On the zeros of some combinatorial polynomials

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Abstract

Let $a_n(q)$ denote the distribution on the set of involutions of size n for the statistic which records the number of fixed points. We show for a range of q values that the polynomial $\sum_{i=0}^{n} a_i(q)x^i$ always has the smallest possible number of real zeros, that is, none when the degree is even and one when the degree is odd. On the way, we show that the sequence $a_n(q)$ is log-convex for all $q \geq 1$. Our proof in the case $q = 1$ is elementary, while the proof for the general case makes use of programming to estimate the zeros of some related analytic functions in q. Furthermore, the sequence of real zeros obtained from the odd case is shown to be monotonically convergent. We also consider the polynomial $\sum_{i=0}^{n} d_i x^i$, where d_i denotes the number of derangements of an i-element set, and show that the same holds for it. Our results extend recent ones concerning Fibonacci and Tribonacci coefficient polynomials.

Keywords: zeros of polynomials, linear recurrences, involution, derangement.

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1 Introduction

Garth, Mills, and Mitchell [1] considered the Fibonacci coefficient polynomial defined by $p_n(x) = F_1x^n + F_2x^{n-1} + \cdots + F_nx + F_{n+1}$ and showed that it has no real zeros if n is even and exactly one real zero if n is odd. This result was later extended by Mátyás [4, 5] to polynomials whose coefficients are given by more general second order recurrences (having constant coefficients). Mátyás and Szalay [6] showed that the same also holds true for the Tribonacci coefficient polynomials $q_n(x) = T_2x^n + T_3x^{n-1} + \cdots + T_{n+1}x + T_{n+2}$, a result which has been extended to k-Fibonacci [2] and generalized Tribonacci [3] coefficient polynomials.

One might wonder for what other combinatorial sequences v_i does this result hold for the polynomial $f(x) = \sum_{i=0}^{n} v_ix^i$. Here, we consider this question in two cases when the sequence v_i counts (i) derangements of size i (i.e., permutations all of whose cycles have length two or more), or (ii) involutions of size i (i.e., permutations all of whose cycles have length one or two).

Let d_n denote the sequence defined by

$$d_n = (n-1)(d_{n-1} + d_{n-2}), \qquad n \geq 2,$$

with $d_0 = 1$ and $d_1 = 0$. Note that d_n counts the number of derangements of an n-element set. See, for example, A000166 in [7]. In the next section, we show that the polynomial

$$d(x) = \sum_{i=0}^{n} d_i x^i$$

always has the smallest possible number of real zeros and that the sequence of zeros in the odd case is monotonically convergent.

Let $a_n(q)$ denote the sequence of polynomials defined by

$$a_n(q) = q a_{n-1}(q) + (n-1) a_{n-2}(q), \qquad n \geq 2,$$

with $a_0(q) = 1$ and $a_1(q) = q$. Note that $a_n(q)$ gives the distribution for the statistic on the set of involutions of size n which records the number of fixed points; see, e.g., A099174 in OEIS [7]. The special cases $q = 1$ and $q = 2$ of $a_n(q)$ correspond, respectively, to sequences A000085 and A005425 in [7] and occur in several settings in enumerative combinatorics.

In this paper, we will show that the polynomial $a(x; q)$ defined by

$$a(x; q) = \sum_{i=0}^{n} a_i(q) x^i$$

has one real zero when the degree is odd and no real zeros when the degree is even for all q in the interval $[1, \lambda]$, where $\lambda \approx 6.276$. Furthermore, we show that the sequence of real zeros in the odd case is increasing and convergent to zero. We first prove the case $q = 1$ and then extend the arguments to obtain the result for all q in $[1, \lambda]$, wherein we use programming (such as Maple) to estimate real zeros of some analytic functions in q which are needed in our proof. We remark that $[1, \lambda]$ seems to be the largest interval in q for which the technique presented here applies.

We note that the sequences under consideration in the current paper are given by second order linear recurrences, but with variable instead of constant coefficients. This will require a different technique than that used in earlier papers on Fibonacci coefficient polynomials and their relatives. Instead of multiplying $f(x) = \sum_{i=0}^{n} v_i x^i$ by a characteristic polynomial to transform it into another polynomial whose coefficients are mostly zero (as was done in [1] and in subsequent papers), we first apply a linear operator of a differential nature to f. This yields a first order differential equation for f which can then be used to express it in a more convenient integral form. In the case of $v_i = a_i(q)$, which is considered in the third section, we will also need some algebraic properties of these polynomials in order to complete our argument using an integral form of $a(x; q)$ (see Lemma 3.2 below).

2 Derangement polynomials

We consider in this section the real zeros of the polynomial $d(x) = \sum_{i=0}^{n} d_i x^i$. We first will need the following integral representation of $d(x)$.

Lemma 2.1. *Let $-1 < x_0 < 0$ be fixed. Then we have*

$$d(x) = \frac{\int_{x_0}^{x} \frac{v(t)}{t(1+t)} e^{1/t} dt + c}{|x| e^{1/x}}, \qquad -1 < x < 0, \tag{1}$$

where $v(t) = -(n+1) d_n t^{n+2} - d_{n+1} t^{n+1} + 1$ and $c = |x_0| d(x_0) e^{1/x_0}$.

Proof. First observe that

$$(1 - x^2) d(x) - x^2 d'(x) - x^3 d'(x) = -(n+1) d_n x^{n+2} - d_{n+1} x^{n+1} + 1. \tag{2}$$

To see this, note that the coefficient of x^i on the left-hand side of (2) is zero if $2 \le i \le n$, by the recurrence for d_n, with the coefficient of x^{n+1} given by $-nd_n - nd_{n-1} = -d_{n+1}$. Rewriting (2) as

$$d'(x) + \frac{x-1}{x^2} d(x) = -\frac{v(x)}{x^2(1+x)}, \qquad -1 < x < 0,$$

and solving this first-order linear differential equation by the usual method, gives (1). □

We now prove the main result of this section.

Theorem 2.2. *If $n \ge 2$, then the polynomial $\sum_{i=0}^{n} d_i x^i$ has exactly one real zero if n is odd and no real zeros if n is even.*

Proof. Clearly, the polynomial $d(x) = \sum_{i=0}^{n} d_i x^i$ has no positive zeros. If $x \le -1$ and $n \ge 5$ is odd, then

$$\begin{aligned} d(x) = \sum_{i=0}^{n} d_i x^i &= 1 + \sum_{i=1}^{\frac{n-1}{2}} (d_{2i+1} x^{2i+1} + d_{2i} x^{2i}) \\ &= 2x^3 + x^2 + 1 + \sum_{i=2}^{\frac{n-1}{2}} (d_{2i+1} x^{2i+1} + d_{2i} x^{2i}) < 0, \end{aligned}$$

being the sum of non-positive terms at least one of which is negative, since $d_{2i+1}x^{2i+1} < -d_{2i}x^{2i}$ if $i \ge 2$ and $2x^3 + x^2 + 1 \le 0$. If $n = 3$, then $d(x) = 2x^3 + x^2 + 1$, which has a single real zero at $x = -1$. If $x \le -1$ and n is even, then

$$d(x) = \sum_{i=0}^{n} d_i x^i = 1 + \sum_{i=1}^{\frac{n}{2}} (d_{2i-1} x^{2i-1} + d_{2i} x^{2i}) > 0.$$

So we consider the zeros of $d(x)$ for $-1 < x < 0$. To do so, we write $d(x) = \frac{j(x)}{|x|e^{1/x}}$, where $j(x)$ is the numerator in (1). Note that the polynomial $v(t) = -(n+1)d_n t^{n+2} - d_{n+1} t^{n+1} + 1$ has a zero at $t = -1$, by the identity $d_{n+1} = (n+1)d_n + (-1)^{n+1}$ (see [8, p. 67]), whence the integrand is bounded on the interval $(-1, 0)$.

Now assume $n \ge 5$ is odd. By Descartes' rule of signs, the polynomial $v(x)$ has two negative zeros. Since $v(x)$ achieves its minimum value over $(-\infty, 0)$ at $x = -\frac{d_{n+1}}{(n+2)d_n}$, with

$$-\frac{d_{n+1}}{(n+2)d_n} = -\frac{(n+1)d_n + (-1)^{n+1}}{(n+2)d_n} > -1,$$

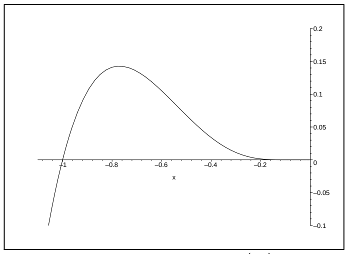
we see that the other negative zero of $v(x)$, which we'll denote by r, occurs in the interval $(-1, 0)$. Note that $v(x) < 0$ if $-1 < x < r$ and $v(x) > 0$ if $r < x < 0$.

By the continuity of $d(x)$, as $d(0) > 0$, we may choose x_0, with $r < x_0 < 0$, such that $d(x) > 0$ on $[x_0, 0]$. Note that

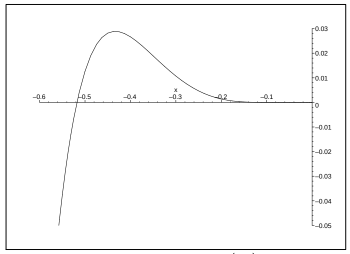
$$j'(x) = \frac{v(x)}{x(1+x)} e^{1/x},$$

which is positive for $-1 < x < r$ and negative for $r < x < 0$. Since $d(x_0) > 0$, it follows that $j(x)$ is positive on the interval $[r, x_0)$, and thus so is $d(x)$. Therefore, $d(x) > 0$ on $[r, 0)$. Furthermore, $d(-1) < 0$ since $n \ge 5$ is odd, which implies $j(-1) < 0$, and thus $j(x)$ has a single zero on the interval $(-1, r)$ since it is increasing there with $j(r) > 0$. It follows that $d(x)$ has a single zero on the interval $(-1, r)$ and thus on $(-1, 0)$. Furthermore, this zero is seen to have multiplicity one. (See graphs of $j(x)$ and $v(x)$ below when $n = 3, 5, 7$).

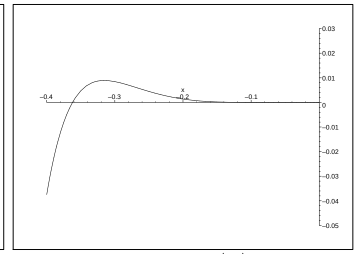
Now assume n is even. Then $v(x)$ has single negative zero at $x = -1$, with $v(x) > 0$ if $-1 < x < 0$. Then $j'(x) < 0$ if $-1 < x < 0$, with $j(x_0) > 0$, where x_0 is chosen as in the odd case above. Thus, $j(x) > 0$ for all $x \in (-1, x_0]$, which implies $d(x) > 0$ for all x in this interval. Therefore, $d(x) > 0$ if $-1 < x < 0$, which implies $d(x)$ has no real zeros when n is even. □



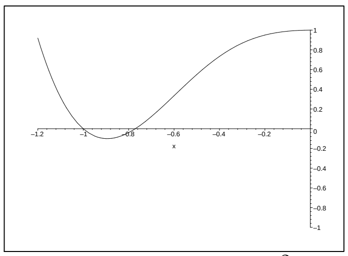
Graph of $j_3(x)$



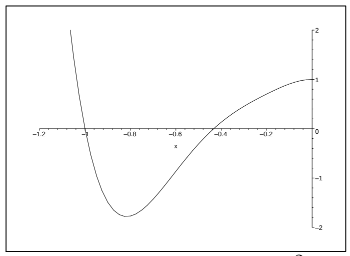
Graph of $j_5(x)$



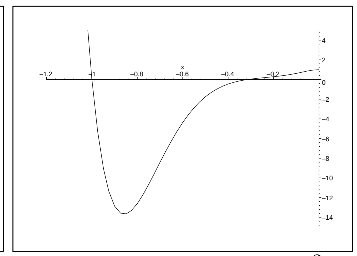
Graph of $j_7(x)$



Graph of $v_3(x)/(1+x^2)$



Graph of $v_5(x)/(1+10x^2)$



Graph of $v_7(x)/(1+60x^2)$

3 Involution polynomials

We will need the log-convexity of the polynomials $a_n(q)$.

Lemma 3.1. *If $q \geq 1$, then*

$$a_{n-1}(q)a_{n+1}(q) > a_n^2(q), \qquad n \geq 3. \tag{3}$$

Proof. Here, we will denote $a_n(q)$ by a_n. To prove (3), we will show by induction the following pair of inequalities:

$$\frac{n+1}{n}a_n^2 > a_{n-1}a_{n+1} > a_n^2, \qquad n \geq 3. \tag{4}$$

The $n = 3$ case of (4) is easily verified. We first prove the right-hand inequality in (4) with n replaced by $n+1$. If $n \geq 3$, then $a_n a_{n+2} > a_{n+1}^2$ if and only if

$$a_n(qa_{n+1} + (n+1)a_n) > a_{n+1}^2,$$

i.e.,

$$(n+1)a_n^2 > a_{n+1}(a_{n+1} - qa_n) = na_{n-1}a_{n+1},$$

which holds by the induction hypothesis.

For the left-hand inequality in (4), first note that $\frac{n+2}{n+1}a_{n+1}^2 > a_n a_{n+2}$ if and only if

$$\frac{n+2}{n+1}a_{n+1}^2 > a_n(qa_{n+1} + (n+1)a_n),$$

i.e.,

$$(n+2)a_{n+1}(qa_n + na_{n-1}) > (n+1)a_n(qa_{n+1} + (n+1)a_n),$$

which reduces to

$$n(n+2)a_{n-1}a_{n+1} > (n+1)^2 a_n^2 - q a_n a_{n+1}. \tag{5}$$

To show (5), note that $n(n+2)a_n^2 > (n+1)^2 a_n^2 - q a_n a_{n+1}$ since $q a_n a_{n+1} > a_n^2$. Thus, by the induction hypothesis, we have

$$n(n+2)a_{n-1}a_{n+1} > n(n+2)a_n^2 > (n+1)^2 a_n^2 - q a_n a_{n+1},$$

which gives (5) and completes the proof. □

We will also need the following integral representation of $a(x;q)$.

Lemma 3.2. *Let $x_0 < 0$ and q be fixed. Then for all $x < 0$, we have*

$$a(x;q) = \frac{\int_{x_0}^{x} \frac{u(t;q)}{t^2} e^{-q/t+1/2t^2} dt + c}{|x| e^{-q/x+1/2x^2}}, \tag{6}$$

where $u(t;q) = -(n+1)a_n(q)t^{n+2} - a_{n+1}(q)t^{n+1} + 1$ and $c = |x_0| a(x_0;q) e^{-q/x_0+1/2x_0^2}$.

Proof. First observe that

$$(1 - qx - x^2)a(x;q) - x^3 a'(x;q) = -(n+1)a_n(q)x^{n+2} - a_{n+1}(q)x^{n+1} + 1. \tag{7}$$

Note that the coefficient of x^i on the left-hand side of (7) is zero if $2 \le i \le n$ since $a_i(q) = q a_{i-1}(q) + (i-1)a_{i-2}(q)$. One may rewrite (7) as

$$a'(x;q) + \frac{x^2 + qx - 1}{x^3} a(x;q) = -\frac{u(x;q)}{x^3},$$

where $u(x;q)$ is as given. Solving this first-order linear differential equation using the standard technique gives (6). □

Remark 3.3. *Note that substituting $x = 0$ in formula (6) produces the indeterminate $\frac{\infty}{\infty}$. However, the formula is still valid in this case, since*

$$\begin{aligned}
\lim_{x \to 0^-} a(x;q) &= \lim_{x \to 0^-} \left(\frac{\int_{x_0}^{x} \frac{u(t;q)}{t^2} e^{-1/t+1/2t^2} dt + c}{|x| e^{-1/x+1/2x^2}} \right) \\
&= \lim_{x \to 0^-} \left(\frac{\frac{u(x;q)}{x^2} e^{-1/x+1/2x^2}}{-x\left(\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3}\right) e^{-1/x+1/2x^2}} \right) \\
&= \lim_{x \to 0^-} \left(\frac{1}{-x^2 - x + 1} \right) = 1 = a(0;q).
\end{aligned}$$

3.1 The case $q = 1$

In this subsection, we focus on the real zeros of the polynomials $a(x;q)$ in the case $q = 1$. Here, we will denote the $q = 1$ case of $a_n(q)$ by a_n. We now show the following inequality for the sequence a_n.

Lemma 3.4. *We have*

$$a_n^{n+1} < (n+1)^n a_{n-1}^n, \qquad n \ge 1. \tag{8}$$

Proof. Before we prove (8), let us first make some preliminary observations. Let $b_n := \frac{a_n}{a_{n-1}}$. Note that $b_n < b_{n+1}$ for all $n \ge 3$, by Lemma 3.1. The recurrence for a_n may be written equivalently as

$$b_{n+1} = 1 + \frac{n}{b_n}, \qquad n \ge 1,$$

so that

$$b_n < 1 + \frac{n}{b_n}, \qquad n \geq 3.$$

Then $b_n^2 - b_n - n < 0$ implies

$$b_n < \frac{1 + \sqrt{1+4n}}{2} < \frac{1}{2} + \sqrt{n+1}, \qquad n \geq 3. \tag{9}$$

Thus, by (9),

$$a_n = \prod_{i=1}^{n} b_i < \prod_{i=1}^{n} \left(\frac{1}{2} + \sqrt{i+1} \right), \qquad n \geq 1. \tag{10}$$

We now show (8). One may verify (8) directly for $1 \leq n \leq 14$, so let us assume $n \geq 15$ for the remainder of the proof. Note that (8) may be written equivalently as

$$a_n b_n^n < (n+1)^n. \tag{11}$$

For (11), it is enough to show

$$\left(\frac{1}{2} + \sqrt{n+1} \right)^n \prod_{i=1}^{n} \left(\frac{1}{2} + \sqrt{i+1} \right) < (n+1)^n, \tag{12}$$

by (9) and (10). Since the left-hand side of (12) may be written as

$$\left(\frac{1}{2} + \sqrt{n+1} \right)^n \prod_{i=1}^{n} \left(\frac{1}{2} + \sqrt{i+1} \right)^{\frac{1}{2}} \left(\frac{1}{2} + \sqrt{n+2-i} \right)^{\frac{1}{2}},$$

and since

$$\left(\frac{1}{2} + \sqrt{i+1} \right) \left(\frac{1}{2} + \sqrt{n+2-i} \right) \leq \left(\frac{1}{2} + \sqrt{\frac{n+3}{2}} \right)^2, \qquad 1 \leq i \leq n,$$

it follows that the left-hand side of (12) is at most

$$\left[\left(\frac{1}{2} + \sqrt{n+1} \right) \left(\frac{1}{2} + \sqrt{\frac{n+3}{2}} \right) \right]^n.$$

So to show (12), it suffices to show

$$\left(\frac{1}{2} + \sqrt{n+1} \right) \left(\frac{1}{2} + \sqrt{\frac{n+3}{2}} \right) < n+1. \tag{13}$$

Note that

$$(\sqrt{2}+1)\sqrt{n+3} < 2(\sqrt{2}-1)n + \frac{3\sqrt{2}}{2} - 4, \qquad n \geq 15, \tag{14}$$

since it is true for $n = 15$ and hence for all $n > 15$ as

$$\frac{\sqrt{2}+1}{\sqrt{n+4} + \sqrt{n+3}} < 2\sqrt{2} - 2, \qquad n \geq 15,$$

where the last inequality was obtained by replacing n with $n+1$ on both sides of (14) and subtracting the n case.

By (14), we then have

$$\begin{aligned} \left(\frac{1}{2} + \sqrt{n+1} \right) \left(\frac{1}{2} + \sqrt{\frac{n+3}{2}} \right) &= \frac{1}{4} + \frac{1}{2} \left(\sqrt{n+1} + \sqrt{\frac{n+3}{2}} \right) + \sqrt{\frac{(n+1)(n+3)}{2}} \\ &< \frac{1}{4} + \frac{\sqrt{n+3}}{2} \left(1 + \frac{1}{\sqrt{2}} \right) + \frac{n+2}{\sqrt{2}} < n+1, \qquad n \geq 15, \end{aligned}$$

which gives (13) and completes the proof. □

Let $a(x) = a(x; 1)$. We now prove the main result in the case $q = 1$.

Theorem 3.5. *The polynomial $\sum_{i=0}^{n} a_i x^i$ has exactly one real zero if n is odd and no real zeros if n is even.*

Proof. Clearly, $a(x) = \sum_{i=0}^{n} a_i x^i$ has no positive zeros as it has positive coefficients. So assume $x < 0$. We show that $a(x)$ has one negative zero when n is odd and that $a(x) > 0$ for all x when n is even. First assume n is odd. Let $u(x) = u(x; 1)$, where $u(x; q)$ is as defined in Lemma 3.2. We first show $u(x) > 0$ if $x < 0$, that is,

$$-(n+1)a_n x^{n+2} - a_{n+1}x^{n+1} + 1 > 0, \qquad x < 0. \tag{15}$$

Since $u'(x) = -(n+1)x^n((n+2)a_n x + a_{n+1})$, we see that $u(x)$ achieves a minimum value over $x < 0$ at $x = \frac{-a_{n+1}}{(n+2)a_n}$. So to show (15) is to show $u\left(\frac{-a_{n+1}}{(n+2)a_n}\right) > 0$ when n is odd, i.e.,

$$1 > (n+1)a_n \left(\frac{-a_{n+1}}{(n+2)a_n}\right)^{n+2} + a_{n+1}\left(\frac{-a_{n+1}}{(n+2)a_n}\right)^{n+1},$$

which may be rewritten as

$$(n+2)^{n+2} a_n^{n+1} > a_{n+1}^{n+2}, \qquad n \text{ odd}. \tag{16}$$

Note that (16) holds by Lemma 3.4, which establishes (15).

By Lemma 3.2, we have

$$a(x) = \frac{h(x)}{|x|e^{-1/x+1/2x^2}}, \qquad x < 0, \tag{17}$$

where

$$h(x) = \int_{x_0}^{x} \frac{u(t)}{t^2} e^{-1/t+1/2t^2}\, dt + |x_0| a(x_0) e^{-1/x_0+1/2x_0^2}.$$

By continuity, we may choose $-1 < x_0 < 0$ sufficiently close to zero such that $a(x_0) > 0$ since $a(0) = 1$, whence $h(x_0) > 0$. Note that

$$h'(x) = \frac{u(x)}{x^2} e^{-1/x+1/2x^2} > 0, \qquad x < 0,$$

by (15), with $h(-1) \le 0$ by (17) since

$$a(-1) = \sum_{i=0}^{n} a_i(-1)^i = (-a_n + a_{n-1}) + (-a_{n-2} + a_{n-3}) + \cdots + (-a_1 + a_0) \le 0$$

for n odd (with equality only if $n = 1$). It follows that $h(x)$, and thus $a(x)$, has one negative zero whenever n is odd, and it belongs to the interval $[-1, 0)$. Since $h'(x) > 0$ for all $x < 0$ with $h(x)$ analytic on the interval $(-\infty, 0)$, this zero necessarily has multiplicity one.

Now suppose n is even. We will show that $a(x) > 0$ for all x by induction, the $n = 2$ case clear since $a(x) = 2x^2 + x + 1$ in that case. By Descartes' rule of signs, the polynomial $u(x)$ has one negative zero when n is even, which we denote by r. We first show $a(r) > 0$. To do so, note initially that if $x < -\frac{a_{n-1}}{a_n}$, then

$$a(x) = \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n-2} a_i x^i + x^{n-1}(a_{n-1} + a_n x) > 0,$$

since $\sum_{i=0}^{n-2} a_i x^i > 0$, by the induction hypothesis, and since $x^{n-1}(a_{n-1} + a_n x) > 0$. So to show $a(r) > 0$, it suffices to show $r < -\frac{a_{n-1}}{a_n}$. Since $u(0) = 1$ and $u'(x)$ has one sign change for $x < 0$

when n is even, and it is from positive to negative, we need only verify $u\left(-\frac{a_{n-1}}{a_n}\right) > 0$, which holds for n even if and only if

$$(n+1)a_n a_{n-1}^{n+2} - a_n a_{n+1} a_{n-1}^{n+1} < a_n^{n+2},$$

or

$$(n+1)a_n a_{n-1}^{n+2} - a_n(a_n + na_{n-1})a_{n-1}^{n+1} < a_n^{n+2}.$$

The latter inequality reduces to

$$a_n a_{n-1}^{n+2} < a_n^{n+2} + a_n^2 a_{n-1}^{n+1},$$

which is seen to hold as $a_n a_{n-1}^{n+2} < a_n^2 a_{n-1}^{n+1}$. Thus $a(r) > 0$, as desired.

We now represent $a(x)$ once again as in (17) above. This time, however, we choose $x_0 = r$. Then $h'(x) > 0$ for $r < x < 0$ and $h'(x) < 0$ for $x < r$, which implies that the minimum value of $h(x)$ for $x < 0$ is achieved at $x = r$. Now $a(r) > 0$ implies $h(r) > 0$, whence $h(x) > 0$ for all $x < 0$. Thus, $a(x) > 0$ for all x by (17), which completes the induction and the proof in the even case. □

3.2 A generalization

Here, we consider the problem of determining the nature of the real zeros of the polynomials

$$a(x;q) = \sum_{i=0}^{n} a_i(q)x^i.$$

We will show that $a(x;q)$ has the smallest possible number of real zeros over a range of q-values by generalizing the argument given above in the case $q = 1$. Let $b_n(q) := \frac{a_n(q)}{a_{n-1}(q)}$ if $n \geq 1$. At times, we will suppress the q argument in either $a_n(q)$ or $b_n(q)$ when the context in clear.

The next two lemmas concern properties of the rational functions $b_n(q)$.

Lemma 3.6. *If $n \geq 1$, then $b_n(q)^2 > n$ for all $q > 1$.*

Proof. We will show for all $q > 1$ that

$$\sqrt{n}a_{n-1} < a_n < (q + \sqrt{n-1})a_{n-1}, \qquad n \geq 2, \tag{18}$$

by induction, the $n = 2$ case clear. If $n \geq 2$, then

$$a_{n+1} = qa_n + na_{n-1} < qa_n + \frac{n}{\sqrt{n}}a_n = (q + \sqrt{n})a_n,$$

by the induction hypothesis, which gives the right-hand inequality in (18) in the $n + 1$ case.

On the other hand, we have

$$a_{n+1} = qa_n + na_{n-1} > qa_n + \frac{n}{q + \sqrt{n-1}}a_n \geq \sqrt{n+1}a_n, \qquad n \geq 2, \tag{19}$$

which gives the left-hand inequality in (18) in the $n + 1$ case and completes the proof of (18). Note that the second inequality in (19) is seen to hold since it is equivalent to

$$q^2 + \left(n - \sqrt{n^2 - 1}\right) \geq \frac{2q}{\sqrt{n+1} + \sqrt{n-1}}, \qquad n \geq 2,$$

which is true, since for $n \geq 2$, one has

$$q^2 > q > \frac{2q}{\sqrt{3} + 1} \geq \frac{2q}{\sqrt{n+1} + \sqrt{n-1}}.$$

□

Lemma 3.7. *If $n \geq 1$, then $b_n'(q) > 0$ for all $q > 1$.*

Proof. Let $b_n' = b_n'(q)$ and $q > 1$. We will show $0 < b_n' \leq 1$ for all $n \geq 1$ by induction, the $n = 1$ and $n = 2$ cases clear. If $n \geq 2$, then differentiating both sides of the recurrence $b_{n+1}b_n = qb_n + n$, and rearranging, gives

$$b_n'(b_{n+1} - q) = b_n(1 - b_{n+1}'),$$

i.e.,

$$nb_n' = b_n^2(1 - b_{n+1}'), \tag{20}$$

since $\frac{n}{b_n} = b_{n+1} - q$. By the induction hypothesis, we have $b_n' > 0$, which taken together with (20), implies $b_{n+1}' \leq 1$. On the other hand, if $b_{n+1}' \leq 0$, then $b_n' \leq 1$ implies

$$b_n^2 = n\left(\frac{b_n'}{1 - b_{n+1}'}\right) \leq n,$$

which contradicts Lemma 3.6. Thus, $0 < b_{n+1}' \leq 1$, which completes the induction and the proof. □

We will often denote the sum $\sum_{i=0}^{m} a_i(q)x^i$ more specifically by $a_m(x;q)$. Since the even case concerning the real zeros of $a_m(x;q)$ will follow in much the same way as it did when $q = 1$, we will concern ourselves primarily with the case when m is odd. Furthermore, since it is easily seen that $a_m(x;q) > 0$ if $x > 0$ and $a_m(x;q) < 0$ if $x < -1$ for all odd m if $q \geq 1$, then we may restrict our attention to the interval $[-1, 0]$.

Let us first require that $a_m(x;q)$ be increasing on the interval $(-1, 0)$ since $a_m(-1;q) \leq 0$ and $a_m(0;q) > 0$ if $q \geq 1$. Write

$$\frac{d}{dx}a_m(x;q) = \sum_{i=0}^{m-1}(i+1)a_{i+1}(q)x^i = q + \sum_{i=1}^{n}\ell_i,$$

where $m = 2n + 1$, $n \geq 0$, and

$$\ell_i = 2ia_{2i}(q)x^{2i-1}\left(1 + \frac{(2i+1)a_{2i+1}(q)}{2ia_{2i}(q)}x\right), \qquad 1 \leq i \leq n.$$

In order to ensure that $\frac{d}{dx}a_m(x;q)$ be positive on the interval $(-1, 0)$, let us first find for each ℓ_i a uniform lower bound on $(-1, 0)$ as a function of q. Note that if $\ell_i < 0$, then making the substitution $x = -\frac{2ia_{2i}(q)}{(2i+1)a_{2i+1}(q)}y$, where $0 \leq y \leq \frac{(2i+1)a_{2i+1}(q)}{2ia_{2i}(q)}$, implies

$$\begin{aligned}
-\ell_i &= \frac{(2ia_{2i}(q))^{2i}}{((2i+1)a_{2i+1}(q))^{2i-1}}y^{2i-1}(1-y) \\
&\leq \frac{(2ia_{2i}(q))^{2i}}{((2i+1)a_{2i+1}(q))^{2i-1}}\left(\frac{2i-1}{2i}\right)^{2i-1}\left(\frac{1}{2i}\right) = \left(\frac{2i-1}{2i+1}\right)^{2i-1}\frac{a_{2i}(q)^{2i}}{a_{2i+1}(q)^{2i-1}}.
\end{aligned}$$

Thus, regardless of whether ℓ_i is positive or negative, it is always the case that

$$\ell_i \geq -\left(\frac{2i-1}{2i+1}\right)^{2i-1}\frac{a_{2i}(q)^{2i}}{a_{2i+1}(q)^{2i-1}}.$$

Therefore, for all $n \geq 0$ and $q \geq 1$, we have

$$\frac{d}{dx}a_{2n+1}(x;q) \geq f_n(q), \qquad -1 < x < 0, \tag{21}$$

where

$$f_n(q) := q - \sum_{i=1}^{n} \left(\frac{2i-1}{2i+1}\right)^{2i-1} \frac{a_{2i}(q)^{2i}}{a_{2i+1}(q)^{2i-1}}.$$

For each n, we seek the largest interval of the form $[1,t]$ such that $f_n(q) \geq 0$ for all $q \in [1,t]$. For such q, note that $a_{2n+1}(x;q)$ is increasing on the interval $(-1,0)$, by (21). To this end, let

$$q_n^* := \max\{t \geq 1 : \ f_n(q) \geq 0 \text{ for all } q \in [1,t]\},$$

where we take $q_n^* = \infty$ if $f_n(q) \geq 0$ for all $q \geq 1$. Note that q_n^* is a decreasing function of n since $f_n(q) > f_{n+1}(q)$ for all positive q. By Maple, we have that q_n^* is finite if and only if $n \geq 5$. If $0 \leq j \leq n$ and $1 \leq q \leq q_n^*$, then we are guaranteed that

$$\frac{d}{dx} a_{2j+1}(x;q) \geq f_j(q) \geq f_n(q) \geq 0, \qquad -1 < x < 0, \tag{22}$$

and thus $a_{2j+1}(x;q)$ has exactly one zero on the interval $[-1,0)$ for all such j and q since $a_{2j+1}(-1;q) \leq 0$ and $a_{2j+1}(0;q) > 0$. Note that we require $q \geq 1$ here and elsewhere since we do not have a generalization of Lemma 3.1 for $q < 1$ and a meaningful one may not indeed be possible (note also that the sequence $a_n(q)$ need not be increasing for such q).

It will also be necessary that inequality (16) is suitably generalized. To this end, given $n \geq 0$, let $\widetilde{q}_n$ denote the $q \geq 1$ such that

$$a_{2n+4}(q) b_{2n+4}(q)^{2n+4} = (2n+5)^{2n+5}. \tag{23}$$

Note that, by Lemmas 3.7 and 3.4, the left-hand side of (23) is an increasing function of q for each fixed n and is strictly less than the right-hand side when $q = 1$, whence $\widetilde{q}_n$ is uniquely determined. Thus, if $1 \leq q \leq \widetilde{q}_n$, then the left-hand side of (23) is less than or equal $(2n+5)^{2n+5}$, which provides a generalization of (16) for such q.

One could then attempt to verify the inequalities

$$a_{2j+4}(q) b_{2j+4}(q)^{2j+4} \leq (2j+5)^{2j+5}, \qquad j \geq n, \tag{24}$$

for all $1 \leq q \leq \widetilde{q}_n$, which would then show (see below) that $a_{2j+3}(x;q)$ has exactly one real zero for all such j and q. By Lemma 3.7, it would suffice to verify (24) in the case $q = \widetilde{q}_n$.

Thus, given any fixed $n \geq 0$ and $1 \leq q \leq \min\{q_n^*, \widetilde{q}_n\}$, we know that $a_{2j+1}(x;q)$ has exactly one real zero if $0 \leq j \leq n$, and so to show that $a_m(x;q)$ always has the smallest possible number of real zeros for q in the given range, it will suffice to verify (24) for all $j \geq n$. So in order to maximize the range in q for which our technique will work, we choose n such that $\min\{q_n^*, \widetilde{q}_n\}$ is largest among all $n \geq 0$. This value, which we denote by λ, will yield the best possible q by the present method.

Lemma 3.8. *The maximum value of* $\min\{q_n^*, \widetilde{q}_n\}$ *over* $n \geq 0$ *occurs when* $n = 140$ *and is given by* $\lambda = q_{140}^* \approx 6.2758547971$.

Proof. Using Maple, we determine the approximate values of $\widetilde{q}_n$ for $0 \leq n \leq 150$ and find that $\widetilde{q}_n$ is increasing over such n, with $\widetilde{q}_{139} < q_{139}^*$ and $\widetilde{q}_{140} > q_{140}^*$. Since q_n^* is a decreasing function of n and since $\widetilde{q}_{139} \approx 6.2579383791$ is less than $q_{140}^* \approx 6.2758547971$, it follows that the largest value of $\min\{q_n^*, \widetilde{q}_n\}$ is achieved when $n = 140$ and is given by q_{140}^*. □

Lemma 3.9. *If* $0 \leq j \leq 140$ *and* $1 \leq q \leq \lambda$*, then the polynomial* $a_{2j+1}(x;q)$ *has exactly one real zero, and it belongs to the interval* $[-1,0)$.

Proof. This follows from (22) and the subsequent discussion, along with Lemma 3.8. □

We now show the inequalities in (24) in the case when $n = 140$.

Lemma 3.10. *If $j \geq 140$ and $1 \leq q \leq \lambda$, then*

$$a_{2j+4}(q)b_{2j+4}(q)^{2j+4} < (2j+5)^{2j+5}. \tag{25}$$

Proof. Using Maple, one may verify that the sequence $\widetilde{q}_j$ is increasing for $0 \leq j \leq 250$. Since $\lambda < \widetilde{q}_j$ if $j \geq 140$, it follows from Lemma 3.7 that (25) is satisfied when $140 \leq j \leq 250$. So let us assume $j > 250$. Similar to the proof of Lemma 3.4 above, we have for all $n \geq 1$,

$$b_n(q) \leq \frac{q}{2} + \sqrt{n + \frac{q^2}{4}}, \qquad q \geq 1,$$

and

$$\begin{aligned} a_n(q) = \prod_{i=1}^{n} b_i(q) &\leq \prod_{i=1}^{n} \left(\frac{q}{2} + \sqrt{i + \frac{q^2}{4}} \right) \\ &= \prod_{i=1}^{n} \left(\frac{q}{2} + \sqrt{i + \frac{q^2}{4}} \right)^{\frac{1}{2}} \left(\frac{q}{2} + \sqrt{n+1-i+\frac{q^2}{4}} \right)^{\frac{1}{2}} \\ &\leq \left(\frac{q}{2} + \sqrt{\frac{n+1}{2} + \frac{q^2}{4}} \right)^{n}, \qquad q \geq 1. \end{aligned}$$

Thus, to show (25), it is enough to show

$$\left(\frac{q}{2} + \sqrt{2j+4+\frac{q^2}{4}} \right) \left(\frac{q}{2} + \sqrt{\frac{2j+5}{2} + \frac{q^2}{4}} \right) < 2j+5, \qquad j > 250, \tag{26}$$

where $1 \leq q \leq \lambda$. Since $\lambda < 6.5$ and the left-hand side of (26) is an increasing function of q, it is enough to verify (26) in the case when $q = 6.5$, i.e.,

$$\left(\frac{13}{4} + \sqrt{2j+4+\frac{169}{16}} \right) \left(\frac{13}{4} + \sqrt{\frac{2j+5}{2} + \frac{169}{16}} \right) < 2j+5, \qquad j > 250. \tag{27}$$

Let p_j denote the left-hand side of (27). Note that

$$\begin{aligned} p_{j+1} - p_j &< \frac{6.5}{\sqrt{2j+6+\frac{169}{16}} + \sqrt{2j+4+\frac{169}{16}}} + \frac{3.25}{\sqrt{\frac{2j+7}{2} + \frac{169}{16}} + \sqrt{\frac{2j+5}{2} + \frac{169}{16}}} \\ &\quad + \frac{4j + \frac{683}{16}}{\sqrt{2}\left(2j+4+\frac{169}{16}\right)} \\ &< 2, \qquad j > 250, \end{aligned} \tag{28}$$

since the first inequality holds for all $j > 0$ and since the middle quantity is a decreasing function of j (being the sum of three decreasing functions) which is less than 2 when $j = 251$. Then (27) follows from (28) since it holds when $j = 251$, which completes the proof of (25). □

We now can give a generalization of Theorem 3.5.

Theorem 3.11. *If $1 \leq q \leq \lambda$, then the polynomial $\sum_{i=0}^{n} a_i(q)x^i$ has exactly one real zero if n is odd and no real zeros if n is even.*

Proof. Clearly, $a(x;q)$ has no positive zeros or zeros less than -1, so we may restrict attention to the interval $[-1, 0]$. First assume $n = 2j+1$ is odd. By Lemma 3.9, we may assume $j \geq 141$.

Let $u(x;q) = -(n+1)a_n(q)x^{n+2} - a_{n+1}(q)x^{n+1} + 1$. To show that $u(x;q) > 0$ if $x < 0$, we must show that $u\left(\frac{-a_{n+1}(q)}{(n+2)a_n(q)}; q\right) > 0$, or, equivalently,

$$a_{n+1}(q)b_{n+1}(q)^{n+1} < (n+2)^{n+2}, \qquad n \geq 283 \;\text{ odd.} \tag{29}$$

Inequality (29) holds by Lemma 3.10, which implies $u(x;q) > 0$ if $x < 0$. The remainder of the proof in the odd case uses Lemma 3.2 and follows now in much the same way as it did when $q = 1$.

The argument for the even case also follows in a similar manner as before and in fact applies to all $q \geq 1$. □

4 Convergence of zeros

In this section, we prove that the sequence of real zeros of $a(x;q)$ and $d(x)$ for n odd is monotonically convergent.

Proposition 4.1. *If $1 \leq q \leq \lambda$, then the sequence of real zeros of the polynomials $a(x;q)$ for n odd is monotonically increasing and converges to* 0.

Proof. We treat the $q = 1$ case, as the proof for the general case is similar. We first show that the sequence of zeros increases. Where needed, we will denote the sum $\sum_{i=0}^{n} a_i x^i$ more explicitly by $a_n(x)$ if $n \geq 0$, where $a_i = a_i(1)$. Let s denote the real zero of $a_{2m-1}(x)$. Note that $s = -1$ when $m = 1$, so let us assume $m \geq 2$. The proof of Theorem 3.5 above shows that $-1 < s < 0$ in this case. Observe that $a_{2m-1}(s) = 0$ may be written as

$$1 + s + \sum_{i=1}^{m-1} \left(\frac{a_{2i}}{a_{2i+1}} + s\right) a_{2i+1}s^{2i} = 0. \tag{30}$$

Then we have $\frac{a_{2m-2}}{a_{2m-1}} + s < 0$, for if not, and $\frac{a_{2m-2}}{a_{2m-1}} \geq -s$, then $\frac{a_{2i}}{a_{2i+1}} \geq -s$ for all $1 \leq i \leq m-1$, since the ratio $\frac{a_n}{a_{n+1}}$ is decreasing, by Lemma 3.1. But then the left-hand side of (30) would be positive as $1 + s > 0$ and $\sum_{i=1}^{m-1}\left(\frac{a_{2i}}{a_{2i+1}} + s\right) a_{2i+1}s^{2i} \geq 0$ (being a sum of non-negative terms). Then $\frac{a_{2m-2}}{a_{2m-1}} + s < 0$ implies $\frac{a_{2m}}{a_{2m+1}} + s < 0$, again since the ratio is decreasing. Therefore,

$$\begin{aligned} a_{2m+1}(s) &= \left(\frac{a_{2m}}{a_{2m+1}} + s\right) a_{2m+1}s^{2m} + a_{2m-1}(s) \\ &= \left(\frac{a_{2m}}{a_{2m+1}} + s\right) a_{2m+1}s^{2m} < 0, \end{aligned}$$

which implies that the zero of $a_{2m+1}(x)$ is greater than that of $a_{2m-1}(x)$.

To show that the sequence of real zeros converges to zero, we fix y, $-1 < y < 0$, and show that $a_{2m+1}(y) < 0$ for m sufficiently large. To do so, we first bound the ratio $b_n := \frac{a_n}{a_{n-1}}$ from below. Since $b_{n+1} = 1 + \frac{n}{b_n}$ and $b_{n+1} \leq \frac{n+1}{n} b_n$ if $n \geq 3$, by (4), we have

$$(n+1)b_n^2 - nb_n - n^2 \geq 0,$$

which gives

$$b_n \geq \frac{n}{2n+2}\left(1 + \sqrt{4n+5}\right), \qquad n \geq 1. \tag{31}$$

From (31), we see that $\frac{a_{n-1}}{a_n} \to 0$ as $n \to \infty$ and that $a_n = \prod_{i=1}^{n} b_i$ is larger than C^n for any positive constant C for all n sufficiently large. This implies

$$a_{2m+1}(y) = \sum_{i=0}^{m} \left(\frac{a_{2i}}{a_{2i+1}} + y\right) a_{2i+1}y^{2i} < 0,$$

for all m sufficiently large, which completes the proof. □

Modifying the above proof somewhat yields the same result for the polynomials $d(x)$.

Proposition 4.2. *The sequence of real zeros of the polynomials $d(x)$ for $n \geq 3$ odd is monotonically increasing and converges to* 0.

Proof. First note that

$$\begin{aligned} d_{2m-2}d_{2m+1} &= \frac{1}{2m-1}(d_{2m-1}+1)((2m+1)d_{2m}-1) \\ &= \frac{1}{2m-1}((2m+1)d_{2m-1}d_{2m}+(2m+1)d_{2m}-d_{2m-1}-1) \\ &> \frac{2m+1}{2m-1}d_{2m-1}d_{2m} > d_{2m-1}d_{2m}, \end{aligned}$$

whence $d_{2m-2}d_{2m+1} > d_{2m-1}d_{2m}$ for all $m \geq 1$. The monotonicity of the real zeros of the $d(x)$ now follows as in the first part of the proof of Proposition 4.1 above.

Recall that $d_n \sim \frac{n!}{e}$ for large n (see [8, p. 67]) and note that $\frac{d_{n-1}}{d_n} \to 0$ as $n \to \infty$. Using Stirling's formula for $n!$, the convergence of the real zeros now follows in a manner similar to the second part of the proof of Proposition 4.1 above. □

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