GENERALIZED BELL NUMBERS AND ALGEBRAIC DIFFERENTIAL EQUATIONS

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Abstract

It is shown that the exponential generating function of the generalized Bell numbers introduced recently by the authors satisfies an algebraic differential equation in analogy to the case of the conventional Bell numbers. Several examples are discussed explicitly.

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1. Introduction

The Bell numbers (sequence A000110 in [20]) are among the most important combinatorial numbers and have been studied from many different point of views, see, e.g., [5, 20, 21, 25] and the many references given therein. Denoting the n-th Bell number by $B(n)$, one can introduce $B(x) := \sum_{n=0}^{\infty} B(n)x^n$, the ordinary generating function as well as $\widehat{B}(x) := \sum_{n=0}^{\infty} B(n)\frac{x^n}{n!}$, the exponential generating function. It is a classical result that

$$\widehat{B}(x) = e^{e^x-1}. \tag{1.1}$$

Denoting the derivative with respect to x by a prime, i.e., $f'(x) = \frac{df}{dx}(x)$, this shows that

$$\widehat{B}'(x) = e^x\widehat{B}(x). \tag{1.2}$$

Using a simple argument - which can be found, e.g., in Klazar [12] - one derives from this relation an algebraic differential equation for $\widehat{B}$ as follows. First note that (1.2) can be written as $\widehat{B}'(x)/\widehat{B}(x) = e^x$. Now, taking a derivative with respect to x yields, due to $(e^x)' = e^x$, that

$$\left(\frac{\widehat{B}'(x)}{\widehat{B}(x)}\right)' = e^x.$$

Using on the right-hand side $\widehat{B}'(x)/\widehat{B}(x) = e^x$, one obtains the algebraic differential equation

$$\widehat{B}''\widehat{B} - (\widehat{B}')^2 - \widehat{B}'\widehat{B} = 0. \tag{1.3}$$

This relation for $\widehat{B}$ can be found in exactly this form in, e.g., Klazar [12]. Before turning to the generalized Bell numbers, we would like to point out that the main point of [12] was to show that the ordinary generating function B (and some related functions) cannot satisfy an algebraic differential equation. Gould and Quaintance considered in [9] so-called variant sequences whose exponential generating function f satisfies the functional differential equation $f'(x) = e^{bx} f(ax)$ generalizing (1.2).

By considering variables U, V satisfying the commutation relation $UV = VU + hV^s$ for $s \in \mathbb{N}_0$ and $h \in \mathbb{C} \setminus \{0\}$, the present authors introduced in [14] generalized Stirling numbers $\mathfrak{S}_{s;h}(n,k)$ by considering the normal ordered expression of $(VU)^n$. Burde considered in [2] closely related numbers, see also Lang [13]; note that the case $s = 0$ and $h = 1$ represents the conventional Stirling numbers. In [16] the generalized Stirling and Bell numbers associated with $s \in \mathbb{R}$ and $h \in \mathbb{C}\setminus\{0\}$ were considered in detail and many properties were shown. For example, if s is a natural number, then the corresponding generalized Stirling numbers can be interpreted as s-rook numbers of a staircase board, generalizing the conventional case corresponding to $s = 0$ in a beautiful fashion. These generalized Stirling numbers form a particular subfamily of the generalized Stirling numbers introduced by Hsu and Shiue [10]. For the associated generalized Bell numbers, many results have been derived in [16], generalizing those of the conventional case.

The main result of the present paper is to show that the exponential generating function of the generalized Bell numbers satisfies an algebraic differential equation for all $s \in \mathbb{R}$. In the particular case $s \in \mathbb{Z}$, one has a nice analogue to (1.3). The properties of the ordinary generating function are left to future studies.

2. Some properties of the generalized Stirling and Bell numbers

The generalized Stirling numbers are defined for $s \in \mathbb{R}$ and $h \in \mathbb{C} \setminus \{0\}$ by the recurrence relation

$$\mathfrak{S}_{s;h}(n+1,k) = \mathfrak{S}_{s;h}(n,k-1) + h\{k + s(n-k)\}\mathfrak{S}_{s;h}(n,k),$$

for all $n \geq 0$ and $k \geq 1$, with initial values $\mathfrak{S}_{s;h}(n,0) = \delta_{n,0}$ and $\mathfrak{S}_{s;h}(0,k) = \delta_{0,k}$ for all $n, k \in \mathbb{N}_0$. The associated Bell numbers are defined by $\mathfrak{B}_{s;h}(n) := \sum_{k=0}^{n} \mathfrak{S}_{s;h}(n,k)$. Clearly, the conventional case corresponds to the choice $s = 0$ and $h = 1$; further special values will be considered later. If we denote the generalized Stirling numbers of Hsu and Shiue by $S(n,k;\alpha,\beta,r)$ [10], then one has, due to the recursion relation, the identification $\mathfrak{S}_{s;h}(n,k) = S(n,k;-hs,h(1-s),0)$. Conversely, if $r = 0$ and $\alpha \neq \beta$, then [15]

$$S(n,k;\alpha,\beta,0) = \mathfrak{S}_{\frac{\alpha}{\alpha-\beta};\beta-\alpha}(n,k). \tag{2.1}$$

The exponential generating function of the generalized Bell numbers is defined by

$$\widehat{\mathfrak{B}}_{s;h}(x) := \sum_{n=0}^{\infty} \mathfrak{B}_{s;h}(n)\frac{x^n}{n!}.$$

An important role is played by the exponential generating function of the generalized Stirling numbers with $k = 1$, i.e., of $\mathfrak{S}_{s;h}(n,1)$, defined by

$$\mathfrak{S}\mathfrak{e}_{s;h}(x) := \sum_{n\geq 1} \mathfrak{S}_{s;h}(n,1)\frac{x^n}{n!}.$$

Proposition 2.1 ([16], Proposition 3.6)**.** *Fix $h \neq 0$. The function $\mathfrak{Se}_{s;h}$ satisfies for $s \in \mathbb{R} \setminus \{0,1\}$ the differential equation $\mathfrak{Se}'_{s;h}(x) = (1-hsx)^{-\frac{1}{s}}$. Consequently, it is given explicitly by*

$$\mathfrak{Se}_{s;h}(x) = \frac{1}{h(s-1)} \left\{ 1 - (1-hsx)^{\frac{s-1}{s}} \right\}.$$

In the case $s=0$, it is given by

$$\mathfrak{Se}_{0;h}(x) = \frac{1}{h}(e^{hx}-1).$$

In the case $s=1$, it is given by

$$\mathfrak{Se}_{1;h}(x) = \log\left(\frac{1}{(1-hx)^{1/h}}\right).$$

We need to recall one more result before we can turn to the differential equation. This result is the generalization of (1.2).

Theorem 2.2 ([16], Theorem 4.6)**.** *Fix $h \neq 0$ and let $s \in \mathbb{R}$. The exponential generating function $\widehat{\mathfrak{B}}_{s;h}$ satisfies the differential equation*

$$\widehat{\mathfrak{B}}'_{s;h}(x) = \mathfrak{Se}'_{s;h}(x)\widehat{\mathfrak{B}}_{s;h}(x).$$

Remark 2.3. *The explicit form of $\mathfrak{Se}'_{s;h}$ resulting from Proposition 2.1 shows that only in the conventional case $s=0$ - where one has $\widehat{\mathfrak{B}}'_{0;h}(x) = e^{hx}\widehat{\mathfrak{B}}_{0;h}(x)$ - the sequence of $\mathfrak{B}_{s;h}(n)$ is a variant sequence in the sense of* [9].

Before we turn to the derivation of the analogue of (1.3), we want to point out that it is possible to derive from Theorem 2.2 a much simpler algebraic differential equation than the one given in the next section. In the following proposition we restrict to the case $s>0$, but the case $s<0$ is very similar.

Proposition 2.4. *Fix $h \neq 0$. Let $s \in \mathbb{Q}$ be given as $s = \frac{p}{q}$ with $\gcd(p,q)=1$ and $p,q>0$. Then $\widehat{\mathfrak{B}}_{\frac{p}{q};h}$ satisfies the algebraic differential equation*

$$\left(1 - \frac{hpx}{q}\right)^q (\widehat{\mathfrak{B}}'_{\frac{p}{q};h}(x))^p - (\widehat{\mathfrak{B}}_{\frac{p}{q};h}(x))^p = 0. \tag{2.2}$$

Proof. Observe that one has from Proposition 2.1 that $\mathfrak{Se}'_{s;h}(x) = (1-hsx)^{-\frac{1}{s}}$ for all $s \in \mathbb{R}$ (and $h \neq 0$). Combining this with Theorem 2.2, one finds $(1-hsx)^{\frac{1}{s}}\widehat{\mathfrak{B}}'_{s;h}(x) = \widehat{\mathfrak{B}}_{s;h}(x)$. If s is rational, i.e., $s = \frac{p}{q} \in \mathbb{Q}$, then one has

$$\left(1 - \frac{hpx}{q}\right)^{\frac{q}{p}} \widehat{\mathfrak{B}}'_{\frac{p}{q};h}(x) = \widehat{\mathfrak{B}}_{\frac{p}{q};h}(x).$$

Taking this to the p-th power yields the assertion. □

If $q=1$, i.e., $s \in \mathbb{Z}$, then the above relation simplifies. However, as mentioned above, (2.2) is not the analogue of (1.3) we are looking for.

3. The differential equation of second order for $\widehat{\mathfrak{B}}_{s;h}$ with rational s

After having recollected the necessary results from [16], we can turn to the derivation of the differential equation. Before we do this, let us point out that the main ingredient in the derivation of the differential equation in the conventional case was the fact that $(e^x)' = e^x$ which allowed one to connect $(\widehat{B}'/\widehat{B})'$ and $\widehat{B}'/\widehat{B}$. In the general case one has $\mathfrak{Se}'_{s;h}(x)$ instead of e^x (see Theorem 2.2), so the first step will be to establish a connection between $\mathfrak{Se}''_{s;h}$ and $\mathfrak{Se}'_{s;h}$. Note that Theorem 2.2 shows that one has for all $s \in \mathbb{R}$,

$$\frac{\widehat{\mathfrak{B}}'_{s;h}(x)}{\widehat{\mathfrak{B}}_{s;h}(x)} = \mathfrak{Se}'_{s;h}(x), \tag{3.1}$$

and, consequently,

$$\left(\frac{\widehat{\mathfrak{B}}'_{s;h}(x)}{\widehat{\mathfrak{B}}_{s;h}(x)}\right)' = \mathfrak{Se}''_{s;h}(x). \tag{3.2}$$

Thus, we can use the same argument as in the conventional case provided we can express $\mathfrak{Se}''_{s;h}$ as a function of $\mathfrak{Se}'_{s;h}$.

Lemma 3.1. *Fix $h \neq 0$. The function $\mathfrak{Se}_{s;h}$ satisfies for all $s \in \mathbb{R}$ the differential equation*

$$\mathfrak{Se}''_{s;h}(x) = h(\mathfrak{Se}'_{s;h}(x))^{s+1}. \tag{3.3}$$

Proof. For this we consider the three cases $s \in \mathbb{R} \setminus \{0, 1\}$, $s = 0$ and $s = 1$ subsequently. Let us turn to the first case $s \in \mathbb{R} \setminus \{0, 1\}$. Here we can directly start from $\mathfrak{Se}'_{s;h}(x) = (1 - hsx)^{-\frac{1}{s}}$ given in Proposition 2.1 to obtain

$$\begin{aligned} \mathfrak{Se}''_{s;h}(x) &= h(1 - hsx)^{-\frac{1}{s}-1} \\ &= h\left((1 - hsx)^{-\frac{1}{s}}\right)^{s+1} \\ &= h(\mathfrak{Se}'_{s;h}(x))^{s+1}, \end{aligned}$$

as requested. Let us turn to the second case $s = 0$. Here we start from $\mathfrak{Se}_{0;h}(x) = \frac{1}{h}(e^{hx} - 1)$ given in Proposition 2.1 to obtain $\mathfrak{Se}'_{0;h}(x) = e^{hx}$ and, consequently, $\mathfrak{Se}''_{0;h}(x) = he^{hx}$. It follows that

$$\mathfrak{Se}''_{0;h}(x) = h\mathfrak{Se}'_{0;h}(x),$$

as claimed. Finally, we consider the remaining case $s = 1$. Here we start from

$$\mathfrak{Se}_{1;h}(x) = \log\left(\frac{1}{(1 - hx)^{1/h}}\right)$$

given in Proposition 2.1 and find $\mathfrak{Se}'_{1;h}(x) = (1 - hx)^{-1}$, hence $\mathfrak{Se}''_{1;h}(x) = h(1 - hx)^{-2}$. It follows that

$$\mathfrak{Se}''_{1;h}(x) = h(\mathfrak{Se}'_{1;h}(x))^2$$

and the assertion is now proven for all cases. □

Lemma 3.1 provides the main input for the following proposition.

Proposition 3.2. *Fix $h \neq 0$. The exponential generating function $\widehat{\mathfrak{B}}_{s;h}$ of the generalized Bell numbers satisfies for all $s \in \mathbb{R}$ the differential equation*

$$\widehat{\mathfrak{B}}''_{s;h}\widehat{\mathfrak{B}}_{s;h} - (\widehat{\mathfrak{B}}'_{s;h})^2 - h(\widehat{\mathfrak{B}}'_{s;h})^{s+1}(\widehat{\mathfrak{B}}_{s;h})^{1-s} = 0. \tag{3.4}$$

Proof. It remains to collect the above results. By combining (3.2) with Lemma 3.1, we obtain

$$\frac{\widehat{\mathfrak{B}}''_{s;h}\widehat{\mathfrak{B}}_{s;h} - (\widehat{\mathfrak{B}}'_{s;h})^2}{\widehat{\mathfrak{B}}^2_{s;h}} = h(\mathfrak{S}\mathfrak{e}'_{s;h})^{s+1}.$$

Using (3.1), this can be written as

$$\widehat{\mathfrak{B}}''_{s;h}\widehat{\mathfrak{B}}_{s;h} - (\widehat{\mathfrak{B}}'_{s;h})^2 = h\left(\frac{\widehat{\mathfrak{B}}'_{s;h}}{\widehat{\mathfrak{B}}_{s;h}}\right)^{s+1}\widehat{\mathfrak{B}}^2_{s;h},$$

yielding the assertion. □

The differential equation (3.4) is the analogue of (1.3) and reduces to it for $s = 0$ and $h = 1$. However, there appear fractions in (3.4) if $s \neq 0$. Since we want to obtain an algebraic differential equation for $\widehat{\mathfrak{B}}_{s;h}$, we consider the different cases explicitly.

Theorem 3.3. *Fix $h \neq 0$. If $s \in \mathbb{N}$, then $\widehat{\mathfrak{B}}_{s;h}$ satisfies the algebraic differential equation*

$$\widehat{\mathfrak{B}}''_{s;h}(\widehat{\mathfrak{B}}_{s;h})^s - (\widehat{\mathfrak{B}}'_{s;h})^2(\widehat{\mathfrak{B}}_{s;h})^{s-1} - h(\widehat{\mathfrak{B}}'_{s;h})^{s+1} = 0. \tag{3.5}$$

If $s = 0$, then the function $\widehat{\mathfrak{B}}_{0;h}$ satisfies the algebraic differential equation

$$\widehat{\mathfrak{B}}''_{0;h}\widehat{\mathfrak{B}}_{0;h} - (\widehat{\mathfrak{B}}'_{0;h})^2 - h\widehat{\mathfrak{B}}'_{0;h}\widehat{\mathfrak{B}}_{0;h} = 0. \tag{3.6}$$

If $s \in \mathbb{Z} \setminus \mathbb{N}_0$, then $\widehat{\mathfrak{B}}_{s;h}$ satisfies the algebraic differential equation

$$\widehat{\mathfrak{B}}''_{s;h}(\widehat{\mathfrak{B}}'_{s;h})^{-s-1}\widehat{\mathfrak{B}}_{s;h} - (\widehat{\mathfrak{B}}'_{s;h})^{1-s} - h(\widehat{\mathfrak{B}}_{s;h})^{1-s} = 0. \tag{3.7}$$

Proof. The case $s = 0$ is clear. If $s > 0$, then we have to multiply (3.4) with $(\widehat{\mathfrak{B}}_{s;h})^{s-1}$, while for $s < 0$ we have to multiply (3.4) with $(\widehat{\mathfrak{B}}'_{s;h})^{-s-1}$. □

It is interesting to note that the conventional case $s = 0$ and $h = 1$ is a particular case of the above theorem which separates two different families of cases. In the following examples, we briefly consider several other choices of the parameters s, h.

Example 3.4. *Let $s = 1$ and $h = 1$. The generalized Stirling numbers are given by the unsigned Stirling numbers of first kind, i.e., $\mathfrak{S}_{1;1}(n,k) = (-1)^{n-k}s(n,k)$ [16]. According to Theorem 3.3, the exponential generating function of the corresponding generalized Bell numbers satisfies*

$$\widehat{\mathfrak{B}}''_{1;1}\widehat{\mathfrak{B}}_{1;1} - 2(\widehat{\mathfrak{B}}'_{1;1})^2 = 0.$$

It was shown in [16] that $\widehat{\mathfrak{B}}_{1;1}(x) = \frac{1}{1-x}$ and one can easily check that it satisfies the differential equation.

Example 3.5. *Let $s = 2$ and $h = 1$. This case corresponds to the meromorphic Weyl algebra* [6, 7] *and was discussed in* [16, 17]. *From Theorem 3.3 we obtain in this case the algebraic differential equation*

$$\widehat{\mathfrak{B}}''_{2;1}(\widehat{\mathfrak{B}}_{2;1})^2 - (\widehat{\mathfrak{B}}'_{2;1})^2\widehat{\mathfrak{B}}_{2;1} - (\widehat{\mathfrak{B}}'_{2;1})^3 = 0.$$

It was shown in [16] *that $\widehat{\mathfrak{B}}_{2;1}(x) = e^{1-\sqrt{1-2x}}$. Note that $1 - \sqrt{1-2x} = \sum_{n\geq 1} a(n)\frac{x^n}{n!}$, where $a(n) = 1 \cdot 3 \cdot 5 \cdots (2n-3)$ is the number of complete binary trees having n labelled endpoints (and where $a(0) = 0$ has been set), see* [22, Example 5.2.6].

Example 3.6. *Let $s = -1$ and $h = 1$. Theorem 3.3 yields in this case*

$$\widehat{\mathfrak{B}}''_{-1;1}\widehat{\mathfrak{B}}_{-1;1} - (\widehat{\mathfrak{B}}'_{-1;1})^2 - (\widehat{\mathfrak{B}}_{-1;1})^2 = 0.$$

It was shown in [16] *that $\widehat{\mathfrak{B}}_{-1;1}(x) = e^{x+\frac{1}{2}x^2}$ - which is a variant of the exponential generating function of Hermite polynomials - and that $\mathfrak{B}_{-1;1}(n)$ is the number of involutions of $\{1, 2, \ldots, n\}$, see* [22].

Note that in Theorem 3.3 (and the above examples) we have restricted the parameter s to the integers, though it would be interesting to consider non-integer values of s as well. For example, it was discussed in [16] that by choosing $s = \frac{1}{2}$ and $h = 2$ one obtains

$$\mathfrak{S}_{\frac{1}{2};2}(n, k) = L(n, k),$$

where we have denoted by $L(n, k)$ the (unsigned) Lah numbers (sequence A008297 in [20]). Defining $\mathfrak{L}(n) := \sum_{k=0}^{n} L(n, k)$, one has $\mathfrak{B}_{\frac{1}{2};2}(n) = \mathfrak{L}(n)$ and for the exponential generating function

$$\widehat{\mathfrak{B}}_{\frac{1}{2};2}(x) = \sum_{n=0}^{\infty} \mathfrak{L}(n)\frac{x^n}{n!} \equiv \widehat{\mathfrak{L}}(x).$$

Thus, at least for the case $s = \frac{1}{2}$ it seems interesting to find an algebraic differential equation too. The way to do this is in principle straightforward. Recall that the relation given in Lemma 3.1 holds for all $s \in \mathbb{R}$. This relation is the key for connecting (3.1) and (3.2). For rational s, we can use the same relation to connect appropriate *powers* of (3.1) and (3.2). Thus, let $s \in \mathbb{Q}$ and write it as $s = \frac{p}{q}$ (no common factors in p and q). Lemma 3.1 yields $\mathfrak{Se}''_{\frac{p}{q};h}(x) = h(\mathfrak{Se}'_{\frac{p}{q};h}(x))^{\frac{p}{q}+1}$. Taking this to the power q gives

$$(\mathfrak{Se}''_{\frac{p}{q};h}(x))^q = h^q(\mathfrak{Se}'_{\frac{p}{q};h}(x))^{p+q},$$

which is now the connecting relation to be used. Thus,

$$\begin{aligned} \left\{ \left(\frac{\widehat{\mathfrak{B}}'_{\frac{p}{q};h}(x)}{\widehat{\mathfrak{B}}_{\frac{p}{q};h}(x)} \right)' \right\}^q &= (\mathfrak{Se}''_{\frac{p}{q};h}(x))^q \\ &= h^q(\mathfrak{Se}'_{\frac{p}{q};h}(x))^{p+q} \\ &= h^q \left(\frac{\widehat{\mathfrak{B}}'_{\frac{p}{q};h}(x)}{\widehat{\mathfrak{B}}_{\frac{p}{q};h}(x)} \right)^{p+q}, \end{aligned}$$

where we have used in the first line (3.2) and in the last line (3.1). Simplifying this expression yields the following proposition.

Proposition 3.7. *Fix $h \neq 0$. Let $s \in \mathbb{Q}$ be given as $s = \frac{p}{q}$ with $\gcd(p,q) = 1$. Then $\widehat{\mathfrak{B}}_{\frac{p}{q};h}$ satisfies the differential equation*

$$\left(\widehat{\mathfrak{B}}''_{\frac{p}{q};h}\widehat{\mathfrak{B}}_{\frac{p}{q};h} - (\widehat{\mathfrak{B}}'_{\frac{p}{q};h})^2\right)^q - h^q(\widehat{\mathfrak{B}}'_{\frac{p}{q};h})^{q+p}(\widehat{\mathfrak{B}}_{\frac{p}{q};h})^{q-p} = 0. \tag{3.8}$$

Clearly, if $q = 1$, then (3.8) gives back (3.4). To obtain an algebraic differential equation, one has to consider the different cases of combinations of (p, q) for the exponents $q + p$ and $q - p$ explicitly.

Theorem 3.8. *Fix $h \neq 0$. Let $s \in \mathbb{Q}$ be given as $s = \frac{p}{q}$ with $\gcd(p,q) = 1$ and $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. If $p < -q$, then $q + p < 0$ and $q - p > 0$ and $\widehat{\mathfrak{B}}_{\frac{p}{q};h}$ satisfies the algebraic differential equation*

$$\left(\widehat{\mathfrak{B}}''_{\frac{p}{q};h}\widehat{\mathfrak{B}}_{\frac{p}{q};h} - (\widehat{\mathfrak{B}}'_{\frac{p}{q};h})^2\right)^q (\widehat{\mathfrak{B}}'_{\frac{p}{q};h})^{-(q+p)} - h^q(\widehat{\mathfrak{B}}_{\frac{p}{q};h})^{q-p} = 0. \tag{3.9}$$

If $-q < p < q$, then $q + p > 0$ and $q - p > 0$ and $\widehat{\mathfrak{B}}_{\frac{p}{q};h}$ satisfies the algebraic differential equation

$$\left(\widehat{\mathfrak{B}}''_{\frac{p}{q};h}\widehat{\mathfrak{B}}_{\frac{p}{q};h} - (\widehat{\mathfrak{B}}'_{\frac{p}{q};h})^2\right)^q - h^q(\widehat{\mathfrak{B}}'_{\frac{p}{q};h})^{q+p}(\widehat{\mathfrak{B}}_{\frac{p}{q};h})^{q-p} = 0. \tag{3.10}$$

If $q < p$, then $q + p > 0$ and $q - p < 0$ and $\widehat{\mathfrak{B}}_{\frac{p}{q};h}$ satisfies the algebraic differential equation

$$\left(\widehat{\mathfrak{B}}''_{\frac{p}{q};h}\widehat{\mathfrak{B}}_{\frac{p}{q};h} - (\widehat{\mathfrak{B}}'_{\frac{p}{q};h})^2\right)^q (\widehat{\mathfrak{B}}_{\frac{p}{q};h})^{p-q} - h^q(\widehat{\mathfrak{B}}'_{\frac{p}{q};h})^{q+p} = 0. \tag{3.11}$$

(The two "boundary cases" $p = -q$ and $p = q$ are in fact cases which were treated above for integers s.)

Proof. Consider (3.8) for the different cases of combinations of p and q and clear fractions. If $p = -q$, then $s = -1$ (hence $p = -1, q = 1$); similarly, if $p = q$, then $s = 1$ (hence $p = q = 1$). □

Example 3.9. *Consider the Lah numbers from above, i.e., the case $s = \frac{1}{2}$ and $h = 2$. It follows from* (3.10) *in Theorem 3.8 that the exponential generating function $\widehat{\mathfrak{L}}$ - which is given explicitly as $\widehat{\mathfrak{L}}(x) = \exp(\frac{x}{1-x})$, see sequence A008297 in* [20] *- satisfies the nontrivial algebraic differential equation*

$$\left(\widehat{\mathfrak{L}}''\widehat{\mathfrak{L}} - (\widehat{\mathfrak{L}}')^2\right)^2 - 4(\widehat{\mathfrak{L}}')^3\widehat{\mathfrak{L}} = 0.$$

Remark 3.10. *From* (3.8) *one sees that for rational $s = p/q$ large denominators q make the differential equation "ugly" since the "worst" term in the equation is given by $(\widehat{\mathfrak{B}}''_{\frac{p}{q};h}\widehat{\mathfrak{B}}_{\frac{p}{q};h})^q$. The numerator p does not have the same influence since it acts only on terms having at maximum one derivative.*

For irrational s, i.e., $s \in \mathbb{R} \setminus \mathbb{Q}$, the above method fails because one cannot use the key Lemma 3.1 to obtain an *algebraic* relation between $\mathfrak{Se}''_{s;h}$ and $\mathfrak{Se}'_{s;h}$. However, taking one more derivative will handle this case, see next section. On the other hand, one can use the same method to derive a differential equation for the variant sequences of Gould and Quaintance [9] mentioned in the Introduction.

Proposition 3.11. *Let f be the exponential function of a variant sequence, i.e., there exist $a, b \in \mathbb{R}$ such that $f'(x) = e^{bx}f(ax)$. Then f satisfies the algebraic functional differential equation*

$$f''(x)f(ax) - (af'(ax) + bf(ax))\,f'(x) = 0. \tag{3.12}$$

Proof. Differentiating the defining equation gives $(f'(x)/f(ax))' = (e^{bx})' = be^{bx} = b(f'(x)/f(ax))$. Simplifying the expression yields the claim. □

Choosing $a = 1 = b$ in (3.12) one obtains (1.3), as it should. Note that one can write (3.12) also as

$$\frac{f''(x)}{f'(x)} = b + a\frac{f'(ax)}{f(ax)}.$$

4. The differential equation of third order for $\widehat{\mathfrak{B}}_{s;h}$ with arbitrary s

In this section we derive an algebraic differential equation for $\widehat{\mathfrak{B}}_{s;h}$ with arbitrary s, in particular $s \in \mathbb{R} \setminus \mathbb{Q}$. The price we have to pay is that it is of third order. The key to this differential equation is the following simple observation.

Proposition 4.1. *If the function g satisfies an algebraic differential equation and one has*

$$f'(x) = g(x)f(x), \tag{4.1}$$

then f satisfies an algebraic differential equation, too.

Proof. Assume that g satisfies $P(x, g, g', g^{(2)}, \dots, g^{(n)}) = 0$ for some polynomial P. Using $g = f'/f$, hence $g^{(k)} = (f'/f)^{(k)}$, one obtains the equation

$$P\left(x, \frac{f'}{f}, \left(\frac{f'}{f}\right)', \left(\frac{f'}{f}\right)^{(2)}, \dots, \left(\frac{f'}{f}\right)^{(n)}\right) = 0.$$

Recall that one can write

$$\left(\frac{f'}{f}\right)^{(k)} = \frac{Q_k(f, f', \dots, f^{(k+1)})}{f^{2^k}}$$

where $Q_k(f, f', \dots, f^{(k+1)})$ is a polynomial in $f, f', \dots, f^{(k+1)}$ satisfying

$$Q_{k+1}(f, \dots, f^{(k+2)}) = f^{2^k} Q_k'(f, \dots, f^{(k+1)}) - 2^k f^{2^k-1} f' Q_k(f, \dots, f^{(k+1)})$$

with initial value $Q_0(f, f') = f'$. Thus, the above differential equation has the form

$$P\left(x, \frac{Q_0(f, f')}{f}, \frac{Q_1(f, f', f^{(2)})}{f^2}, \dots, \frac{Q_n(f, \dots, f^{(n+1)})}{f^{2^n}}\right) = 0,$$

where the $Q_k(f, \dots, f^{(k+1)})$ are polynomials in $f, \dots, f^{(k+1)}$. Clearing fractions, one obtains an algebraic differential equation for f. □

Combining Theorem 2.2 and Proposition 4.1, we can conclude that $\widehat{\mathfrak{B}}_{s;h}$ satisfies an algebraic differential equation provided $\mathfrak{Se}'_{s;h}$ satisfies an algebraic differential equation.

Lemma 4.2. *Fix $h \neq 0$. Then the function $\mathfrak{Se}'_{s;h}$ satisfies for all $s \in \mathbb{R}$ the algebraic differential equation*

$$\mathfrak{Se}'''_{s;h}\mathfrak{Se}'_{s;h} - (s+1)(\mathfrak{Se}''_{s;h})^2 = 0. \tag{4.2}$$

In the particular case $s = -1$ this equation holds true since $\mathfrak{Se}'''_{-1;h}(x) = 0$.

Proof. Differentiating (3.3) on both sides yields $\mathfrak{Se}'''_{s;h} = h(s+1)(\mathfrak{Se}'_{s;h})^s\mathfrak{Se}''_{s;h}$. From (3.3) we have $(\mathfrak{Se}'_{s;h})^s = (\mathfrak{Se}'_{s;h})^{s+1}/\mathfrak{Se}'_{s;h} = \mathfrak{Se}''_{s;h}/h\mathfrak{Se}'_{s;h}$. Combining these two equations shows the first claim. From Proposition 2.1 one has $\mathfrak{Se}_{-1;h}(x) = -\frac{1}{2h}\left\{1 - (1+hx)^2\right\}$, showing that $\mathfrak{Se}'''_{-1;h}(x) = 0$. □

Lemma 4.2 implies that $\widehat{\mathfrak{B}}_{s;h}$ satisfies for all $s \in \mathbb{R}$ an algebraic differential equation. Using (4.2), we can give this equation explicitly.

Theorem 4.3. *Fix $h \neq 0$. The exponential generating function $\widehat{\mathfrak{B}}_{s;h}$ satisfies for all $s \in \mathbb{R}$ the algebraic differential equation*

$$\widehat{\mathfrak{B}}'''_{s;h}\widehat{\mathfrak{B}}'_{s;h}(\widehat{\mathfrak{B}}_{s;h})^2 + (2s-1)\widehat{\mathfrak{B}}''_{s;h}(\widehat{\mathfrak{B}}'_{s;h})^2\widehat{\mathfrak{B}}_{s;h} - (s+1)(\widehat{\mathfrak{B}}''_{s;h})^2(\widehat{\mathfrak{B}}_{s;h})^2 + (1-s)(\widehat{\mathfrak{B}}'_{s;h})^4 = 0. \tag{4.3}$$

In the particular case $s = -1$, the differential equation simplifies to

$$\widehat{\mathfrak{B}}'''_{-1;h}(\widehat{\mathfrak{B}}_{-1;h})^2 - 3\widehat{\mathfrak{B}}''_{-1;h}\widehat{\mathfrak{B}}'_{-1;h}\widehat{\mathfrak{B}}_{-1;h} + 2(\widehat{\mathfrak{B}}'_{-1;h})^3 = 0.$$

Proof. Combining Theorem 2.2 with (4.2), we find

$$\left(\frac{\widehat{\mathfrak{B}}'_{s;h}}{\widehat{\mathfrak{B}}_{s;h}}\right)'' \frac{\widehat{\mathfrak{B}}'_{s;h}}{\widehat{\mathfrak{B}}_{s;h}} - (s+1)\left(\left(\frac{\widehat{\mathfrak{B}}'_{s;h}}{\widehat{\mathfrak{B}}_{s;h}}\right)'\right)^2 = 0,$$

which gives after some routine calculations the claimed equation. For $s = -1$ one can directly start from $\mathfrak{Se}'''_{-1;h}(x) = 0$ (see Lemma 4.2), or $(\widehat{\mathfrak{B}}'_{-1;h}/\widehat{\mathfrak{B}}_{-1;h}))'' = 0$. □

Example 4.4. *Let us consider again the Lah numbers, i.e., $s = 1/2$ and $h = 2$ (see Example 3.9). From* (4.3) *one obtains for $\widehat{\mathfrak{B}}_{\frac{1}{2};2} = \widehat{\mathfrak{L}}$ the algebraic differential equation*

$$2\widehat{\mathfrak{L}}'''\widehat{\mathfrak{L}}'(\widehat{\mathfrak{L}})^2 - 3(\widehat{\mathfrak{L}}'')^2(\widehat{\mathfrak{L}})^2 + (\widehat{\mathfrak{L}}')^4 = 0,$$

which is an alternative to the one given in Example 3.9.

Note that (4.3) reduces for $s = 0$ - corresponding to the conventional Bell numbers - to

$$\widehat{\mathfrak{B}}'''_{0;h}\widehat{\mathfrak{B}}'_{0;h}(\widehat{\mathfrak{B}}_{0;h})^2 - \widehat{\mathfrak{B}}''_{0;h}(\widehat{\mathfrak{B}}'_{0;h})^2\widehat{\mathfrak{B}}_{0;h} - (\widehat{\mathfrak{B}}''_{0;h})^2(\widehat{\mathfrak{B}}_{0;h})^2 + (\widehat{\mathfrak{B}}'_{0;h})^4 = 0.$$

On the other hand, we obtain by differentiating (1.3) and multiplying with $\widehat{B}'\widehat{B}$ the equation

$$\widehat{B}'''\widehat{B}'(\widehat{B})^2 - \widehat{B}''(\widehat{B}')^2\widehat{B} - \widehat{B}''\widehat{B}'(\widehat{B})^2 - \widehat{B}(\widehat{B}')^3 = 0.$$

Writing $\widehat{B}''\widehat{B}'(\widehat{B})^2 + \widehat{B}(\widehat{B}')^3 = (\widehat{B}''\widehat{B} + (\widehat{B}')^2)\widehat{B}'\widehat{B}$, we can use (1.3) and obtain $(\widehat{B}''\widehat{B} + (\widehat{B}')^2)(\widehat{B}''\widehat{B} - (\widehat{B}')^2) = (\widehat{B}'')^2(\widehat{B})^2 - (\widehat{B}')^4$. Inserting this into the differential equation gives

$$\widehat{B}'''\widehat{B}'(\widehat{B})^2 - \widehat{B}''(\widehat{B}')^2\widehat{B} - (\widehat{B}'')^2(\widehat{B})^2 + (\widehat{B}')^4 = 0,$$

which is exactly the differential equation for $\widehat{\mathfrak{B}}_{0;h}$ ($= \widehat{B}$ if $h = 1$) displayed above. Thus, in the case $s = 0$, the algebraic differential equation (4.3) of third order is in fact a consequence of the algebraic differential equation (1.3) of second order. The same is true for all $s \in \mathbb{Z}$.

Proposition 4.5. *Fix $h \neq 0$ and let $s \in \mathbb{Z}$. Then the algebraic differential equation* (4.3) *is a consequence of the algebraic differential equation given in Theorem 3.3.*

Proof. The case $s = 0$ was shown above. Consider $s \in \mathbb{N}$. Differentiating (3.5) given in Theorem 3.3 and multiplying the result with $\widehat{\mathfrak{B}}'_{s;h}$ yields

$$\widehat{\mathfrak{B}}'''_{s;h}\widehat{\mathfrak{B}}'_{s;h}(\widehat{\mathfrak{B}}_{s;h})^s + (s-2)\widehat{\mathfrak{B}}''_{s;h}(\widehat{\mathfrak{B}}'_{s;h})^2(\widehat{\mathfrak{B}}_{s;h})^{s-1} + (1-s)(\widehat{\mathfrak{B}}'_{s;h})^4(\widehat{\mathfrak{B}}_{s;h})^{s-2} - h(s+1)\widehat{\mathfrak{B}}''_{s;h}(\widehat{\mathfrak{B}}'_{s;h})^{s+1} = 0.$$

From (3.5) we have that $h(\widehat{\mathfrak{B}}'_{s;h})^{s+1} = \widehat{\mathfrak{B}}''_{s;h}(\widehat{\mathfrak{B}}_{s;h})^s - (\widehat{\mathfrak{B}}'_{s;h})^2(\widehat{\mathfrak{B}}_{s;h})^{s-1}$. Inserting this, simplifying the expression and noting that the common factor $(\widehat{\mathfrak{B}}_{s;h})^{s-2}$ never vanishes, one arrives at (4.3). The case for s a negative integer is very similar (and the roles of $\widehat{\mathfrak{B}}_{s;h}$ and $\widehat{\mathfrak{B}}'_{s;h}$ are exchanged): This time one has to differentiate (3.7) given in Theorem 3.3 and multiply the result with $\widehat{\mathfrak{B}}_{s;h}$. Using then (3.7) for $h(\widehat{\mathfrak{B}}_{s;h})^{s-1}$ and simplifying the expression, one may then observe that the common factor $(\widehat{\mathfrak{B}}'_{s;h})^{-s-2}$ never vanishes to arrive at (4.3). □

Remark 4.6. *Recall that there is a close connection* (2.1) *between the generalized Stirling numbers* $\mathfrak{S}_{s;h}(n,k)$ *treated above and those of Hsu and Shiue. Clearly, if* $r = 0$ *(and* $\alpha \neq \beta$*), then we can identify both generalized Stirling numbers by appropriate choices of parameters, i.e.,* $S(n,k;\alpha,\beta,0) = \mathfrak{S}_{s(\alpha,\beta);h(\alpha,\beta)}(n,k)$ *with* $s(\alpha,\beta) = \frac{\alpha}{\alpha-\beta}$ *and* $h(\alpha,\beta) = \beta - \alpha$*. It follows that if* α *and* β *are rational, then* $s(\alpha,\beta)$ *is rational, too. Therefore, one can transfer the above discussion to* $S(n,k;\alpha,\beta,0)$*. Accordingly, the main step of generalization consists in allowing* $r \neq 0$ *for* $S(n,k;\alpha,\beta,r)$ *in the analogous considerations. If we denote the generalized Bell numbers of Hsu and Shiue by* $W_n := \sum_k S(n,k;\alpha,\beta,r)$*, then one has for* $\widehat{\mathfrak{W}}(x) := \sum_n W_n \frac{x^n}{n!}$ *the explicit expression* [10, Equation (15)]

$$\widehat{\mathfrak{W}}(x) = (1+\alpha x)^{\frac{r}{\alpha}} \exp\left[\frac{1}{\beta}\left((1+\alpha x)^{\frac{\beta}{\alpha}} - 1\right)\right]. \tag{4.4}$$

Following the strategy from above, we first determine $\widehat{\mathfrak{W}}'(x)/\widehat{\mathfrak{W}}(x)$*, yielding*

$$\frac{\widehat{\mathfrak{W}}'(x)}{\widehat{\mathfrak{W}}(x)} = \frac{r + (1+\alpha x)^{\frac{\beta}{\alpha}}}{(1+\alpha x)}. \tag{4.5}$$

This is the analogue of (3.1)*. Denoting the right-hand side of* (4.5) *by* $F(x)$*, one should try to obtain an algebraic differential relation for* F *in the form* $F^{(k)} = P(F, F', F^{(2)}, \ldots, F^{(k-1)})$ *for some* $k \in \mathbb{N}$*, generalizing* (3.3)*. Using* (4.5)*, this would give an algebraic relation of the form*

$$\left(\frac{\widehat{\mathfrak{W}}'(x)}{\widehat{\mathfrak{W}}(x)}\right)^{(k)} = P\left(\frac{\widehat{\mathfrak{W}}'(x)}{\widehat{\mathfrak{W}}(x)}, \left(\frac{\widehat{\mathfrak{W}}'(x)}{\widehat{\mathfrak{W}}(x)}\right)', \ldots, \left(\frac{\widehat{\mathfrak{W}}'(x)}{\widehat{\mathfrak{W}}(x)}\right)^{(k-1)}\right),$$

which could then be discussed further. However, taking a derivative of F *with respect to* x *gives*

$$F'(x) = \frac{\beta - \alpha}{(1+\alpha x)} F(x) - \frac{r\beta}{(1+\alpha x)^2},$$

and it seems that the influence of the nonvanishing r *makes it much more difficult to realize the sketched strategy for* $\widehat{\mathfrak{W}}(x)$ *than for* $\widehat{\mathfrak{B}}_{s;h}(x)$ *considered above.*

Remark 4.7. *There do exist several other generalizations of the classical Stirling numbers. For example, Flajolet and Prodinger introduced Stirling numbers for complex arguments* [8] *(see also* [11] *for more properties of these numbers). A different ansatz for Stirling numbers with complex arguments was proposed by Richmond and Merlini slightly earlier* [19]*. Carlitz* [3, 4] *and Toscano* [23, 24] *introduced some classical generalizations of the Stirling numbers which have been discussed more recently again, see, e.g.,* [1, 13, 18]*. For all these generalized Stirling numbers one may consider the associated Bell numbers and discuss the differential equation their exponential generating function satisfies.*

5. Conclusion

We have shown that the exponential generating function of the generalized Bell numbers $\widehat{\mathfrak{B}}_{s;h}(n)$ satisfies for all $s \in \mathbb{R}$ an algebraic differential equation of third order (Theorem 4.3). In the case $s \in \mathbb{Z}$, one even has an algebraic differential equation of second order (Theorem 3.3), which is the natural analogue to the differential equation of the conventional Bell numbers, and it can be shown that the equation of third order is a consequence of the second order equation (Proposition 4.5). For $s = p/q \in \mathbb{Q}$, one can also find an algebraic differential equation of second order (Theorem 3.8), but this may involve large powers of $\widehat{\mathfrak{B}}''_{s;h}$ if the denominator q is large. Thus, from a more practical point of view, it depends whether the differential equation of second order is more useful than the one of third order.

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