Restricted partitions and Motzkin left factors

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Abstract

Let \( L_n, n \geq 1 \), count the number of paths from the origin to the line \( x = n - 1 \) using \((1, 1), (1, -1), \) and \((1, 0)\) steps that never dip below the \(x\)-axis (called Motzkin left factors). The \( L_n \) count, among other things, certain restricted subsets of permutations and Catalan paths. In this note, we provide new combinatorial interpretations for these numbers in terms of finite set partitions. In particular, we identify four classes of the partitions of size \( n \), all of which have cardinality \( L_n \) and each avoiding a set of two classical patterns of length four. We obtain a further generalization in one of the cases by considering a pair of statistics on the partition class. In a couple of cases, to show the result, we make use of the kernel method to solve a functional equation arising after a certain parameter has been introduced.

Keywords: pattern avoidance, Motzkin number, Motzkin left factor, \( q \)-generalization.

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1 Introduction

Let \( \mathcal{M}_n \) be the set of all paths from \((0, 0)\) to \((n, 0)\) using \((1, 1), (1, -1), \) and \((1, 0)\) steps, which we will denote by \( u, d, \) and \( \ell \), respectively, with no steps lying below the \(x\)-axis (called Motzkin paths). The cardinality of \( \mathcal{M}_n \) defines the Motzkin number \( M_n \), which has been widely studied (see, e.g., A001006 of [15] and the references contained therein). Let \( \mathcal{L}_n \) denote the set of all paths of length \( n \) using \( u, d \) and \( \ell \) steps starting from the origin and not dipping below the \(x\)-axis. Such paths are called Motzkin left factors; see, e.g., p. 111 of [1] or p. 9 of [9]. Let \( L_n = |\mathcal{L}_{n-1}| \) if \( n \geq 1 \), with \( L_0 = 1 \). The \( L_n \) are also given by the generating function

\[
\sum_{n \geq 0} L_n x^n = \frac{1 - 3x + \sqrt{1 - 2x + 3x^2}}{2(1 - 3x)} \tag{1}
\]

and satisfy the relation

\[
L_{n+1} = M_n + \sum_{k=0}^{n-1} M_k L_{n-k}, \quad n \geq 1, \tag{2}
\]

with \( L_0 = L_1 = 1 \).

Among the other lattice path interpretations for the numbers \( L_n \) is that they count the symmetric Catalan paths of semilength \( 2n - 1 \) with no peaks at even level as well as the Catalan paths of semilength \( n \) with no occurrence of \( duuu \). The \( L_n \) also enumerate a variety of other structures, ranging from the set of directed animals [6] of size \( n \) to the permutations of \( \{1, 2, \ldots, n\} \) avoiding...
the patterns 321 and 41523 (see [3]) to the set of base 3 n-digit numbers whose digit sum is also n. See A005773 of cite [15] for further information on these numbers. Here, we provide new combinatorial interpretations for the $L_n$ in terms of finite set partitions, showing, in particular, that they enumerate certain two-pattern avoidance classes.

If $n \geq 1$, then a partition of $[n] = \{1, 2, \ldots, n\}$ is any collection of non-empty, pairwise disjoint subsets, called blocks, whose union is $[n]$. (If $n = 0$, then there is a single empty partition which has no blocks.) A partition $\Pi$ having exactly $k$ blocks is also called a $k$-partition and will be denoted by $\Pi = B_1/B_2/\ldots/B_k$, where the blocks are arranged in ascending order according to the size of the smallest elements. We will denote the set of $k$-partitions of $[n]$ by $P_{n,k}$ and the set of all partitions of $[n]$ by $P_n$. One may represent the partition $\Pi = B_1/B_2/\ldots/B_k \in P_{n,k}$, equivalently, by the canonical sequential form $\pi = \pi_1\pi_2\cdots\pi_n$, wherein $j \in B_{\pi_j}$, $1 \leq j \leq n$, and in such case we will write $\Pi = \pi$. For example, the partition $\Pi = 1, 5, 7/2, 3, 4, 8/6 \in P_{4,4}$ has the canonical sequential form $\pi = 12231413$. Note that $\pi = \pi_1\pi_2\cdots\pi_n \in P_{n,k}$ is a restricted growth function from $[n]$ to $[k]$ (see, e.g., [16] for details), meaning that it satisfies the following three properties: (i) $\pi_1 = 1$, (ii) $\pi$ is onto $[k]$, and (iii) $\pi_i+1 \leq \max\{\pi_1, \pi_2, \ldots, \pi_i\} + 1$ for all $i$, $1 \leq i \leq n-1$. In what follows, we will represent set partitions as words using their canonical sequential forms and consider some particular cases of the general problem of counting the members of a partition class having various restrictions imposed on the order of the letters.

A classical pattern $\tau$ is a member of $[\ell]^m$ which contains all of the letters in $[\ell]$. We say that a word $\sigma \in [k]^n$ contains the classical pattern $\tau$ if $\sigma$ contains a subsequence isomorphic to $\tau$. Otherwise, we say that $\sigma$ avoids $\tau$. For example, a word $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ avoids the pattern 132 if it has no subsequence $\sigma_i\sigma_j\sigma_k$ with $i < j < k$ and $\sigma_i < \sigma_k < \sigma_j$ and avoids the pattern 1212 if it has no subsequence $\sigma_i\sigma_j\sigma_k\sigma_l$ with $\sigma_i = \sigma_k < \sigma_j = \sigma_l$. The pattern avoidance question has been the topic of many papers in enumerative combinatorics, starting with Knuth [11] and Simion and Schmidt [14] on permutations and considered, more recently, on words, compositions, and finite set partitions. For the avoidance problem on partitions, we refer the reader to the papers by Klazar [10], Sagan [12], and Jelínek and Mansour [8] and to the references therein.

In this paper, we identify four classes of partitions each avoiding a set of two classical patterns of length four and each enumerated by the number $L_n$. In addition to providing new interpretations for the numbers $L_n$, this addresses specific cases of a general question raised by Goyt in the final section of [7] concerning the avoidance by set partitions of two or more patterns of length four. Our main result is the following theorem which we prove in the next section as a series of propositions.

**Theorem 1.1.** If $n \geq 0$, then $p_n(u,v) = L_n$ for the following sets $(u,v)$:

1. (1222, 1212)  
2. (1112, 1212)  
3. (1211, 1221)  
4. (1222, 1221).

We remark that in the first two cases our proofs are more or less combinatorial, while, in the last two, they are algebraic and involve applications of the kernel method [2] to solve the functional equations which arise once a certain parameter has been introduced. In addition, we prove a refinement of Theorem 1.1 as well as obtain $p, q$-generalizations of the numbers $M_n$ and $L_n$ by considering pairs of statistics on the sets $P_n(111, 1212)$ and $P_n(1222, 1212)$, respectively.

We will use the following notation. If $\{w_1, w_2, \ldots\}$ is a set of classical patterns, then let $P_n(w_1, w_2, \ldots)$ and $P_{n,k}(w_1, w_2, \ldots)$ denote, respectively, the subsets of $P_n$ and $P_{n,k}$ which avoid all of the patterns. We will denote the cardinalities of $P_n(w_1, w_2, \ldots)$ and $P_{n,k}(w_1, w_2, \ldots)$ by $p_n(w_1, w_2, \ldots)$ and $p_{n,k}(w_1, w_2, \ldots)$, respectively.

## 2 Proof of main result

Theorem 1.1 above will follow from combining the propositions in the sections below. We first consider the patterns $\{1222, 1212\}$ and $\{1112, 1212\}$.
2.1 The cases $\{1222, 1212\}$ and $\{1112, 1212\}$

Throughout, we will denote the sets $P_n(1222, 1212)$ and $P_n(1112, 1212)$, by $\mathcal{A}_n$ and $B_n$, respectively, and similarly use $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$. Let $\mathcal{R}_n$ denote the set $P_n(111, 1212)$.

**Proposition 2.1.** If $n \geq 0$, then $p_n(1222, 1212) = L_n$.

**Proof.** We will define an explicit bijection between $\mathcal{A}_n$ and $L_{n-1}$ for all $n \geq 1$. First observe that any member $\pi \in \mathcal{A}_n$ may be expressed as $\pi = 1\pi_11\pi_2 \cdots 1\pi_r$ for some $r \geq 1$, where $\pi_i$ does not contain 1 and is such that $\text{stan}(\pi_i)$ belongs to $P_n(111, 1212)$ for some $n_i \geq 0$ (by $\text{stan}(\pi_i)$, we mean the equivalent partition on the letters $\{1, 2, \ldots\}$, called the standardization, gotten by replacing the $j$-th smallest letter of $\pi_i$ with $j$). Furthermore, note that it must be the case that every letter of $\pi_j$ is larger than every letter of $\pi_i$ if $j > i$ in order to avoid 1212.

We now define, in a recursive fashion, a bijection between $\mathcal{R}_m$ and $\mathcal{M}_m$ for all $m \geq 0$, where $f_0(\emptyset) = \emptyset$ and $f_1(1) = \ell$. If $m \geq 2$ and $\lambda \in P_m(111, 1212)$, then either

$$\begin{align*}
(i) \quad &\lambda = 1\lambda' \quad \text{or} \\
(ii) \quad &\lambda = 1\lambda'\lambda''\,
\end{align*}$$

where 1 does not belong to $\lambda'$ or $\lambda''$ and all of the letters of $\lambda''$ are larger than all of those in $\lambda'$, with $\lambda'$ and $\lambda''$ both avoiding the patterns 111 and 1212. If $m \geq 2$, we define $f_m$, recursively, by setting $f_m(\lambda) = ff_{m-1}(\lambda)$ in case (i) and setting $f_m(\lambda) = uf_{m_1}(\lambda')df_{m_2}(\lambda'')$, where $m_1$ and $m_2$ denote the respective lengths of $\lambda'$ and $\lambda''$. The bijection is reversed upon considering whether or not a path in $\mathcal{M}_m$ starts with $\ell$ or $u$, and in the latter case, considering the position of the first $d$ returning to the $x$-axis. In what follows, we will write $f$ to denote $f_m$, suppressing the subscript whenever the cardinality of the underlying structure is understood.

We now define a bijection $g$ between $\mathcal{A}_n$ and $L_{n-1}$. If $\pi = 1\pi_11\pi_2 \cdots 1\pi_r$ is as above, then let

$$g(\pi) = f(\pi_1)uf(\pi_2)uf(\pi_r).$$

To reverse $g$, suppose $\alpha \in L_{n-1}$ ends at the point $(n-1, r-1)$ for some $r \geq 1$. Given $0 \leq i \leq r-1$, let $s_i$ denote the right most step of $\alpha$ which lies along the line $x = i$ as an $\ell$ or either touches it as a $d$ from above or as a $u$ from below (in the case when $i = 0$, only the first two conditions apply). Decompose $\alpha$ as $\alpha = \alpha_0\alpha_1 \cdots \alpha_{r-1}$, where $\alpha_0$ counts all steps of $\alpha$ up to and including $s_0$ and $\alpha_i$, $1 \leq i \leq r-1$, is the sequence of steps starting with the $u$ directly following step $s_{i-1}$ and ending at step $s_i$. Note that $\alpha_i = u\alpha_i'$ if $i \geq 1$, with $\alpha_i'$ and $\alpha_0$ possibly empty Motzkin paths. Then define $g^{-1}(\alpha)$ by

$$g^{-1}(\alpha) = 1f^{-1}(\alpha_0)1f^{-1}(\alpha_1) \cdots 1f^{-1}(\alpha_{r-1}).$$

Figure 1 below illustrates the path $g(\pi)$ corresponding to $\pi = 12334215511617898 \in R_{17}$.

![Figure 1: The Motzkin left factor $g(\pi) \in L_{16}$.](image)

**Remark:** One can give a full bijection between $\mathcal{R}_n$ and $\mathcal{M}_n$ as follows. First recall the equivalence between Catalan paths of semilength $m$ and perfect matchings of $[2m]$ that avoid the pattern
1212 (called non-crossing matchings, see, e.g., [8]) gotten by drawing horizontal lines to the right of each up step in a Catalan path, noting the position of the first down step encountered, and partitioning the steps into the m position pairs. The resulting perfect matching on [2m] avoids 1212 and, conversely, starting with such a matching, one may construct a Catalan path of semilength m whose paired up and down steps correspond to the blocks of the matching.

Now suppose \( \pi \in \mathcal{R}_n \) and let \( S \subset [n] \) comprise the singletons of \( \pi \) with \( k = |S| \). Let \( \pi' \) denote the standardization of the partition \( \pi \cap ([n] - S) \). Since \( \pi' \) is a perfect matching which avoids 1212, one may construct a Catalan path \( p(\pi') \) of semilength \( n - k \) as described above. Then insert level steps \( \ell \) into \( p(\pi') \) such that their positions correspond to the elements of \( S \) to yield a Motzkin path of length \( n \). This process is seen to be reversible.

We now consider avoidance of the patterns 1112 and 1212.

**Proposition 2.2.** The generating function for the number of partitions of \([n]\), \( n \geq 0 \), that avoid the patterns 1112 and 1212 is given by

\[
\frac{1 - 3x + \sqrt{1 - 2x - 3x^2}}{2(1 - 3x)}.
\]

**Proof.** If \( n \geq 3 \) and \( \pi \in \mathcal{B}_n \), then we may decompose \( \pi \) as either

(i) \( \pi = 1\alpha 11\cdots 1 \) \( s \) times \hspace{1cm} or \hspace{1cm} (ii) \( \pi = 1\beta 1\gamma 11\cdots 1 \) \( s \) times

where \( s \geq 0 \), \( \alpha \) is a possibly empty partition on the letters \( \{2, 3, \ldots \} \) avoiding the patterns 1112 and 1212, \( \beta \) is a possibly empty partition on \( \{2, 3, \ldots, i\} \) \( \) for some \( i \) avoiding 111 and 1212, and \( \gamma \) is a non-empty partition on the letters \( \{i + 1, i + 2, \ldots \} \) avoiding 1112 and 1212. Note that \( \beta \) must avoid 111 since \( \gamma \) is assumed non-empty. Let \( M(x) \) denote the generating function for the Motzkin numbers \( M_n \), i.e., \( M(x) = \sum_{n \geq 0} M_n x^n \). If \( h(x) = \sum_{n \geq 0} |\mathcal{B}_n| x^n \), then we see from the prior proof and the above decompositions that it must satisfy

\[
h(x) = 1 + \frac{x}{1 - x} h(x) + \frac{x^2}{1 - x} M(x)(h(x) - 1),
\]

i.e.,

\[
h(x) = \frac{1 - x - x^2 M(x)}{1 - 2x - x^2 M(x)}.
\]

Upon simplifying, the requested result now follows from the last equation and the fact \( M(x) = \frac{1 - x}{1 - \sqrt{1 - 2x - 3x^2}} \). \( \square \)

**Remark:** It is possible to construct a bijection between \( \mathcal{B}_n \) and \( \mathcal{L}_{n-1} \). Note that the members of \( \mathcal{B}_n \) of the form in the first decomposition above correspond to paths in \( \mathcal{L}_{n-1} \) starting with \( \ell^nu \) and not returning to the \( x \)-axis when \( \alpha \) is non-empty and to the path \( \ell^{n-1} \) when \( \alpha \) is empty. Members of \( \mathcal{B}_n \) of the form in (ii) correspond to paths in \( \mathcal{L}_{n-1} \) of the form \( \ell^uX'\ell^{n-1} \lambda' \), where \( \lambda' \) and \( \lambda'' \) denote Motzkin and left factor Motzkin paths, respectively.

### 2.2 The case \{1211, 1221\}

Let \( \pi = \pi_1 \pi_2 \cdots \pi_n \) denote a partition of \([n]\), represented canonically. Recall that empty sums take the value zero, by convention. To establish this case, we divide up the set of partitions in question according to a certain statistic, namely, the one which records the length of the maximal increasing initial run. To do so, given \( k \geq 1 \), let \( f_k(x) \) denote the generating function for the number of partitions \( \pi \) of \([n]\) having at least \( k \) letters and avoiding the patterns 1211 and 1221 such that \( \pi_1 \pi_2 \cdots \pi_k = 12\cdots k \) with \( \pi_{k+1} \leq k \) (if there is a \((k+1)\)-st letter). We have the following relation involving the generating functions \( f_k(x) \).
Lemma 2.3. If \( k \geq 1 \), then

\[
f_k(x) = x^k + x^k \mathcal{T}_1(x) + \sum_{j=1}^{k-1} x^{j+1} \mathcal{T}_{k-j}(x),
\]

with initial condition \( f_0(x) = 1 \), where \( \mathcal{T}_k(x) = \sum_{i \geq k} f_i(x) \).

Proof. Note that \( f_1(x) = x + x \mathcal{T}_1(x) \), since a partition in this case may just have one letter or start 11. If \( k \geq 1 \), in general, then a partition \( \pi \) enumerated by \( f_k(x) \) must be of one of the following three forms:

(i) \( 12 \cdots k \),
(ii) \( 12 \cdots kj \pi' \), where \( 1 \leq j \leq k-1 \),
(iii) \( 12 \cdots kk \pi'' \).

The first case contributes \( x^k \). Note that in the second case, the word \( \pi' \) contains no letters in \([j]\), for otherwise there would be an occurrence of 1221 if it contained a letter in \([j-1]\) or an occurrence of 1211 if it contained the letter \( j \). Thus, the letters \((j+1)(j+2) \cdots k \pi'\), taken together, comprise a partition of the form enumerated by \( \mathcal{T}_{k-j}(x) \), which implies the contribution in this case is \( x^{j+1} \mathcal{T}_{k-j}(x) \). Similar reasoning in the third case yields a contribution of \( x^k \mathcal{T}_1(x) \) since \( \pi'' \) can contain no letters in \([k-1]\). Combining the three cases yields (3). \( \square \)

We now prove the third case in Theorem 1.1 above.

Proposition 2.4. The generating function for the number of partitions of \([n]\), \( n \geq 0 \), that avoid the patterns 1211 and 1221 is given by

\[
\frac{1 - 3x + \sqrt{1 - 2x - 3x^2}}{2(1 - 3x)}.
\]

Proof. Define the generating function \( f(x, y) = \sum_{k \geq 0} f_k(x) y^k \). Multiplying (3) by \( y^k \) and summing over \( k \geq 1 \) yields

\[
f(x, y) = 1 + \frac{xy}{1 - xy} + \frac{xy}{1 - xy} \mathcal{T}_1(x) + \sum_{k \geq 2} \left( \sum_{j=1}^{k-1} x^{j+1} \mathcal{T}_{k-j}(x) \right) y^k
\]

\[
= 1 + \frac{xy}{1 - xy} + \frac{xy}{1 - xy} (f(x, 1) - 1) + \sum_{j \geq 1} x^{j+1} \sum_{k \geq j+1} \mathcal{T}_{k-j}(x) y^k
\]

\[
= 1 + \frac{xy}{1 - xy} f(x, 1) + \sum_{j \geq 1} x^{j+1} y^j \sum_{k \geq 1} \mathcal{T}_{k-j}(x) y^k
\]

\[
= 1 + \frac{xy}{1 - xy} f(x, 1) + \frac{x^2 y}{1 - xy} \sum_{k \geq 1} y^k \sum_{i \geq k} f_i(x)
\]

\[
= 1 + \frac{xy}{1 - xy} f(x, 1) + \frac{x^2 y}{1 - xy} \sum_{i \geq 1} f_i(x) \sum_{k \geq i} y^k
\]

\[
= 1 + \frac{xy}{1 - xy} f(x, 1) + \frac{x^2 y^2}{(1 - xy)(1 - y)} \sum_{i \geq 1} f_i(x) (1 - y^i)
\]

\[
= 1 + \frac{xy}{1 - xy} f(x, 1) + \frac{x^2 y^2}{(1 - xy)(1 - y)} (f(x, 1) - f(x, y)),
\]
which implies

\[
\left(1 + \frac{x^2y^2}{(1-xy)(1-y)}\right) f(x, y) = 1 + \left( \frac{xy}{1-xy} + \frac{x^2y^2}{(1-xy)(1-y)} \right) f(x, 1). \tag{4}
\]

This type of functional equation can be solved systematically using the kernel method (see [2]). In this case, if we assume that \( y = y_0 \) in (4), where \( y_0 \) satisfies \( 1 + \frac{x^2y_0^2}{(1-xy_0)(1-y_0)} = 0 \), i.e.,

\[
y_0 = \frac{1 + x - \sqrt{1 - 2x - 3x^2}}{2x(1 + x)},
\]

then

\[
\sum_{n \geq 0} P_n(1211, 1221)x^n = f(x, 1) = \frac{1 - \frac{1}{1-y_0}}{1 - \frac{x}{1-x-y_0}} = \frac{1 - 3x + \sqrt{1 - 2x - 3x^2}}{2(1 - 3x)},
\]

as required. (Note that \( f(0, 1) = 1 \) dictates our choice of root for \( y_0 \).)

**Remark:** Substituting the expression above for \( f(x, 1) \) into (4) recovers the expression for \( f(x, y) \), from which one can compute an explicit formula for the coefficient of \( x^n y^k \).

### 2.3 The case \{1222, 1221\}

We once again divide up the set of partitions in question according to the statistic which records the length of the maximal increasing initial run. If \( k \geq 1 \), then let \( f_k(x) \) (respectively, \( g_k(x) \)) denote the generating function for the number of partitions \( \pi \) of \([n]\) having at least \( k \) letters and avoiding the patterns 1222 (respectively, 111) and 1221 such that \( \pi_1 \pi_2 \cdots \pi_k = 12 \cdots k \) with \( \pi_{k+1} \leq k \) (if there is a \((k+1)\)-st letter). We have the following relations involving the generating functions \( f_k(x) \) and \( g_k(x) \).

**Lemma 2.5.** If \( k \geq 1 \), then

\[
g_k(x) = x^k + \sum_{j=1}^{k} x^{j+1} \overline{g}_{k-j}(x), \tag{5}
\]

with initial condition \( g_0(x) = 1 \), where \( \overline{g}_k(x) = \sum_{i \geq k} g_i(x) \).

**Proof.** If \( k \geq 1 \), in general, then a partition \( \pi \) enumerated by \( g_k(x) \) must be of one of the following two forms:

\[(i)12 \cdots k, \]

\[(ii)12 \cdots kj\pi', \text{ where } 1 \leq j \leq k.\]

The first case contributes \( x^k \). Note that in the second case, the word \( \pi' \) contains no letters in \([j]\), for otherwise there would be an occurrence of 1221 if it contained a letter in \([j-1]\) or an occurrence of 111 if it contained the letter \( j \). Thus, the letters \((j+1)(j+2) \cdots k\pi'\), taken together, comprise a partition of the form enumerated by \( \overline{g}_{k-j}(x) \), which implies the contribution in this case is \( x^{j+1} \overline{g}_{k-j}(x) \). Combining the two cases yields (5). \( \square \)

**Lemma 2.6.** If \( k \geq 1 \), then

\[
f_k(x) = x^k + x \overline{f}_k(x) + \sum_{j=2}^{k} x^{j+1} \overline{g}_{k-j}(x), \tag{6}
\]

with initial condition \( f_0(x) = 1 \), where \( \overline{f}_k(x) = \sum_{i \geq k} f_i(x) \).
Proof. Note that \( f_1(x) = x + x \mathcal{T}_1(x) \), since a partition in this case may just have one letter or start 11. If \( k \geq 1 \), in general, then a partition \( \pi \) enumerated by \( f_k(x) \) must be of one of the following three forms:

\[
\begin{align*}
(i) & \quad 12 \cdots k, \\
(ii) & \quad 12 \cdots k1\pi', \\
(iii) & \quad 12 \cdots kj\pi'', \quad \text{where} \ 2 \leq j \leq k.
\end{align*}
\]

The second case is seen to contribute \( x \mathcal{T}_k(x) \). Note that in the third case, the word \( \pi'' \) contains no letter in \([j-1]\), for otherwise there would be an occurrence of 1221 if it contained a letter in \([j-1]\) or an occurrence of 1222 if it contained the letter \( j \). Thus, the letters \((j+1)(j+2)\cdots k\pi'\)
taken together, comprise a partition of the form enumerated by \( \mathcal{Y}_{k-j}(x) \), which implies the contribution in this case is \( x^{j+1}\mathcal{Y}_{k-j}(x) \). Combining the three cases yields (6). \( \square \)

The final case of Theorem 1.1 now follows from the two previous lemmas.

**Proposition 2.7.** The generating function for the number of partitions of \([n]\), \( n \geq 0 \), that avoid the patterns 1222 and 1221 is given by

\[
\frac{1 - 3x + \sqrt{1 - 2x - 3x^2}}{2(1 - 3x)}.
\]

**Proof.** Define the generating functions \( g(x, y) = \sum_{k \geq 0} g_k(x)y^k \) and \( f(x, y) = \sum_{k \geq 0} f_k(x)y^k \).

Multiplying (5) and (6) by \( y^k \) and summing over \( k \geq 1 \) yields

\[
g(x, y) = \frac{1}{1 - xy} + \sum_{k \geq 1} y^k \sum_{j=1}^{k} x^{j+1} \mathcal{Y}_{k-j}(x)
\]

\[
= \frac{1}{1 - xy} + \frac{x^2y}{1 - xy} \sum_{k \geq 0} y^k \mathcal{Y}_k(x)
\]

\[
= \frac{1}{1 - xy} + \frac{x^2y}{(1 - xy)(1 - y)} (g(x, 1) - yg(x, y)), \quad \text{and}
\]

\[
f(x, y) = \frac{1}{1 - xy} + x \sum_{k \geq 1} y^k \mathcal{T}_k(x) + \sum_{k \geq 2} y^k \sum_{j=2}^{k} x^{j+1} \mathcal{Y}_{k-j}(x)
\]

\[
= \frac{1}{1 - xy} + \frac{xy}{1 - y} \sum_{i \geq 1} (1 - y^i) f_i(x) + \frac{x^3y^2}{(1 - xy)(1 - y)} \sum_{i \geq 0} (1 - y^{i+1}) g_i(x)
\]

\[
= \frac{1}{1 - xy} + \frac{xy}{1 - y} (f(x, 1) - f(x, y)) + \frac{x^3y^2}{(1 - xy)(1 - y)} (g(x, 1) - yg(x, y)).
\]

This implies

\[
\left(1 + \frac{x^2y^2}{(1 - xy)(1 - y)}\right) g(x, y) = \frac{1}{1 - xy} + \frac{x^2y}{(1 - xy)(1 - y)} g(x, 1)
\]

\[
\left(1 + \frac{xy}{1 - y}\right) f(x, y) = \frac{1}{1 - xy} + \frac{xy}{1 - y} f(x, 1) + \frac{x^3y^2}{(1 - xy)(1 - y)} (g(x, 1) - yg(x, y)).
\]

To solve these functional equations, we use the kernel method. In this case, if we assume that \( y = y_0 \) in (7), where \( y_0 \) satisfies \( 1 + \frac{x^2y_0^2}{(1 - x_0 y_0) (1 - y_0)} = 0 \), i.e.,

\[
y_0 = \frac{1 + x - \sqrt{1 - 2x - 3x^2}}{2x(1 + x)},
\]

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then
\[ \sum_{n \geq 0} P_n(111, 1221)x^n = g(x, 1) = \frac{y_0}{1 - xy_0} = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}. \]

Moreover, (7) gives
\[ g(x, 1/(1 - x)) = \frac{1 - x}{1 - 3x}(1 - xg(x, 1)). \] (9)

Now, if we assume that \( y = y_1 = \frac{1}{1-x} \) in (8), then\[ f(x, 1) = \frac{1 - x}{1 - 2x} - \frac{1}{1 - 2x} \left( g(x, 1) - \frac{1}{1-x}g(x, 1/(1 - x)) \right), \]
which, by (9), implies\[ f(x, 1) = \frac{1 - 3x + \sqrt{1 - 2x - 3x^2}}{2(1 - 3x)}, \]
as required. \( \square \)

### 3 Other results

In this section, we prove some results related to Theorem 1.1 above.

#### 3.1 A refinement and an identity

One can refine Theorem 1.1 by adding a parameter which records the number of blocks of a partition.

**Theorem 3.1.** If \( n, k \geq 0 \), then \( p_{n,k}(u, v) \) is the same for the following pairs \((u, v)\):

1. \((1222, 1212)\)
2. \((1112, 1212)\)
3. \((1211, 1221)\)
4. \((1222, 1221)\).

Furthermore, the common generating function \( h(x, y) = \sum_{n,k \geq 0} p_{n,k}(u, v)x^ny^k \) is given by\[ h(x, y) = \frac{(2 - y) + x(y^2 - 2y - 2) + y\sqrt{(1 - xy)^2 - 4x^2y}}{2(1 - (2y + 1)x)} \]. (10)

**Proof.** We prove the second statement. For this, we first compute the generating function \( M(x, y) \) defined by\[ M(x, y) = \sum_{n,k \geq 0} p_{n,k}(111, 1212)x^ny^k. \]

From the reasoning in the second paragraph of the proof of Proposition 2.1 above, we see that it must satisfy\[ M(x, y) = 1 + xyM(x, y) + x^2yM(x, y)^2 \]
and is thus given by\[ M(x, y) = \frac{1 - xy - \sqrt{(1 - xy)^2 - 4x^2y}}{2x^2y}. \]

From the proof of Proposition 2.1, we also see that \( h(x, y) \) in the first case is given by\[ 1 + y \sum_{i \geq 1} (xM(x, y))^i = 1 + \frac{xyM(x, y)}{1 - xM(x, y)}. \]
which yields (10), upon substituting the expression for $M(x, y)$ and simplifying. A similar proof applies in the second case, upon adjusting the argument given for Proposition 2.2 above. The last two cases follow from the proofs of Proposition 2.4 and 2.7, upon adding an extra parameter recording the number of blocks in a partition. 

**Remark:** Substituting $y = 1$ in equation (10) yields (1).

Using the interpretation for the numbers $L_n$ given in Theorem 1.1 and of the Motzkin numbers given in the proof of Proposition 2.1 above, one can perhaps supply combinatorial proofs (in the sense of [4]) of certain identities involving these numbers more easily. We give one such example below. We have not been able to find the following identity in the literature. It follows easily from the generating functions, once stated.

**Proposition 3.2.** The numbers $L_n$ and $M_n$ satisfy the relation

$$L_n = 3L_{n-1} - M_{n-2}, \quad n \geq 2, \quad (11)$$

with $L_0 = L_1 = 1$.

**Proof.** By Theorem 1.1, the left side of (11) counts the members of $A_n$. To show that the right side also achieves this, first note that there are $L_{n-1}$ members of $A_n$ that end in a 1 as well as $L_{n-1}$ members whose final letter occurs nowhere else. So we must show that the members of $A_n$ whose final letter is greater than 1 and occurs one other time number $L_{n-1} - M_{n-2}$. We’ll denote this subset by $A_n'$. Let $A_n'' \subseteq A_n$ consist of those partitions having at least two occurrences of 1. Note that $|A_n''| = L_n - M_{n-1}$, upon subtracting the members of $A_n$ having a single 1, of which there are $M_{n-1}$ (note that they are of the form 1α, where $α \in R_{n-1}$).

To complete the proof, it suffices to define a bijection between $A_n'$ and $A_n''$. Note that $π ∈ A_n'$ implies that it can be expressed as $π = π'απ''a$, where $a > 1$, each element of $[a-1]$ occurs in $π'$, and all of the letters of $π''$ are greater than $a$. Define $f : A_n' → A_n''$ by $f(π) = π'1π''$. One can verify that $f$ is a bijection; note that $f$ is well-defined since $a > 1$ implies $π'$ is non-empty. 

### 3.2 Statistics on partitions and paths

Recall that $P_n(111, 1212)$ is denoted by $R_n$. In Section 2.1 above, we saw the equivalence of the set of partitions $R_n$ and the set of Motzkin paths $M_n$. Here, we take this equivalence a step further and consider a pair equally distributed statistics on the two sets. We also consider extensions of these statistics to the sets $A_n = P_n(1222, 1212)$ and $L_{n-1}$, thereby obtaining a $p, q$-analogue of the sequence $L_n$.

Given $π = π_1π_2⋯π_n ∈ R_n$, expressed canonically, let $des(π)$ denote the number of descents of $π$, i.e., the number of indices $i$, $1 ≤ i ≤ n − 1$, such that $π_i > π_{i+1}$, and let $inv(π)$ denote the number of inversions of $π$, i.e., the number of ordered pairs $(i, j)$ with $1 ≤ i < j ≤ n$ and $π_i > π_j$. Define the distribution polynomial $M_n(p, q)$ by

$$M_n(p, q) = \sum_{π ∈ R_n} p^{des(π)}q^{inv(π)}, \quad n ≥ 1,$$

with $M_0(p, q) = 1$. For example, if $n = 4$, then

$$R_4 = \{1122, 1123, 1223, 1233, 1234, 1213, 1232, 1221, 1231\},$$

which implies $M_4(p, q) = 5 + 2pq + 2pq^2$.

Note that $π ∈ R_n$ may be expressed as either (i) 1α or (ii) 1β1γ, where $α$, $β$, and $γ$ do not contain 1 and all of the letters of $γ$ are greater than all of the letters of $β$. Conditioning on the length $i$ of $β$, we where we differentiate between the $i = 0$ and $i > 0$ cases, we obtain the recurrence

$$M_n(p, q) = M_{n-1}(p, q) + (1-p)M_{n-2}(p, q) + p\sum_{i=0}^{n-2} q^i M_i(p, q)M_{n-2-i}(p, q), \quad n ≥ 2, \quad (12)$$

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with the initial conditions \( M_0(p, q) = M_1(p, q) = 1 \).

Given \( \lambda \in \mathcal{M}_n \), consider the matching \( u, d \) pairs obtained by drawing a horizontal line to the right of each \( u \) in \( \lambda \) and noting the \( d \) where it first intersects \( \lambda \) again. For each such pair, consider the number of steps of \( \lambda \) (including \( \ell \)'s) strictly between the \( u \) and the \( d \) and then add up the resulting numbers for all of the pairs to obtain a value which we will denote by \( \text{sum}(\lambda) \). Let \( \text{num}(\lambda) \) denote the number of matching pairs in which the \( u \) and \( d \) are separated by at least one step. We leave it as an exercise for the reader to verify that the joint distribution of \((\text{num}, \text{sum})\) on \( \mathcal{M}_n \) is equal to the distribution of \((\text{des}, \text{inv})\) on \( \mathcal{R}_n \). The next proposition summarizes the above observations.

**Proposition 3.3.** Let \( M_n(p, q) \) be given by recurrence (12) above. Then for all \( n \geq 0 \), we have

\[
M_n(p, q) = \sum_{\pi \in \mathcal{R}_n} p^{\text{des}(\pi)} q^{\text{inv}(\pi)} = \sum_{\lambda \in \mathcal{M}_n} p^{\text{num}(\lambda)} q^{\text{sum}(\lambda)}.
\]

Remark: The statistic \( \text{num} \) is seen to give the number of occurrences of \( \ell d \) or \( d \ell \) within a member of \( \mathcal{M}_n \). Thus, \( \text{num} \) (and hence \( \text{des} \) on \( \mathcal{R}_n \)) is seen to be equivalent to the “right double fall” statistic considered in [13], upon comparing the definitions. On the other hand, we were unable to find in the literature the \( \text{sum} \) statistic on \( \mathcal{M}_n \). Also, our formula below for the total \( \text{num} \) on \( \mathcal{M}_n \) as well as its extension to \( \mathcal{L}_n \) seem to be new.

When \( p = 0 \) or \( q = 0 \), note that (12) reduces to the Fibonacci recurrence and we have \( M_n(0, q) = M_n(p, 0) = F_n \) for all \( n \geq 0 \). This may be realized combinatorially by observing that those members of \( \mathcal{R}_n \) having either no descents or no inversions are precisely those in which each block is of the form \( \{i\} \) or \( \{i, i+1\} \) for some \( i \). Such partitions are clearly synonymous with square-and-domino tilings of length \( n \) and they are counted by \( F_n \). A similar interpretation can be given for this using Motzkin paths.

Let \( M(x; p, q) = \sum_{n \geq 0} M_n(p, q)x^n \). Multiplying (12) by \( x^n \) and summing over \( n \geq 2 \) yields the relation

\[
M(x; p, q) = 1 + x(1 + (1 - p)x)M(x; p, q) + px^2 M(qx; p, q)M(x; p, q).
\]

While it does not seem possible to find an explicit expression for \( M(x; p, q) \) for general \( p \) and \( q \), one can give the following continued fraction expansion.

**Proposition 3.4.** We have

\[
M(x; p, q) = \frac{1}{1 - x(1 + (1 - p)x) - px^2 \sum_{n=0}^{\infty} \frac{px^2}{1 - qx(1 + (1 - p)qx) - px^2} \cdots}.
\]

Proof. By (13), we have

\[
M(x; p, q) = \frac{1}{1 - x(1 + (1 - p)x) - px^2 M(qx; p, q)}.
\]

Applying this recurrence an infinite number of times yields the requested result. \( \square \)

Remark: An explicit formula for \( M_n(p, q) \) in the sense of [5, Prop. 3A] may be given using the above continued fraction expansion, though it involves multiple sums.

When \( q = 1 \), one can solve (13) explicitly to get

\[
M(x; p, 1) = \frac{1 - x + (p - 1)x^2 - \sqrt{(1 - x + (p - 1)x^2)^2 - 4px^2}}{2px^2}.
\]

(14)
Using (12) when \( q = 1 \) and the Lagrange inversion formula, one can show that the number of partitions of \( P_n(111, 1212) \), \( n \geq 1 \), having exactly \( m \) descents is given by

\[
\sum_{j=m}^{n} \frac{1}{{j \choose m+1}{j \choose m}{j-m \choose n-j}}.
\]

See [13] for a similar, though different, formula for the number of paths in \( M_n \) having a prescribed number of levels and right double falls.

Let \( b_n \) denote the total number descents in all of the members of \( P_n(111, 1212) \). Differentiating both sides of (14) with respect to \( p \), and letting \( p = 1 \), implies

\[
\frac{d}{dp} M(x; p, 1) \bigg|_{p=1} = \frac{x^2 + x - 1 + (x^2 - 3x + 1)\sqrt{\frac{1+x}{1-x}}}{2x^2},
\]

which is known to be the generating function for the number of compact-rooted directed animals of size \( n \) having three source points (see, e.g., [6] and A005775 of [15]). Thus, we have that this number equals \( b_n \) for all \( n \geq 3 \) and it would be interesting to find a bijective proof.

On the other hand, when \( p = 1 \), it does not seem that (13) can be solved explicitly for general \( q \). Nor are we able to find a closed form for \( M_n(1, q) \). However, we do have the following result when \( q = -1 \).

**Proposition 3.5.** If \( n \geq 0 \), then

\[
M_{2n}(1, -1) = M_{2n+1}(1, -1) = \sum_{k=0}^{n} \binom{n}{k} C_k,
\]

where \( C_k = \frac{1}{k+1} \binom{2k}{k} \) denotes the \( k \)-th Catalan number.

**Proof.** We supply both algebraic and combinatorial proofs of this result. Taking \( p = 1 \) and \( q = -1 \) in (13) gives

\[
M(x; 1, -1) = 1 + xM(x; 1, -1) + x^2 M(-x; 1, -1) M(x; 1, -1).
\]

Replacing \( x \) with \(-x\) in this equation, solving the resulting system in the variables \( M(x; 1, -1) \) and \( M(-x; 1, -1) \), and noting \( M(0; 1, -1) = 1 \) yields

\[
M(x; 1, -1) = \frac{1 - x^2 - \sqrt{(1-x^2)^2 - 4x^2(1-x^2)}}{2x^2(1-x)}
\]

which may be rewritten as

\[
M(x; 1, -1) = \frac{1}{1-x} \left( \frac{1-\sqrt{1-4u}}{2u} \right),
\]

where \( u = \frac{x^2}{1-x} \). The result now follows from a short calculation using the facts \( \sum_{n \geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x} \) and \( \sum_{n \geq k} \binom{n}{k} x^k = \frac{x^k}{(1-x)^{k+1}} \).

To give a bijective proof of (15), let \( R^+ \) and \( R^- \) denote the subsets of \( R_n \) whose members have even and odd \( inv \)-parity, respectively. It suffices to identify a subset of \( R^+_n \) of \( n \) having cardinality \( \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} C_k \), along with an \( inv \)-parity changing involution of \( R_n - R^+_n \).

Let us first consider the even case. Let \( R^+_{2n} \) comprise those partitions \( \pi = \pi_1 \pi_2 \cdots \pi_{2n} \) such that for all \( 1 \leq i \leq n \) either \( \pi_{2i-1} \) and \( \pi_{2i} \) are both the only letters of their kind in \( \pi \) or neither of them are. It is seen that all members of \( R^+_{2n} \) have an even number of inversions and their
cardinality is $\sum_{k=0}^{n} \binom{n}{k} C_k$, upon choosing the $n - k$ indices $i$ such that both $\pi_{2i-1}$ and $\pi_{2i}$ correspond to singletons.

We now define an inv-parity involution of $R_{2n} - R_{2n}^2$. Suppose that $\pi = \pi_1 \pi_2 \cdots \pi_{2n} \in R_{2n} - R_{2n}^2$ and that $i_0$ is the smallest index $i$ such that either $\pi_{2i-1}$ or $\pi_{2i}$ is the only letter of its kind in $\pi$, but not both. Let $\pi_{2i_0-1}\pi_{2i_0} = ab$ and let us first assume that there is an additional $a$ or $b$ occurring to the right of $b$ in $\pi$. We change that letter to the other option, noting that this changes the inv-parity since $b = a + 1$ in this case. Otherwise, we must have either (i) $a < b$, where there is a second $a$ to the left of this one, or (ii) $a > b$, where there is a second $b$ to the left of the $a$. If (i) or (ii) occurs, then switch the order of the letters $\pi_{2i_0-1}$ and $\pi_{2i_0}$, leaving the rest of $\pi$ undisturbed. Combining the two mappings yields the desired involution of $R_{2n} - R_{2n}^2$.

For the odd case, apply the involution described above in the even case to the first $2n$ letters of $\pi = \pi_1 \pi_2 \cdots \pi_{2n+1} \in R_{2n+1}$ if $\pi_{2n+1}$ is the only letter of its kind or if it does not equal $a = \pi_{2i_0-1}$ or $b = \pi_{2i_0}$. Extend this involution by changing $\pi_{2n+1}$ to the other option if it happens that it equals either $a$ or $b$. The set of survivors are precisely those partitions of the form $\pi'(k + 1)$ for some $k$, where $\pi' \in R_{2n}$ has exactly $k$ distinct letters. □

One can also consider statistics on the set $A_n = P_n(1222, 1212)$, or, equivalently, statistics on $L_{n-1}$, where $n \geq 1$, and thereby obtain polynomial generalizations of $L_n$. Let $\text{des}(\pi)$ count the number of descents of $\pi = \pi_1 \pi_2 \cdots \pi_n \in A_n$ and let $\text{inv}'(\pi)$ count the number of ordered pairs $(i, j), 1 < i < j < n$, with $\pi_i > \pi_j$ (note that $i > 1$ is required here). One can also give equivalent statistics on $L_{n-1}$, though the descriptions are lengthier than those described above on $A_n$. Define the polynomials $L_n(p, q)$ by

$$L_n(p, q) = \sum_{\pi \in A_n} p^{\text{des}(\pi)} q^{\text{inv}'(\pi)}, \quad n \geq 1,$$

with $L_0(p, q) = 1$, and the generating function $L(x; p, q)$ by

$$L(x; p, q) = \sum_{n \geq 0} L_n(p, q)x^n.$$

The following result generalizes Proposition 2.1 and is equivalent to it in the case $p = q = 1$.

**Theorem 3.6.** Let $M(x; p, q)$ be given by (13) above. We have

$$L(x; p, q) = 1 + \frac{xM(x; p, q)}{1 - x(1 - p + pM(x; p, q))}. \quad (16)$$

**Proof.** We decompose non-empty $\pi \in A_n$ as $\pi = 1(\pi_11)(\pi_21)\cdots(\pi_{r-1}1)\pi_r$, where $r \geq 1$ and the $\pi_i$ contain no 1’s and avoid the patterns 111 and 1212. Considering whether or not $\pi$ is empty, we see from Proposition 3.3 that each section $\pi_i1$ has the same generating function $x(1 - p + pM(x; p, q))$ for $1 \leq i \leq r - 1$. The numerator $xM(x; p, q)$ accounts for the remaining letters, namely, $1\pi_r$. □

**Remark:** One can also show (16) by first arguing directly that $L_n(p, q)$ satisfies the relation

$$L_{n+1}(p, q) = M_n(p, q) + L_n(p, q) + p \sum_{i=1}^{n-1} M_i(p, q)L_{n-i}(p, q), \quad n \geq 1, \quad (17)$$

which generalizes (2). Multiplying (17) by $x^n$, summing over $n \geq 1$, and solving for $L(x; p, q)$ then yields (16).

**Remark:** Using (14) and (16), one give an explicit formula for $L(x; p, 1)$.

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References


