REMARKS ON GORENSTEIN WEAK INJECTIVE AND WEAK FLAT MODULES

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Abstract. In this paper, we introduce the notions of Gorenstein weak injective and weak flat modules respectively in terms of weak injective and weak flat modules, which is larger than classical classes of Gorenstein injective and flat modules. In this new setting, we characterize rings over which all modules are Gorenstein weak injective. Moreover, we also discuss a relation between weak cosyzygy and Gorenstein weak cosyzygy of a module, and the stability of Gorenstein weak injective modules.

1. Introduction

Throughout R is an associative ring with identity and all modules are unitary. Unless stated otherwise, an R-module will be understood to be a left R-module. Given an R-module M, we denote by $pd_R(M)$ and $fd_R(M)$ the projective and flat dimensions respectively, and by the character module $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. For unexplained concepts and notations, we refer the readers to [7, 18].

In 1970, Stenström [19] introduced the notion of FP-injective modules, and generalized the homological properties from Noetherian rings to coherent rings, and in this process, finitely generated modules were replaced by finitely presented modules. Recently, as extending work of Stenström's viewpoint, Gao and Wang [13] introduced the notion of weak injective modules. This class of modules was also investigated by Bravo, Gillespie, and Hovey [3] independently. In this process, finitely presented modules were replaced by super finitely presented modules (see [12] or Section 2 for the definition). The fact shows that weak injective modules play a crucial role in the process of generalizing homological properties from special rings to arbitrary rings (see [10, 13, 21, 22, 23] for more details).

In 1965, Eilenberg and Moore first introduced the theory of relative homological algebra in [8]. Since then the relative homological algebra, especially the Gorenstein homological algebra, got a rapid development. Nowadays, it has been developed to an advanced level (e.g. [1, 4, 5, 7, 14, 15, 16, 20]). However, in most results of Gorenstein homological algebra, the condition 'noetherian' is essential. In order to make similar properties of Gorenstein homological algebra hold over general rings, one of methods is to define a class of modules using an exact sequence of injective modules which is also exact after applying the covariant Hom functor with respect to weak injective modules. But these modules are in fact stronger than the Gorenstein injective modules (see [7, Def. 10.1.1] for the definition), and weak injective modules are not contained in these modules in general. So our attempt is to find a new class of modules satisfying the following properties at the same time:

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- this new class of modules are weaker than that of Gorenstein injective modules, and
- every weak injective module is Gorenstein weak injective.

To solve this question, we investigate a class of Gorenstein weak injective modules inspired by [11], that is, an R-module M is called *Gorenstein weak injective* if there exists an exact sequence of weak injective R-modules

$$\cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

such that $M = \mathrm{Coker}(E_1 \to E_0)$ and the functor $\mathrm{Hom}_R(P, -)$ leaves this sequence exact whenever P is a super finitely presented R-module with $pd_R(P) < \infty$.

This paper is organised as follows. In Section 2, we first introduce the notions of Gorenstein weak injective and weak flat modules in terms of weak injective and weak flat modules respectively, and give some basic properties of them. Then we characterize rings over which all modules are Gorenstein weak injective and others. In Section 3, we discuss a relation between weak cosyzygy and Gorenstein weak cosyzygy of a module, and investigate the stability of Gorenstein weak injective modules.

2. Gorenstein weak injective and weak flat modules

In this section, we give the definitions of Gorenstein weak injective and weak flat modules and discuss some of properties of these modules. We first recall some terminologies and preliminary. For more details, we refer the readers to [7, 12, 13].

Definition 2.1. ([7, Def. 6.1.1]) Let $\mathcal{F}$ be a class of R-modules. By an *$\mathcal{F}$-preenvelope* of an R-module M, we mean a morphism $\varphi : M \to F$ where $F \in \mathcal{F}$ such that for any morphism $f : M \to F^{'}$ with $F^{'} \in \mathcal{F}$, there exists a morphism $g : F \to F^{'}$ such that $g\varphi = f$, that is, there is the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & F \\ & {}_{f}\searrow & \downarrow{\scriptstyle g} \\ & & F' \end{array}$$

If furthermore, when $F^{'} = F$ and $f = \varphi$, the only such g are automorphisms of F, then $\varphi : M \to F$ is called an *$\mathcal{F}$-envelope* of M.

Dually, one may give the notion of *$\mathcal{F}$-(pre)cover* of an R-module.

We note that $\mathcal{F}$-envelopes and $\mathcal{F}$-covers may not exist in general, but if they exist, they are unique up to isomorphism.

We also note that if the class $\mathcal{F}$ contains all injective R-modules, then $\mathcal{F}$-preenvelopes are monic, and if the class $\mathcal{F}$ contains all projective R-modules, then $\mathcal{F}$-precovers are epic.

In the process of generalizing homological results from special rings to arbitrary rings, the notion of super finitely presented modules plays a crucial role. Recall from [12] that an R-module M is called *super finitely presented* if there exists an exact sequence $\cdots \to F_1 \to F_0 \to M \to 0$, where each F_i is finitely generated and projective. Following this, Gao and Wang [13] gave the notions of weak injective and weak flat modules in terms of super finitely presented modules, which are generalizations of FP-injective and flat modules. Independently, these classes of modules were also considered by Bravo, Gillespie and Hovey [3], and in their paper, they used the notions of FP_∞-injective and level modules rather than weak injective and weak flat modules.

Definition 2.2. ([13, Def. 2.1]) An R-module M is called *weak injective* if $\mathrm{Ext}_R^1(N, M) = 0$ for any super finitely presented R-module N. A right R-module M is called *weak flat* if $\mathrm{Tor}_1^R(M, N) = 0$ for any super finitely presented R-module N.

The following implications are clear:

$$\begin{aligned} \text{injective } R\text{-modules} \Rightarrow &\ \text{FP-injective } R\text{-modules} \\ \Rightarrow &\ \text{weak injective } R\text{-modules.} \\ \text{flat right } R\text{-modules} \Rightarrow &\ \text{weak flat right } R\text{-modules.} \end{aligned}$$

Moreover, the class of FP-injective R-modules coincides with that of weak injective R-modules and the class of flat right R-modules coincides with that of weak flat right R-modules over a left coherent ring R by [13, Rem. 2.2], and the class of injective R-modules coincides with that of FP-injective R-modules over a left Noetherian ring R by [17, Thm. 3]. Consequently, the class of injective R-modules coincides with that of weak injective R-modules over a Noetherian ring R.

We denote by $\mathcal{WI}(R)$ and $\mathcal{WF}(R^{\mathrm{op}})$ the classes of weak injective and weak flat (right) R-modules respectively. By [10, Thm. 3.4], every R-module has a weak injective preenvelope. So for any R-module M, M has a right $\mathcal{WI}(R)$-resolution. Moreover, since every injective R-module is weak injective, every right $\mathcal{WI}(R)$-resolution of M is also exact, that is, there exists an exact sequence

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots ,$$

where each E^i is weak injective.

Moreover, following [10] or [9, Thm. 12], every right R-module has a weak flat (pre)cover. So every right R-module M has a left $\mathcal{WF}(R^{\mathrm{op}})$-resolution. Since every projective right R-module is weak flat, this resolution is also exact, that is, there is an exact sequence

$$\cdots \to W_2 \to W_1 \to W_0 \to M \to 0$$

with each W_i weak flat.

2.1. **Gorenstein weak injective modules.** In homological algebra, finitely generated modules are well-behaved over Noetherian rings, and finitely presented modules are well-behaved over coherent rings. One of main reasons is that it always provides a resolution of a finitely generated module by finitely generated projective modules. Thus in order to study homological properties in general rings, we first choose a class of modules that admit a resolution by finitely generated projective modules. We notice that a module N is super finitely presented with $pd_R(N) < \infty$ if and only if it has a finite resolution by finitely generated projective modules. Moreover, Gao and Wang [11] used finitely presented modules of finite projective dimension to define Gorenstein FP-injective modules, and showed that they are well-behaved over coherent rings. Thus using super finitely presented modules of finite projective dimension, we give the definition of Gorenstein weak injective modules as follows.

Definition 2.3. An R-module M is called *Gorenstein weak injective* if there exists an exact sequence of weak injective R-modules

$$\mathbb{W} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

such that $M = \mathrm{Coker}(W_1 \to W_0)$ and the functor $\mathrm{Hom}_R(N, -)$ leaves this sequence exact whenever N is a super finitely presented R-module with $pd_R(N) < \infty$.

We denote by $\mathcal{GWI}(R)$ the class of Gorenstein weak injective R-modules.

Remark 2.4.

(1) By definition, every weak injective R-module is Gorenstein weak injective.

(2) Since every FP-injective R-module is weak injective, every Gorenstein FP-injective R-module (in the sense of Gao's definition, see [11]) is Gorenstein weak injective. If R is a left coherent ring, then the class of Gorenstein weak injective R-modules coincides with the class of Gorenstein FP-injective R-modules. Moreover, we have the following implications by [11, Prop. 2.5]:

$$\begin{aligned}\text{Gorenstein injective } R\text{-modules} &\Rightarrow \text{ Gorenstein FP-injective } R\text{-modules}\\ &\Rightarrow \text{ Gorenstein weak injective } R\text{-modules.}\end{aligned}$$

If R is an n-Gorenstein ring (i.e. a left and right Noetherian ring with self-injective dimension at most n on both sides for some non-negative integer n), then these three kinds of R-modules coincide.

(3) The class of Gorenstein weak injective R-modules is closed under direct products by definition.

(4) If $\mathbb{W} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$ is an exact sequence of weak injective R-modules such that the functor $\mathrm{Hom}_R(N, -)$ leaves this sequence exact whenever N is a super finitely presented R-module with $pd_R(N) < \infty$, then by symmetry, all the images, the kernels and the cokernels of $\mathbb{W}$ are Gorenstein weak injective.

Proposition 2.5. *Let M be a Gorenstein weak injective R-module. Then we have* $\mathrm{Ext}^i_R(N, M) = 0$ *whenever N is a super finitely presented R-module with $pd_R(N) < \infty$ and $i \geq 1$.*

Proof. Since M is a Gorenstein weak injective R-module, by Definition 2.3, there exists an exact sequence $0 \to M \to W^0 \to M^1 \to 0$ with W^0 weak injective and M^1 Gorenstein weak injective, such that the functor $\mathrm{Hom}_R(N, -)$ leaves this sequence exact whenever N is a super finitely presented R-module with $pd_R(N) < \infty$. Moreover, consider the following exact sequence

$$0 \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, W^0) \to \mathrm{Hom}_R(N, M^1) \to \mathrm{Ext}^1_R(N, M) \to \mathrm{Ext}^1_R(N, W^0)\ .$$

Since W^0 is weak injective, we have $\mathrm{Ext}^1_R(N, W^0) = 0$, and hence $\mathrm{Ext}^1_R(N, M) = 0$. Moreover, since M^1 is Gorenstein weak injective, we also have $\mathrm{Ext}^1_R(N, M^1) = 0$. Consider the following exact sequence

$$0 = \mathrm{Ext}^1_R(N, W^0) \longrightarrow \mathrm{Ext}^1_R(N, M^1) \longrightarrow \mathrm{Ext}^2_R(N, M) \longrightarrow \mathrm{Ext}^2_R(N, W^0)\ .$$

Note that $\mathrm{Ext}^2_R(N, W^0) = 0$ by [13, Prop. 3.1], and hence $\mathrm{Ext}^2_R(N, M) \cong \mathrm{Ext}^1_R(N, M^1) = 0$. We repeat the argument by replacing M^1 with M to get a Gorenstein weak injective R-module M^2 and the isomorphisms $\mathrm{Ext}^3_R(N, M) \cong \mathrm{Ext}^2_R(N, M^1) \cong \mathrm{Ext}^1_R(N, M^2) = 0$. Continuing this process, we may obtain a Gorenstein weak injective R-module M^{i-1} and the isomorphisms $\mathrm{Ext}^i_R(N,M) \cong \mathrm{Ext}^{i-1}_R(N,M^1) \cong \cdots \cong \mathrm{Ext}^1_R(N,M^{i-1}) = 0$ for any $i \geq 1$. □

The following proposition shows that we may simplify the definition of Gorenstein weak injective R-modules.

Proposition 2.6. *The following are equivalent for an R-module M:*

(1) M is Gorenstein weak injective;

(2) There exists an exact sequence of weak injective R-modules

$$\mathbb{W} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

such that $M = \mathrm{Coker}(W_1 \to W_0)$;

(3) There exists an exact sequence $\cdots \to W_1 \to W_0 \to M \to 0$, where each W_i is weak injective;

(4) There exists an exact sequence $0 \to L \to W \to M \to 0$, where W is weak injective and L is Gorenstein weak injective.

Proof. (1) $\Rightarrow$ (4) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (2). Since every R-module has a weak injective preenvelope by [10, Thm. 3.4], we may easily get an exact sequence $0 \to M \to W^0 \to W^1 \to \cdots$, where each W^i is weak injective. Assembling this sequence with the sequence given in (3), we get the following exact sequence

$$\mathbb{W} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

such that $M = \mathrm{Coker}(W_1 \to W_0)$.

(2) $\Rightarrow$(1). By Definition 2.3, it suffices to show that the complex $\mathrm{Hom}_R(N,\mathbb{W})$ is exact whenever N is a super finitely presented R-module with $pd_R(N) < \infty$.

We use induction on $n = pd_R(N) < \infty$. The case $n = 0$ is trivial. Let $n \geq 1$, and assume that the result holds for the case $n-1$. Consider an exact sequence $0 \to K \to P_0 \to N \to 0$, where P_0 is finitely generated projective and K is super finitely presented. Then $pd_R(K) = n-1$. Since each term of $\mathbb{W}$ is weak injective, we may get the following exact sequence of complexes

$$0 \to \mathrm{Hom}_R(N,\mathbb{W}) \to \mathrm{Hom}_R(P_0,\mathbb{W}) \to \mathrm{Hom}_R(K,\mathbb{W}) \to 0.$$

Clearly, the complex $\mathrm{Hom}_R(P_0,\mathbb{W})$ is exact. Moreover, the complex $\mathrm{Hom}_R(K,\mathbb{W})$ is also exact by the induction hypothesis. So the complex $\mathrm{Hom}_R(N,\mathbb{W})$ is exact, and hence M is Gorenstein weak injective. □

Following this and [13, Prop. 2.6], we have

Corollary 2.7. *Let I be a directed set, and $\{M_i\}_{i\in I}$ a direct system of R-modules. If every M_i is Gorenstein weak injective, then the direct limit $\varinjlim M_i$ is Gorenstein weak injective.*

Proposition 2.8. *Given an exact sequence $0 \to L \to M \to N \to 0$ of R-modules.*

(1) If M is weak injective and L is Gorenstein weak injective, then N is Gorenstein weak injective.

(2) If L is weak injective and N is Gorenstein weak injective, then M is Gorenstein weak injective.

Proof. (1) follows from Proposition 2.6.

(2) Assume that N is Gorenstein weak injective. Then, by Proposition 2.6, there exists an exact sequence $0 \to K \to W \to N \to 0$, where K is Gorenstein weak injective and W is weak injective. Consider the following pull-back diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & K & = & K & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{2.1}$$

Since L and W are weak injective, it follows from the middle row in the diagram (2.1) that W' is weak injective. Moreover, by the middle column in the diagram (2.1), we have that M is Gorenstein weak injective. □

Definition 2.9. The *Gorenstein weak injective dimension* of an R-module M, denoted by $Gwid_R(M)$, is defined to be $\inf\{n \mid \text{there is an exact sequence } 0 \to M \to G^0 \to G^1 \to \cdots \to G^n \to 0$ with G^i Gorenstein weak injective for any $0 \leq i \leq n\}$. If no such n exists, set $Gwid_R(M) = \infty$.

2.2. **Gorenstein weak flat modules.** Now we give the definition of Gorenstein weak flat modules.

Definition 2.10. A right R-module M is called *Gorenstein weak flat* if there exists an exact sequence of weak flat right R-modules

$$\mathbb{W} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

such that $M = \mathrm{Coker}(W_1 \to W_0)$ and the functor $- \otimes_R N$ leaves this sequence exact whenever N is a super finitely presented R-module with $pd_R(N) < \infty$.

We denote by $\mathcal{GWF}(R)$ the class of Gorenstein weak flat R-modules.

Remark 2.11.

(1) By definition, every weak flat right R-module is Gorenstein weak flat.

(2) The class of Gorenstein weak flat right R-modules is closed under direct sums and direct products by definition and [13, Thm. 2.13].

(3) If $\mathbb{W} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$ is an exact sequence of weak flat right R-modules such that the functor $- \otimes_R N$ leaves this sequence exact whenever N is a super finitely presented R-module with $pd_R(N) < \infty$, then by symmetry, all the images, the kernels and the cokernels of $\mathbb{W}$ are Gorenstein weak flat.

As a similar argument to that of Proposition 2.6, we have

Proposition 2.12. *The following are equivalent for a right R-module M:*

(1) M is Gorenstein weak flat;
(2) There exists an exact sequence of weak flat right R-modules

$$\mathbb{W} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

such that $M = \mathrm{Coker}(W_1 \to W_0)$;
(3) There exists an exact sequence $0 \to M \to W^0 \to W^1 \to \cdots$, where each W^i is weak flat;
(4) There exists an exact sequence $0 \to M \to W \to N \to 0$, where W is weak flat and N is Gorenstein weak flat.

Recall from [7, Def. 10.3.1] that a right R-module M is called *Gorenstein flat* if there exists an exact sequence of flat right R-modules

$$\mathbb{F} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

such that $M = \mathrm{Coker}(F_1 \to F_0)$ and the functor $- \otimes_R I$ leaves this sequence exact whenever I is an injective R-module.

Recall that a ring R is called *n-FC* if it is left and right coherent and $FP\text{-}id_R({}_RR) \leq n$ and $FP\text{-}id_R(R_R) \leq n$, where the symbol $FP\text{-}id_R(-)$ denotes the FP-injective dimension of modules. The following proposition shows that the class of Gorenstein weak flat modules is larger than that of Gorenstein flat modules, and they have no difference over n-FC rings.

Proposition 2.13. *Every Gorenstein flat right R-module is Gorenstein weak flat. The converse holds if R is an n-FC ring.*

Proof. Let M be a Gorenstein flat right R-module. Then, by definition, there is an exact sequence of flat right R-modules

$$\mathbb{F} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $M = \mathrm{Coker}(F_1 \to F_0)$. Note that each flat right R-module is weak flat. It follows from Proposition 2.12 that M is Gorenstein weak flat.

Conversely, assume that M is a Gorenstein weak flat right R-module. Then by Definition 2.10, there is an exact sequence of weak flat right R-modules

$$\mathbb{W} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

such that $M = \mathrm{Coker}(W_1 \to W_0)$. Note that, over a left coherent ring, the class of weak flat right modules coincides with that of flat right modules. So $\mathbb{W}$ is also an exact sequence of flat right R-modules. We will argue that the complex $\mathbb{W} \otimes_R E$ is exact for any left R-module E with $FP\text{-}\mathrm{id}_R(E) < \infty$. Since R is an n-FC ring, then we have that $FP\text{-}id_R(E) < \infty$ if and only if $fd_R(E) \leq n$ by [6, Prop. 3.6]. So it suffices to show that $\mathbb{W} \otimes_R E$ is exact for any left R-module E with $fd_R(E) \leq n$. We use induction on $n \geq fd_R(E)$. The case $n = 0$ is trivial. Consider an exact sequence $0 \to L \to P \to E \to 0$ with P projective. Then we have the following exact sequence of complex

$$0 \longrightarrow \mathbb{W} \otimes_R L \longrightarrow \mathbb{W} \otimes_R P \longrightarrow \mathbb{W} \otimes_R E \longrightarrow 0$$

Note that since $fd_R(E) \leq n$, we have $fd_R(L) \leq n - 1$. By the hypothesis induction, $\mathbb{W} \otimes_R L$ is exact. Moreover, $\mathbb{W} \otimes_R P$ is exact, so $\mathbb{W} \otimes_R E$ is also exact. In particular, $\mathbb{W} \otimes_R I$ is exact for any injective R-module I. Therefore, M is Gorenstein flat. □

Following [13, Thm. 2.10 and Props. 2.11, 2.12], we can easily get

Proposition 2.14.

(1) If a right R-module M is Gorenstein weak flat, then M^+ is Gorenstein weak injective.
(2) If an R-module M is Gorenstein weak injective, then M^+ is Gorenstein weak flat.
(3) If an R-module M is Gorenstein weak injective, then M^{++} is Gorenstein weak injective.
(4) If a right R-module M is Gorenstein weak flat, then M^{++} is Gorenstein weak flat.

Definition 2.15. The *Gorenstein weak flat dimension* of an R-module M, denoted by $Gwfd_R(M)$, is defined to be $\inf\{n \mid$ there is an exact sequence $0 \to G^n \to \cdots \to G^1 \to G^0 \to M \to 0$ with G^i Gorenstein weak flat for any $0 \le i \le n\}$. If no such n exists, set $Gwfd_R(M) = \infty$.

2.3. **Rings over which every module is Gorenstein weak injective.** We now give a characterization for rings whose every module is Gorenstein weak injective and, meanwhile, every right module is Gorenstein weak flat as follows.

Proposition 2.16. *The following are equivalent:*

(1) Every R-module is Gorenstein weak injective;
(2) Every right R-module is Gorenstein weak flat;
(3) Every projective R-module is weak injective;
(4) Every flat R-module is weak injective;
(5) Every injective right R-module is weak flat;
(6) R is weak injective as an R-module.

Proof. (1) ⇒ (3). Let P be a projective R-module. Then P is Gorenstein weak injective by hypothesis. So there exists an exact sequence $0 \to K \to W \to P \to 0$, where W is weak injective. Since P is projective, this sequence is split, and hence P is weak injective as a direct summand of W by [13, Prop. 2.3].

(3) ⇒ (1). Let M be any R-module. If every projective R-module is weak injective, then by assembling a projective resolution of M with its weak injective resolution, we may get the following exact sequence of weak injective R-modules

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

such that $M = \mathrm{Ker}(W^0 \to W^1)$. Thus M is Gorenstein weak injective by Proposition 2.6.

(2) ⇒ (5). Let I be an injective right R-module. Then I is Gorenstein weak flat by hypothesis. So there exists an exact sequence $0 \to I \to W \to N \to 0$, where W is weak flat. Since I is injective, this sequence is split, and hence I is weak flat as a direct summand of W by [13, Prop. 2.3].

(5) ⇒ (2). Let M be any R-module. If every injective right R-module is weak flat, then by assembling an injective resolution of M with its weak flat resolution, we may get the following exact sequence of weak flat right R-modules

$$\cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

such that $M = \mathrm{Coker}(W_1 \to W_0)$. Thus M is Gorenstein weak flat by Proposition 2.12.

(4) ⇔ (5) ⇔ (6) The proof is similar to that of [13, Prop. 2.17]. (4) ⇒ (3) ⇒ (6) are trivial. □

Corollary 2.17. *Let R be a left Noetherian ring. Then the following are equivalent:*

(1) *R is quasi-Frobenius;*

(2) *Every R-module is Gorenstein weak injective;*

(3) *Every right R-module is Gorenstein weak flat.*

Proof. (1) $\Rightarrow$ (2). It is obvious, since every R-module is Gorenstein injective over quasi-Frobenius rings by [2, Prop. 2.6].

(2) $\Leftrightarrow$ (3) follow from Proposition 2.16.

(3) $\Rightarrow$ (1) follows from Proposition 2.16 and the fact that the injective R-modules coincide with the weak injective R-modules over a left Noetherian ring R. □

In [13], Gao and Wang defined the left super finitely presented dimension of a ring R to be $l.sp.gldim(R) = \sup\{pd_R M \mid M \text{ is a super finitely presented } R\text{-module}\}$. By [13, Thm. 3.8],

$$\begin{aligned} l.sp.gldim(R) &= \sup\{wid_R(M) \mid M \text{ is any } R\text{-module}\} \\ &= \sup\{wfd_R(M) \mid M \text{ is any right } R\text{-module}\}. \end{aligned}$$

Proposition 2.18. *If $l.sp.gldim(R) < \infty$, then we have*

(1) Every Gorenstein weak injective R-module is weak injective;

(2) Every Gorenstein weak flat right R-module is weak flat.

Proof. (1) Assume that $l.sp.gldim(R) = n < \infty$ and let M be a Gorenstein weak injective R-module. The case $n = 0$ is trivial. Let $n \geq 1$. Since M is Gorenstein weak injective, there is an exact sequence

$$\cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow M \longrightarrow 0$$

with each W_i weak injective. Let $K_n = \mathrm{Ker}(W_{n-1} \to W_{n-2})$. Then we get an exact sequence

$$0 \longrightarrow K_n \longrightarrow W_{n-1} \longrightarrow \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow M \longrightarrow 0 \ .$$

By hypothesis, $wid_R(K_n) \leq n$, and hence M is weak injective by [13, Prop. 3.3].

(2) The proof is similar to that of (1). □

The above proposition shows that the class of weak injective R-modules coincides with that of Gorenstein weak injective R-modules, and the class of weak flat right R-modules coincides with that of Gorenstein weak flat right R-modules over a ring R satisfying $l.sp.gldim(R) < \infty$.

The next proposition also gives a description of rings over which all Gorenstein weak injective R-modules are weak injective from the viewpoint of Gorenstein weak injective dimension of modules.

Proposition 2.19. *The following are equivalent:*

(1) Every Gorenstein weak injective R-module is weak injective;

(2) For any R-module M, $Gwid_R(M) = wid_R(M)$.

Proof. (1) $\Rightarrow$ (2). Let M be an R-module. Since every weak injective R-module is Gorenstein weak injective, it is obvious that $Gwid_R(M) \leq wid_R(M)$. Thus it suffices to show that $wid_R(M) \leq Gwid_R(M)$. Without loss of generality, we assume that $Gwid_R(M) = n < \infty$ for some non-negative integer n. Then, by Definition 2.9, there is an exact sequence $0 \to M \to G^0 \to G^1 \to \cdots \to G^n \to 0$ with G^i Gorenstein weak injective for any $0 \leq i \leq n$. Note that each G^i is weak injective by hypothesis. Thus, $wid_R(M) \leq n = Gwid_R(M)$ by [13, Prop. 3.3], as desired.

(2) $\Rightarrow$ (1) is trivial. □

Similarly, we have

Proposition 2.20. *The following are equivalent:*

(1) Every Gorenstein weak flat R-module is weak flat;

(2) For any R-module M, $Gwfd_R(M) = wfd_R(M)$.

3. The cosyzygy and stability of Gorenstein weak injective modules

Let $\mathcal{X}$ and $\mathcal{Y}$ be two classes of R-modules. We write $\mathcal{X} \perp \mathcal{Y}$ if $\mathrm{Ext}^1_R(X, Y) = 0$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Remark 3.1. (1) If $\mathcal{WI}(R) \perp \mathcal{GWI}(R)$, then $\mathcal{GWI}(R)$ is closed under extensions. Indeed, let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules with $L, N \in \mathcal{GWI}(R)$. By Proposition 2.6, there exist exact sequences

$$\cdots \longrightarrow E_1^{'} \xrightarrow{d_1^{'}} E_0^{'} \xrightarrow{d_0^{'}} L \longrightarrow 0 \;\text{ and }\; \cdots \longrightarrow E_1^{''} \xrightarrow{d_1^{''}} E_0^{''} \xrightarrow{d_0^{''}} N \longrightarrow 0$$

with all $E_i^{'}$, $E_i^{''}$ in $\mathcal{WI}(R)$ and all $\mathrm{Ker} d_i^{'}$, $\mathrm{Ker} d_i^{''}$ in $\mathcal{GWI}(R)$. Consider the following diagram

$$\begin{array}{ccccccccc} & & E_0^{'} & & & & E_0^{''} & & \\ & & \downarrow{\scriptstyle d_0^{'}} & & & & \downarrow{\scriptstyle d_0^{''}} & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0. \end{array}$$

Since $\mathrm{Ext}^1_R(E_0^{''}, L) = 0$, we get an epimorphism $\mathrm{Hom}_R(E_0^{''}, M) \twoheadrightarrow \mathrm{Hom}_R(E_0^{''}, N)$ and there exists $\alpha : E_0^{''} \to M$ such that $d_0^{''} = g\alpha$. Putting $E_0 := E_0^{'} \oplus E_0^{''}$ and $d_0 := (fd_0^{'} \ \alpha)$, then we obtain the following commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_0^{'} & \xrightarrow{\binom{1}{0}} & E_0 & \xrightarrow{(0\ 1)} & E_0^{''} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_0^{'}} & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0^{''}} & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array}$$

Repeating this process, we may get an exact sequence

$$\cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0$$

with all $E_i = E_i^{'} \oplus E_i^{''}$ weak injective. Thus $M \in \mathcal{GWI}(R)$ by Proposition 2.6.

(2) If R is a Gorenstein ring, then $\mathcal{GWI}(R)$ is closed under extensions. Indeed, in this case, $\mathcal{GWI}(R)$ coincides with the class of Gorenstein injective R-modules.

In the following, we always assume that the ground ring is a ring R over which the class of Gorenstein weak injective R-modules is closed under extensions.

Consider the following exact sequence

$$0 \longrightarrow M \xrightarrow{d^0} W^0 \xrightarrow{d^1} W^1 \longrightarrow \cdots,$$

where each W^i is weak injective. Let $V^i = \mathrm{Coker}d^{i-1}$ for any $i \geq 1$. Then we call V^i an ith *weak cosyzygy* of M.

Similarly, if each W^i is Gorenstein weak injective in the above sequence, then we call V^i an ith *Gorenstein weak cosyzygy* of M.

We will investigate a relation between weak cosyzygy and Gorenstein weak cosyzygy of an R-module as follows.

Since every weak injective R-module is Gorenstein weak injective, it is obvious that every ith weak cosyzygy of an R-module M is an ith Gorenstein weak cosyzygy of M. The following theorem shows that the converse holds in some cases.

Theorem 3.2. *Let n be a positive integer and V^n an nth Gorenstein weak cosyzygy of an R-module M. Then V^n is an nth weak cosyzygy of some R-module N, and there is an exact sequence $0 \to G \to N \to M \to 0$, where G is Gorenstein weak injective.*

Proof. We use induction on n. For the case $n = 1$, there is an exact sequence $0 \to M \to G^0 \to V^1 \to 0$ with G^0 Gorenstein weak injective. Moreover, there is an exact sequence $0 \to G \to W^0 \to G^0 \to 0$ with W^0 weak injective and G Gorenstein weak injective.

Consider the following pull-back diagram:

(3.1)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G & = & G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & W^0 & \longrightarrow & V^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & G^0 & \longrightarrow & V^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

It follows from the middle row in the diagram (3.1) that V^1 is a 1st weak cosyzygy of an R-module N. Moreover, we get the desired exact sequence $0 \to G \to N \to M \to 0$ from the second column in the diagram (3.1).

Now let $n \geq 2$ and suppose that the result holds for the case $n - 1$. Let V^n be an nth Gorenstein weak cosyzygy of M. Then we have the following exact sequence

$$0 \longrightarrow M \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots \longrightarrow G^{n-1} \longrightarrow V^n \longrightarrow 0 ,$$

where each G^i is Gorenstein weak injective. Since G^{n-1} is Gorenstein weak injective, there is an exact $0 \to G' \to W^{n-1} \to G^{n-1} \to 0$ with G' Gorenstein weak injective and W^{n-1} weak injective.

Consider the following pull-back diagrams:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G' & = & G' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & V' & \longrightarrow & W^{n-1} & \longrightarrow & V^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & V^{n-1} & \longrightarrow & G^{n-1} & \longrightarrow & V^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.2}$$

and

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V^{n-2} & \longrightarrow & G'' & \longrightarrow & V' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V^{n-2} & \longrightarrow & G^{n-2} & \longrightarrow & V^{n-1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.3}$$

where $V^{n-i} = \mathrm{Coker}(G^{n-i-2} \to G^{n-i-1})$ for $i = 1, 2$. For the exact sequence $0 \to G' \to G'' \to G^{n-2} \to 0$ in the diagram (3.3), since G' and G^{n-2} are Gorenstein weak injective, G'' is also Gorenstein weak injective. Moreover, by the middle row in the diagram (3.3), we have that V' is an $(n-1)$st Gorenstein weak cosyzygy of M. Thus V' is an $(n-1)$st weak cosyzygy of some R-module N by the induction hypothesis. In addition, by assembling the middle row in the diagram (3.2), we may get that V^n is an nth weak cosyzygy of N, as desired. $\square$

In the following, we will consider the stability of Gorenstein weak injective R-modules, which shows that an iteration of the procedure used to describe the class of Gorenstein weak injective modules yields exactly the class of Gorenstein weak injective modules.

We begin with the following question, which is inspired by [20] but different.

Question 3.3. Given an exact sequence of Gorenstein weak injective R-modules

$$\mathbb{G} = \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

such that $M = \mathrm{Coker}(G_1 \to G_0)$ and the functor $\mathrm{Hom}_R(N, -)$ leaves this sequence exact whenever N is a super finitely presented R-module with $pd_R(N) < \infty$, is M Gorenstein weak injective?

As a similar argument to the proof of (2) $\Rightarrow$ (1) in Proposition 2.6 and using Proposition 2.5, the above question is equivalent to the following

Question 3.4. Given an exact sequence of Gorenstein weak injective R-modules

$$\mathbb{G} = \ \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

such that $M = \mathrm{Coker}(G_1 \to G_0)$, is M Gorenstein weak injective?

We call an R-module M defined as above *two-degree Gorenstein weak injective*, and denote by $\mathcal{GWI}^2(R)$ the class of two-degree Gorenstein weak injective R-modules. It is obvious that there is a containment $\mathcal{GWI}(R) \subseteq \mathcal{GWI}^2(R)$.

In the following, we show that the answer to the above question is affirmative over our hypothesis that the class $\mathcal{GWI}(R)$ is closed under extensions.

Theorem 3.5. $\mathcal{GWI}(R) = \mathcal{GWI}^2(R)$.

Before giving the proof of Theorem 3.5, we need the following preliminaries.

Definition 3.6. An R-module M is called *strongly two-degree Gorenstein weak injective* if there exists an exact sequence

$$\cdots \longrightarrow G \xrightarrow{d} G \xrightarrow{d} G \xrightarrow{d} G \longrightarrow \cdots,$$

where G is Gorenstein weak injective, such that $M = \mathrm{Coker}d$ and the functor $\mathrm{Hom}_R(N, -)$ leaves this sequence exact whenever N is a super finitely presented R-module with $pd_R(N) < \infty$.

We denote by $\mathcal{SGWI}^2(R)$ the class of strongly two-degree Gorenstein weak injective R-modules. It is obvious that there is a containment $\mathcal{SGWI}^2(R) \subseteq \mathcal{GWI}^2(R)$. As a similar argument to the proof of Proposition 2.6, we have

Lemma 3.7. *Let M be an R-module. Then the following are equivalent:*

(1) M is strongly two-degree Gorenstein weak injective;

(2) There exists an exact sequence

$$\cdots \longrightarrow G \xrightarrow{d} G \xrightarrow{d} G \xrightarrow{d} G \longrightarrow \cdots,$$

where G is Gorenstein weak injective;

(3) There exists an exact sequence $0 \to M \to G \to M \to 0$, where G is Gorenstein weak injective.

Proposition 3.8. *Let M be an R-module. If M is two-degree Gorenstein weak injective, then it is a direct summand of some strongly two-degree Gorenstein weak injective R-module.*

Proof. Since M is two-degree Gorenstein weak injective, there exists an exact sequence of Gorenstein weak injective R-modules

$$\mathbb{G} = \ \cdots \longrightarrow G_1 \xrightarrow{d_1} G_0 \xrightarrow{d_0} G_{-1} \xrightarrow{d_{-1}} G_{-2} \longrightarrow \cdots$$

where $G_{-i} = G^{i-1}$ for each $i \geq 1$, such that $M = \mathrm{Im}d_0$. For all $m \in \mathbb{Z}$, we denote by $\Sigma^m\mathbb{G}$ the exact sequence obtained from $\mathbb{G}$ by increasing all indexes by m: $(\Sigma^m\mathbb{G})_i = G_{i-m}$ and $d_i^{\Sigma^m\mathbb{G}} = d_{i-m}$ for all $i \in \mathbb{Z}$. Then we get the following exact sequence

$$\bigoplus_{m\in\mathbb{Z}}(\Sigma^m\mathbb{G}) = \ \cdots \longrightarrow \bigoplus\nolimits_{i\in\mathbb{Z}} G_i \xrightarrow{d} \bigoplus\nolimits_{i\in\mathbb{Z}} G_i \xrightarrow{d} \bigoplus\nolimits_{i\in\mathbb{Z}} G_i \xrightarrow{d} \cdots,$$

where $d([g_i]) = [d_i(g_i)]$ for any $g_i \in G_i$. Then $d^2 = 0$ and $\bigoplus_{i\in\mathbb{Z}} G_i$ is Gorenstein weak injective by Remark 2.4(3). Let $H = \mathrm{Im}d$. Then $H \subseteq \mathrm{Ker}d$. Now we assume that $d([g_i]) = [d_i(g_i)] = 0$. It is clear that $d_i(g_i) = 0$, and so there exists $g_{i+1} \in G_{i+1}$ such that $d_{i+1}(g_{i+1}) = g_i$. Thus $d([g_{i+1}]) = [g_i]$, and hence $\mathrm{Ker}d \subseteq \mathrm{Im}d = H$. Therefore, we may get an exact sequence $0 \to H \to G \to H \to 0$, where $G := \bigoplus_{i\in\mathbb{Z}} G_i$. Thus H is strongly two-degree Gorenstein weak injective by Lemma 3.7.

Finally, since $H \cong \oplus_{i\in\mathbb{Z}}\mathrm{Im}d_i$, we get that M is a direct summand of H, as desired. □

Recall from [14, 1.1] that a class $\mathcal{C}$ of R-modules is called *injectively resolving* if all injective R-modules are contained in $\mathcal{C}$, and for any exact sequence $0 \to L \to M \to N \to 0$ with $L \in \mathcal{C}$, the conditions $M \in \mathcal{C}$ and $N \in \mathcal{C}$ are equivalent.

Proposition 3.9. *The class $\mathcal{GWI}(R)$ is injectively resolving.*

Proof. Clearly, every injective R-module is Gorenstein weak injective. Moreover, the class $\mathcal{GWI}(R)$ is closed under extensions, by our running hypothesis. So we will prove that for any exact sequence $0 \to L \to M \to N \to 0$, if L and M are Gorenstein weak injective, then so is N. Indeed, since M is Gorenstein weak injective, there exists an exact sequence $0 \to G \to W \to M \to 0$ such that W is weak injective and G is Gorenstein weak injective.

Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G & = & G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & N' & \to & W & \to & N & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & L & \to & M & \to & N & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.4}$$

Since L and G are Gorenstein weak injective, N' is also Gorenstein weak injective. Thus N is Gorenstein weak injective by Proposition 2.6. □

Corollary 3.10. *The class $\mathcal{GWI}(R)$ is closed under direct summands.*

Proof. It follows from [14, Prop. 1.4], Remark 2.4(3) and Proposition 3.9. □

Now we give the proof of our main theorem (Theorem 3.5).

Proof of Theorem 3.5. Since $\mathcal{GWI}(R) \subseteq \mathcal{GWI}^2(R)$, it suffices to show that $\mathcal{GWI}^2(R) \subseteq \mathcal{GWI}(R)$. Since every two-degree Gorenstein weak injective R-module is a direct summand of some strongly two-degree Gorenstein weak injective R-module, and the class of Gorenstein weak injective R-modules is closed under direct summands by Corollary 3.10, so we only need to prove that every strongly two-degree Gorenstein weak injective R-module is Gorenstein weak injective.

Let M be a strongly two-degree Gorenstein weak injective R-module. Then, by Lemma 3.7, there is an exact sequence $0 \to M \to G \to M \to 0$ with G Gorenstein weak injective. Moreover, there is an exact sequence $0 \to G_1 \to W \to G \to 0$ with W weak injective and G_1 Gorenstein weak injective.

Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_1 & = & G_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & W & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.5}$$

From the middle row in the diagram (3.5), we obtain an exact sequence $0 \to N \to W \to M \to 0$ with W weak injective. Thus, in order to show that M is Gorenstein weak injective, it suffices to prove that N is Gorenstein weak injective by Proposition 2.6.

Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G_1 & = & G_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G_2 & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.6}$$

Since G and G_1 are Gorenstein weak injective, G_2 is also Gorenstein weak injective by the middle column in the diagram (3.6). Hence there exists an exact sequence $0 \to G_3 \to W_0 \to G_2 \to 0$ such that W_0 is weak injective and G_3 is Gorenstein weak injective.

Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_3 & = & G_3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N_1 & \longrightarrow & W_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & G_2 & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.7}$$

From the middle row in the diagram (3.7), we obtain an exact sequence $0 \to N_1 \to W_0 \to N \to 0$ with W_0 weak injective. We repeat the argument by replacing N with N_1 to get N_2 and an exact sequence $0 \to N_2 \to W_1 \to N_1 \to 0$ with W_1 weak injective. Continuing this process, we may obtain an exact sequence $\cdots \to W_1 \to W_0 \to N \to 0$, where each W_i is weak injective, which shows that N is Gorenstein weak injective by Proposition 2.6. We have completed the proof.

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