MODIFIED APPROACH TO GENERALIZED STIRLING NUMBERS VIA DIFFERENTIAL OPERATORS

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ABSTRACT. In this paper we give a modified approach to the generalized Stirling numbers of the second kind $S_{r,s}(n, k)$ and $S_{r,s}(k)$. These numbers were firstly defined by Carlitz and recently studied extensively by Blasiak. This approach depends on the previous results obtained by Carlitz, Toscano and Cakić. We show that Blasiak’s results can be investigated from Carlitz’s and Cakić’s results. Some interesting combinatorial identities are obtained.

Key words: Stirling numbers, generalized Stirling numbers, Combinatorial identities, Normal ordering, Boson operators

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1. Introduction

Stirling numbers have a long and interesting history especially in combinatorics (see [21, 22, 5, 11] and [19, Sequences A008275 and A008277]). Many generalizations and extensions are considered (see [8, 7, 16, 14]), and more combinatorial, probabilistic, statistical and physical (coherent states) applications of these numbers are derived (see [7, 2, 1, 20, 13]). Carlitz and Klamkin [5], (see also [22, 11]), defined the Stirling numbers of the second kind by

\[(xD)^n = \sum_{k=1}^{n} S(n, k)x^k D^k.\]

Equation (1) can be written as $(a^\dagger a)^n = \sum_{k=1}^{n} S(n, k)(a^\dagger)^k a^k$, where $a^\dagger$ and $a$ are boson creation and annihilation operators, satisfying the commutation relation $[a, a^\dagger] = 1$. One of the first results of this area can be found in Carlitz paper in 1932. Namely, Carlitz introduced arrays of numbers $a_{n,t}$, in the following way [4] (see also [3, 15]):

\[(x^\lambda + \mu D^\mu)^n = \sum_{t=1}^{\mu(n-1)+1} a_{n,t}(x^{t+\mu-1+n\lambda} D^{t+\mu-1}),\]

Notice that, if putting $t + \mu - 1 = k$ in (2), then we obtain that

\[(x^\lambda + \mu D^\mu)^n = \sum_{k=\mu}^{\mu n} a_{n,k-\mu+1}x^{k+n\lambda} D^{k},\]
also let $\mu = s$ and $\mu + \lambda = r$ whence $\lambda = r - s$, then we have
\[
(x^r D^s)^n = x^{n(r-s)} \sum_{k=s}^{n} a_{n,k-s+1} x^k D^k.
\]
Comparing (3) with (7) (see [1]), we get $S_{r,s}(n, k) = a_{n,k-s+1}$. In fact this show that the numbers $S_{r,s}(n, k)$, derived by Blasiak [1] were firstly handled by Carlitz [4]. In 1956, Chak [5] Equation 1.1] showed that $(x^k D)^n = x^{nk-n} \sum_{i=0}^{n} A^{(\alpha)}_{n,k,i} x^i D^i x^{-\alpha}$. Indeed, the numbers $A^{(\alpha)}_{n,k,i}$ give us Lang’s result [12], as a special case for $\alpha = 0$, and the usual Stirling numbers of the second kind for $\alpha = 0$ and $k = 1$. An interesting result in this line was derived by Charalambides [7]:
\[
\left( x^{b+1} D \right)^n = \sum_{k=0}^{n} b^k C(n, k, -1/b) x^{nb+k} D^k,
\]
where $C(n, k, -1/b)$ are so-called C-numbers.

2. Main results

Blasiak, Penson and Solomon [2] defined $S_{r,s}(n, k)$, the generalized Stirling numbers of the second kind, as follows. Let $r$ and $s$, $r \geq s$ be nonnegative integers, then
\[
(x^r D^s)^n = x^{n(r-s)} \sum_{k=s}^{n} S_{r,s}(n, k) x^k D^k.
\]
Moreover, let $r = (r_1, r_2, \ldots, r_n)$ and $s = (s_1, s_2, \ldots, s_n)$ are two sequences of nonnegative integers, Blasiak [1] defined $S_{r,s}(k)$ as the nonnegative integers appearing in the normally ordered expansion
\[
(a^+)^{r_n}(a^+)\cdots(a^+)^{r_2}(a^+)^{r_1}(a^+) = (a^+)^{d_n} \left( \sum_{k=s_1}^{s_1+s_2+\cdots+s_n} S_{r,s}(k)(a^+)^k(a^k) \right),
\]
where $d_n = \sum_{m=1}^{n} (r_m - s_m)$ and $S_{r,s}(k) = 0$ for $k < s_1$ or $k > s_1 + s_2 + \cdots + s_n$. Also this equation can be formulated in terms of multiplication $x$ and the derivative operator $D$ as
\[
x^{r_n} D^{s_n} \cdots x^{r_2} D^{s_2} x^{r_1} D^{s_1} = x^{d_n} \left( \sum_{k=s_1}^{s_1+s_2+\cdots+s_n} S_{r,s}(k)x^k D^k \right).
\]
The falling factorial is defined by $x^{(n)} = \prod_{j=0}^{n-1} (x - j)$ with $x^{(0)} = 1$.

Lemma 1. For all $n, k$ nonnegative integers,
\[
S_{r,s}(n, k) = \sum_{i_1 + i_2 + \cdots + i_{n-1} = n-s-k}^{n-1} \prod_{j=1}^{n-1} \binom{s}{i_j} \left( jr - \sum_{l=1}^{j-1} i_l \right)^{(i_j)}.
\]
Proof. Cakić [3] (see also [15]) proved that, if \( n, r \) and \( s \) are natural numbers, then

\[
(x^n D^s)^n = \sum_{0 \leq i_1, \ldots, i_n \leq s} \left( \prod_{j=1}^{n-1} \binom{s}{i_j} (jr - e_j)^{(i_j)} \right) x^{n r - e_n} D^{n s - e_n},
\]

where \( e_j = \sum_{\ell=1}^{j-1} i_\ell \). If putting \( n s - e_n = k \), then we get that

\[
\sum_{i_1 + \cdots + i_n = n s - k} \left( \prod_{j=1}^{n-1} \binom{s}{i_j} (jr - e_j)^{(i_j)} \right) x^k D^k.
\]

Hence, comparing this equation with (4), we obtain the desired result. \( \square \)

From [11] Equation (4.60) and (6) we have the following combinatorial identity

\[
\sum_{i_1 + \cdots + i_n = n s - k} \left( \prod_{j=1}^{n-1} \binom{s}{i_j} (jr - e_j)^{(i_j)} \right) = \frac{(-1)^k}{k!} \sum_{j=s}^k (-1)^j \binom{k}{j} \prod_{m=1}^n (j + (m - 1)(r - s))^{(s)},
\]

where \( e_j = \sum_{\ell=1}^{j-1} i_\ell \). Generally, we have the following result.

**Theorem 1.** We have

\[
(x^n D^s) \cdots (x^n D^s)(x^n D^s)
\]

\[
= \sum_{s_1} \sum_{s_2} \cdots \sum_{s_n} \prod_{i=1}^{n-1} \binom{s_{i+1}}{i} \left( \sum_{j=1}^{\ell} r_j - i_j - 1 \right)^{(i_j)} x^{\sum_{j=1}^{\ell} r_j - \sum_{j=1}^{\ell} i_j} D^{\sum_{j=1}^{\ell} i_j - 1},
\]

where \( i_0 = 0 \). Moreover, the numbers \( S_{r,s,k}(n) \) are given by

\[
S_{r,s,k}(n) = \sum_{i_1 + \cdots + i_n = n s - k} \prod_{i=1}^{n-1} \binom{s_{i+1}}{i} \left( \sum_{j=1}^{\ell} r_j - \sum_{j=1}^{\ell} i_j \right)^{(i_j)}.
\]

Proof. The proof of (9) follows by successive application of Leibnitz formula. In order to prove (10), let \( d_n = \sum_{m=1}^n (r_m - s_m) \), \( \alpha_n = \sum_{j=0}^n i_j \), \( \beta_n = \sum_{j=1}^n s_j \) and \( \gamma_n = \sum_{j=1}^n r_j \). Using the substitution \( \sum_{j=1}^{\ell} s_j - \sum_{j=1}^{\ell} i_j = k \) in (9) we obtain that

\[
(x^n D^s) \cdots (x^n D^s)(x^n D^s) = x^{dn} \sum_{k=s_1}^{\beta_n} \prod_{\alpha_{n-1} = 0}^{n-1} \binom{s_{\ell+1}}{i_\ell} \left( \gamma_\ell - \alpha_\ell - 1 \right)^{(i_\ell)} x^k D^k.
\]

Hence, comparing the above equation with (5), see also [11] Equation (4.7), we get (10), as requested.

Using [11] Equation (4.10) together with (10) we get the interesting combinatorial identity

\[
\sum_{i_1 + \cdots + i_n = n s - k} \prod_{i=1}^{n-1} \binom{s_{i+1}}{i} \left( \sum_{j=1}^{\ell} r_j - \sum_{j=1}^{\ell} i_j \right)^{(i_j)} = \frac{1}{k!} \sum_{j=1}^k \binom{k}{j} (-1)^{k-j} \prod_{m=1}^n (d_{m-1} + j)^{(s_m)}.
\]
For example, if we take $s_i = r_i = 1$, $i = 1, \ldots, n$, in (11), hence $i_j \in \{0, 1\}$, $j = 1, \ldots, n-1$, both sides of previous identity are equal to the Stirling numbers of the second kind, i.e. (see [2] and comment by Carlitz therein)

$$S(n, k) = \sum_{i_1 + \cdots + i_n = n - k}^{n-1} \prod_{\ell = 1}^{n-1} \left( \ell - \sum_{j=0}^{\ell-1} i_j \right) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n.$$  

Furthermore, let $r_i = r$ and $s_i = 1, i = 1, \ldots, n$ in (10), then we obtain new explicit expression for the numbers $S_{r,1}(n, k)$ and $S(n, k; r)$ (see [12] [1]) as

$$S_{r,1}(n, k) = S(n, k; r) = \sum_{i_1 + \cdots + i_n = n - k}^{n-1} \prod_{\ell = 1}^{n-1} \left( \ell - \sum_{j=0}^{\ell-1} i_j \right).$$

Now, we handle an interesting special case of (5). Setting $r_i = s_i$, $i = 1, \ldots, n$, we get

$$x^{r_n}D^{r_n} \cdots x^{r_1}D^{r_1} = \delta^{(r_n)} \cdots \delta^{(r_1)} = \sum_{\ell = r_1}^{r_1 + \cdots + r_n} S_{r,r}(\ell) x^\ell D^\ell,$$

where $S_{r,r}(\ell) = 0$, for $\ell < r_1$ or $\ell > r_1 + r_2 + \cdots + r_n$, hence

$$\sum_{i_0}^{r_n} s(r_n, i_n) \delta^{r_n} \cdots \sum_{i_1}^{r_1} s(r_1, i_1) \delta^{r_1} = \sum_{\ell = r_1}^{r_1 + \cdots + r_n} S_{r,r}(\ell) \sum_{k=0}^{\ell} s(\ell, k) \delta^k.$$

Using Cauchy rule of multiplication for series we obtain

$$\sum_{k=0}^{r_1 + \cdots + r_n} \prod_{i_1 + \cdots + i_n = k, i_j \geq 0}^{n} s(r_j, i_j) \delta^k = \sum_{k=0}^{r_1 + \cdots + r_n} \sum_{\ell = k}^{r_1 + \cdots + r_n} S_{r,r}(\ell) s(\ell, k) \delta^k.$$

Equating the coefficients of $\delta^k$ on both sides yields

$$\sum_{\ell = k}^{r_1 + \cdots + r_n} S_{r,r}(\ell) s(\ell, k) = \sum_{i_1 + \cdots + i_n = k, i_j \geq 0}^{n} s(r_j, i_j), \quad 0 \leq k \leq r_1 + \cdots + r_n.$$

Furthermore setting $r_i = r$, $i = 1, \ldots, n$, in (12) see [1] Equation (4.63)], gives

$$\delta^{(r)} = \sum_{\ell = r}^{nr} S_{r,r}(n, \ell) x^\ell D^\ell,$$

where $S_{r,r}(0, 0) = 1, S_{r,r}(n, \ell) = 0$ for $\ell > nr$ and $\ell < r$ and $n > 0$. Then by virtue of (13), where $r_i = r$, we get the interesting identity

$$\sum_{l=k}^{nr} S_{r,r}(n, l) s(l, k) = \sum_{i_1 + \cdots + i_n = k, i_j \geq 0}^{n} s(r_j, i_j), \quad 0 \leq k \leq nr,$$

where $s(n, k)$ are the usual Stirling numbers of the first kind, defined by $x^{(n)} = \sum_{k=1}^{n} s(n, k)x^k$, $s(n, 0) = \delta_{n,0}$ and $s(n, k) = 0$ for $k > n$.

For example, if $n = 2, r = 2, 0 \leq k \leq 4$ then, for $k = 3$, we have (see [1], Table 1)

$$\sum_{\ell = 3}^{4} S_{2,3}(2, \ell) s(\ell, 3) = \sum_{i_1 + i_2 = 3, i_j \geq 0}^{2} \prod_{j=1}^{i_2} s(2, i_j)$$
then $\text{L.H.S} = S_{2,2}(2,3)s(3,3) + S_{2,2}(2,4)s(4,3) = 4 \cdot 1 + 1 \cdot (-6) = -2$ and $\text{R.H.S} = s(2,1)s(2,2) + s(2,2)s(2,1) + s(2,0)s(2,3) + s(2,3)s(2,0) = (-1) \cdot 1 + 1 \cdot (-1) = -2$.

3. The numbers $R(n,k;r)$

In this section, we handle Lang’s results [12], (see also [6; 7]), via a new approach. Firstly, expanding $(x^rD)^n$ in powers of $xD$, then define $R(n,k;r)$ as the coefficient of $(xD)^k$ in the expansion of $(x^rD)^n$, that is, define $R(n,k;r)$ as

\begin{equation}
(x^rD)^n = x^{(r-1)n} \sum_{k=1}^{n} R(n,k;r)(xD)^k, \tag{15}
\end{equation}

with $R(n,k;1) = \delta_{n,k}$, and $R(n,k;r) = 0$ for all $k > n$.

**Lemma 2.** Let $r$ be any real number. Then $R(n,k;r)$ satisfies the recurrence relation

\begin{equation}
R(n,k;r) = R(n-1,k-1;r) + (r-1)(n-1)R(n-1,k;r). \tag{16}
\end{equation}

**Proof.** Since $(x^rD)^n = (x^rD)(x^rD)^{n-1}$, then from (15) we get that

\[
x^{(r-1)n} \sum_{k=1}^{n} R(n,k;r)(xD)^k = (x^rD)x^{(r-1)(n-1)} \sum_{k=1}^{n} R(n-1,k;r)(xD)^k = x^{r-1} \left\{ (r-1)(n-1)x^{(r-1)(n-1)} \sum_{k=1}^{n} R(n-1,k;r)(xD)^k 
+ x^{(r-1)(n-1)} \sum_{k=1}^{n} R(n-1,k;r)(xD)^{k+1} \right\}.
\]

Comparing the coefficients of $(xD)^k$ on the both sides of the above equation, we get the desired result. \hfill \Box

Indeed, the numbers $R(n,k;r)$ are treated before in different techniques, see [9 10 18]. The following theorem gives us a new explicit formula for the numbers $R(n,k;r)$.

**Theorem 2.** For any $n,k$ nonnegative integers and $r$ real number,

\[
R(n,k;r) = (r-1)^{n-k} \sum_{i_1+\cdots+i_{n-1}=n-k, i_j \in \{0,1\}} \prod_{j=1}^{n-1} \binom{j-1 + i_j}{i_j}.
\]

**Proof.** The theorem for $k = 1$ gives that $R(n,k;r) = (r-1)^{n-1}(n-1)!$, which agrees the definition of $R(n,k;r)$. Also, we have that

\[
R(n,k;r) = (r-1)^{n-k} \left[ \sum_{i_1+\cdots+i_{n-2}=(n-1)-(k-1)} \prod_{j=1}^{n-2} \binom{j-1 + i_j}{i_j} 
+ \sum_{i_1+\cdots+i_{n-2}=(n-1)-k} (n-2+1) \prod_{j=1}^{n-2} \binom{j-1 + i_j}{i_j} \right]
\]
which implies that

\[ R(n, k; r) = (r - 1)^{(n-1)-(k-1)} \sum_{i_1 + \cdots + i_{n-2} = (n-1)-(k-1)} \prod_{j=1}^{n-2} \left( j - 1 + i_j \right) \]

\[ + (r - 1)^{n-k}(n - 1) \sum_{i_1 + \cdots + i_{n-2} = (n-1)-k} \prod_{j=1}^{n-2} \left( j - 1 + i_j \right) \]

\[ = R(n - 1, k - 1; r) + (n - 1)(r - 1)R(n - 1, k; r). \]

Therefore, by Lemma 2 and induction we get the desired result.

Moreover from (15), using (1), expanding \((xD)^k\) in terms of \(x^iD^i\), we have

\[(xD)^n = \sum_{k=1}^{n} R(n, k; r) \sum_{i=1}^{k} S(k, i)x^iD^i = \sum_{i=1}^{n} \left\{ \sum_{k=i}^{n} R(n, k; r)S(k, i) \right\} x^iD^i.\]

Using [12 Equation 12] we get the interesting identity

\[(17) \quad S(n, i; r) = \sum_{k=i}^{n} R(n, k; r)S(k, i),\]

where \(S(n, k; r)\), the generalized Stirling numbers of the second kind [12], are defined by

\[(x^rD)^n = x^{(r-1)n} \sum_{k=1}^{n} S(n, k; r)x^kD^k.\]

Now equation (17) can be written in a matrix form as \(S_r = R_rS\), where \(S_r, R_r\) and \(S\) are \(n \times n\) (or infinite) lower triangular matrix defined by \(S_r := (S(i, j; r))_{i,j}, R_r := (R(i, j; r))_{i,j}\) and \(S := (S(i, j))_{i,j}, 1 \leq i, j \leq n\), respectively. Therefore, equation (17) and equivalent matrix form give a new representation of the numbers \(S(n, k; r)\), treated by Lang [12], in terms of the numbers \(R(n, k; r)\) and the usual Stirling numbers of the second kind. For example if \(n = 4\) and \(r = 2\) we obtain as a special case the well-known identity \(L = s_1 \cdot S\), hence

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
6 & 6 & 1 & 0 \\
24 & 36 & 12 & 1
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
2 & 3 & 1 & 0 \\
6 & 11 & 6 & 1
\end{pmatrix} \cdot
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 3 & 1 & 0 \\
1 & 7 & 6 & 1
\end{pmatrix},
\]

where \(L = (L(n, k))\) and \(s_1 = (s_1(n, k))\) are \(n \times n\) lower triangular matrices whose entries are the signless (unsigned) Lah numbers (see [19 Sequence A008297] and [17]) and signless (unsigned) Stirling numbers of the first kind (see [19 Sequence A008275] and [17]), respectively. The Lah number \(L(n, k)\) is given by \(\binom{n-1}{k-1} \frac{n!}{k!}\) and the signless (unsigned) Stirling number of the first kind \(s_1(n, k)\) is defined by \(x(x + 1) \cdots (x + n - 1) = \sum_{k=0}^{n} s_1(n, k)x^k.\)
Note that, for \( r = 2 \), the numbers \( S(n, k; r) \) are reduced to the numbers \( L(n, k) \). Moreover, if \( n = 4 \) and \( r = 3 \) we have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 \\
15 & 9 & 1 & 0 \\
105 & 87 & 18 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
8 & 6 & 1 & 0 \\
48 & 44 & 12 & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 3 & 1 & 0 \\
1 & 7 & 6 & 0
\end{pmatrix}.
\]

Finally, since \( S_r = R_r S \), then we get \( S = R_r^{-1} S_r, R_r = S_r S \) and \( s_r R_r S = I \), where \( s_r \) and \( s \) are \( n \times n \) matrices whose \((i, j)\)th entries are the generalized Stirling numbers of the first kind [12] and the usual Stirling numbers of the first kind, respectively and \( R_r^{-1} \) is the inverse of \( R_r \).

**Colorally 1.** We have \( R(n, k; r) = (r - 1)^{n-k} s_1(n, k) \).

**Proof.** The proof follows directly by using the recurrence relation (16).

**References**


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