, \frac{\partial y}{\partial n})_{\Gamma_1} + \int_{\Omega} |\text{rot } y|^2 \, dx. \tag{5.6} \]

Putting \( w = 0 \), from the proof of Lemma 5.1 we see that

\[
\left( \frac{\partial y}{\partial n} \right)_{\Gamma_1} + (k(x)y, y)_{\Gamma_1} = 0 \quad \text{on } \Gamma_1, \quad k(x) = \left( e_v, \frac{\partial n}{\partial s_v} \right). \tag{5.7} \]

As mentioned in the proof of Lemma 5.1, \( \frac{\partial n}{\partial s_v} \) is parallel to \( e_v \). Since \( \Gamma_1 \) is not concave, \( \frac{\partial n}{\partial s_v} \) has the same direction as \( e_v \), giving \( k(x) \geq 0 \). Taking account of this fact, from (5.6) we have

\[
\| \nabla y \|_{L^2(\Omega)} \leq \| \text{rot } y \|_{L^2(\Omega)}. \tag{5.8} \]

When \( \Gamma_0 \neq \emptyset \), the norms \( \| \nabla y \|_{L^2(\Omega)} \) and \( \| y \|_{V} \) are equivalent, and so we come to the conclusion using (5.8). Let us consider the case where \( \Gamma_0 = \emptyset \). By (5.8), it suffices to prove that \( \| y \|_{L^2(\Omega)} \leq C \| \text{rot } y \|_{L^2(\Omega)} \). Suppose that it is not true; then we have a sequence \( \{y_m\} \) such that \( \| y_m \|_{L^2(\Omega)} = 1 \) and for any integer \( m \)

\[
\| y_m \|_{L^2(\Omega)} \geq m \| \text{rot } y_m \|_{L^2(\Omega)}. \]

Thus, there exists a function \( y^* \) such that \( \| y^* \|_{L^2(\Omega)} = 1 \), \( \text{rot } y^* = 0 \). Therefore, \( y^* = \nabla p^* \) and \( \nabla p^* \times n_{|\partial \Omega} = 0 \). This means that \( \partial \Omega \) is a contour of \( p^* \). On the other hand, \( \text{div } y = 0 \) implies \( \Delta p^* = 0 \). Thus, we have \( y^* = \nabla p^* = 0 \), which contradicts \( \| y^* \|_{L^2(\Omega)} = 1 \). Thus, we come to the conclusion.

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REFERENCES


