A variational-type method of fundamental solutions for a Cauchy problem of Laplace's equation

T. Wei, Y.G. Chen, J.C. Liu

School of Mathematics and Statistics, Lanzhou University, Gansu 730000, PR China
School of Mathematics and Computational Science, China University of Petroleum (East China), Shandong 257061, PR China

Article history:
Received 31 October 2011
Received in revised form 7 March 2012
Accepted 12 March 2012
Available online 22 March 2012

Keywords:
Convergence analysis
Method of fundamental solutions
Cauchy problem for Laplace equation

Abstract
In this paper, we propose a new regularization method based on a finite-dimensional subspace generated from fundamental solutions for solving a Cauchy problem of Laplace's equation in a simply-connected bounded domain. Based on a global conditional stability for the Cauchy problem of Laplace's equation, the convergence analysis is given under a suitable choice for a regularization parameter and an a-priori bound assumption to the solution. Numerical experiments are provided to support the analysis and to show the effectiveness of the proposed method from both accuracy and stability.

1. Introduction
The Cauchy problem for Laplace's equation arises from many branches of science and engineering such as non-destructive testing [1], steady-state inverse heat conduction [2], and electro-cardiology [3]. Due to the ill-posedness of the problem, numerical computations become very difficult if there is no an a-priori information on the solution. Usually the regularization technique is required to obtain a stable approximate solution. During the last decades, various numerical methods have been proposed, such as quasi-reversibility methods [4–7], conjugate gradient method [8], Tikhonov regularization method [9] and finite difference methods [10,11,2], Lavrentiev's regularization [12,13], moment methods [14–16], energy regularization method [17].

One popular method to solve the Cauchy problem in an irregular domain is to transform the original problem into an abstract operator equation and then solve a minimization problem. This kind of method minimizes the defect between the experimental measurement and the calculated response of a direct system. In solving a minimization problem, various algorithms can be adopted such as the conjugate gradient method [8]. Such an approach could be expensive in computations since many iterations are required and at each iteration several forward problems must be solved.

In spite of great many papers on the Cauchy problem for elliptic equations, the computational challenge still arises. In particular, for those problems with a complicated boundary shape, the solution including a singular behavior in the vicinity of inaccessible boundary or with a high level noise in Cauchy data, new algorithms to meet high accuracy and more stability are still required.

Recently, the method of fundamental solutions (MFS) combined with various regularization methods have been proposed for solving the Cauchy problem of elliptic equations, see [18–21] for examples. A briefly review on the applications of the MFS to some inverse problems is issued in [22].
The MFS based on a collocation fitting on boundary (we call it a collocation MFS) is wildly used to solve various direct and inverse problems. It becomes increasingly popular because of its simplicity of implementation and easy treatments to complicated geometries in high-dimensional space. However there still exists a big lacking of convergence analysis. For a well-posed Dirichlet problem of Laplace’s equation on a general simply-connected bounded domain, the polynomial convergence rate was achieved in [23] and the exponential convergence rates were provided in [24] and [25]. For a special disk domain, some issues were given in [26]. On the Dirichlet problems in an annular domain by the MFS, the convergence analysis was given in [30]. In [31], a variational-type MFS was proposed to solve the mixed boundary problem of Laplace’s equation in a simply-connected bounded domain. And the Dirichlet problem on an annular shaped domain was also discussed in [32].

However as we know the convergence results on inverse problems are very few. For the Cauchy problem of Laplace’s equation, Ohe and Ohnaka in [33] gave firstly a convergence rate for a collocation MFS solution when using exact Cauchy data. By our knowledge, there is no any convergence analysis of the MFS solution for the Cauchy problem of Laplace’s equation in a general simply-connected domain or multiply-connected domain. Wei and Zhou [34] used a collocation MFS combined with a discrete Tikhonov regularization to obtain a stable numerical solution by using noisy Cauchy data. Recently, Wei and Zhou [34] used a collocation MFS combined with a discrete Tikhonov regularization to obtain a stable numerical solution by using noisy Cauchy data. By our knowledge, there is no any convergence analysis of the MFS solution for the Cauchy problem of Laplace’s equation in a general simply-connected domain or multiply-connected domain. Based on the idea in [31,32,35] and a global conditional stability in Alessandrini et al. [36], we propose a new variational-type MFS for solving the Cauchy problem of Laplace’s equation in an irregular simply-connected domain. The convergence analysis is a new issue by our judgement. We will present some numerical experiments for verifying our results.

The paper is organized as follows. In Section 2, the formulation of problem and a variational-type MFS are described. The convergence analysis will be displayed in Section 3. In Section 4, the numerical implementation is presented and several numerical examples are investigated. Finally, in Section 5, some concluding remarks are given.

2. The Cauchy problem and a variational-type method of fundamental solutions

Let \( \Omega \) be a simply-connected bounded domain in \( \mathbb{R}^2 \) with a smooth boundary and \( \Gamma \) be an open part of boundary \( \partial \Omega \) such that \( \Gamma \cup \gamma = \partial \Omega \) with \( \gamma \neq \emptyset \). The Cauchy problem for Laplace’s equation is

\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in} \ \Omega, \tag{2.1}
\]

\[
u = f \quad \text{on} \ \Gamma, \tag{2.2}
\]

\[
\frac{\partial u}{\partial n} = g \quad \text{on} \ \Gamma, \tag{2.3}
\]

where \( \gamma \) is the unit outward normal of \( \Gamma \) and \( f, g \) are the Dirichlet data and the Neumann data respectively. We assume \( f \in H^{\frac{1}{2}}(\Gamma) \) and \( g \in H^{2}(\Omega) \) such that the solution for the Cauchy problem is in \( H^p(\Omega) \) for \( p \geq 2 \). Usually for a practical problem, the exact Cauchy data cannot be obtained and we only know the measured data \( f^\dagger \in L^2(\Gamma) \) and \( g^\dagger \in L^2(\Gamma) \). We always assume

\[
\|f^\dagger - f\|_{0,\Gamma} \leq \delta, \quad \|g^\dagger - g\|_{0,\Gamma} \leq \delta, \tag{2.4}
\]

where \( \delta > 0 \) indicates a level of noise and \( \| \cdot \|_{0,\Gamma} := \| \cdot \|_{L^2(\Gamma)} \) is the \( L^2 \) norm. Throughout this paper we denote by \( \| \cdot \|_{0,\Omega} := \| \cdot \|_{L^2(\Omega)} \), \( \| \cdot \|_{1,\Omega} := \| \cdot \|_{H^1(\Omega)} \), the usual Sobolev norms. The task is to seek an approximate solution for the Cauchy problem (2.1)-(2.3) from the noisy data \( f^\dagger \) and \( g^\dagger \).

Without any loss of generality, we suppose that the origin \((0,0)\) is located in \( \Omega \) (see Fig. 1). Denote

\[
\begin{align*}
or & = \max_{\Omega} r, & \text{r} & = \max_{\Omega \cap (0 < r < 2\pi)} r, \\
or & = \max_{\Omega \cap (0 < r < 2\pi)} r, & \text{r} & = \max_{\Omega \cap (0 < r < 2\pi)} r, \\
\Omega & = \{(r, \theta) ; 0 < r \leq r, 0 < \theta \leq 2\pi \}, & \text{r} & = \max_{\Omega \cap (0 < r < 2\pi)} r \leq \Omega, 0 < \theta \leq 2\pi \}, \\
\Omega & = \{(r, \theta) ; 0 < r \leq r, 0 < \theta \leq 2\pi \},
\end{align*}
\]

where \( \Omega \) is a disk inside \( \Omega \), then we have

\[
\Omega \subset \Omega \subset \Omega. \tag{2.5}
\]

Let the source points \( Q_i \) be located uniformly on a circle \( l_k = \{(r, \theta) ; r = R, 0 < \theta < 2\pi \} \) where \( R > r \), i.e.

\[
Q_i = \{(x, y) ; x = R \cos \theta, y = R \sin \theta \}, \quad h = \frac{2\pi}{N}, \quad i = 1, \ldots, N.
\]

The fundamental solutions corresponding to \( Q_i \) are

\[
\phi_i(P) = \frac{1}{2\pi} \ln |PQ_i|, \quad i = 1, 2, \ldots, N.
\]

We choose an approximate solution in the form
where $c_i$ are unknown coefficients to be sought.

Denote $V_N = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\}$. Since $u_N \in V_N$ has satisfied Laplace’s equation in $\Omega$ already, the coefficients $c_i$ only need to be sought by enforcing the boundary conditions (2.2) and (2.3) satisfied approximately. One popular approach is the collocation MFS, namely let (2.2) and (2.3) be satisfied on some collocation points, see [18–21] for solving the Cauchy problems of elliptic equations. However the convergence estimate is difficult to be obtained and numerical accuracy is sensitive to the noise in Cauchy data. In order to treat with a general simply-connected domain and keep the numerical solution more stable, we combine the Trefftz method in [31,32] and a Tikhonov regularization technique to propose a variational-type MFS as follows.

Define a functional

$$I(\nu) = \|\nu - f\|_0^2 + \omega \|\partial_\nu \nu - g\|_0^2 + \alpha \|
u\|_2^2 \quad (2.7)$$

in which $\omega$ is a positive weight and $\alpha > 0$ is called a regularization parameter. Then we define a regularized solution for the Cauchy problem (2.1)-(2.3) as $u^*_N$ given by solving

$$I(u^*_N) = \min_{\nu \in V_N} I(\nu), \quad (2.8)$$

namely

$$u^*_N := \arg \min_{\nu \in V_N} I(\nu). \quad (2.9)$$

3. Convergence estimate

In this section, we will give an error estimate for $\|u - u^*_N\|_{0, \partial}$ and a convergence rate is easily achieved under a suitable choice for the regularization parameter $\alpha$.

The following function

$$P_n(r, \theta) = \frac{a_0}{2} + \sum_{i=1}^{n} r^i (a_i \cos i \theta + b_i \sin i \theta) \quad (3.1)$$

is called a harmonic polynomial of order $n$ in which the coefficients $a_i, b_i$ are constants.

We cite Theorem 3.1 in [37] as the following lemma.

**Lemma 3.1.** Let $u \in H^p(\Omega)$ be a harmonic function, there exists a sequence of harmonic polynomials $Z_n$ with degree $n$ such that

$$\|u - Z_n\|_{k, \Omega} \leq Cn^{-p} \|u\|_{p, \Omega}. \quad (3.2)$$
where \(0 \leq k \leq p, k \text{ and } p \text{ are not necessarily integers, } C \text{ is a constant independent of } n \text{ and } u.

We can obtain the following lemma which is a slight modification of Theorem 4.1 in [23] and [32], its proof is given in Appendix A.

**Lemma 3.2.** Let \(\Omega\) be a bounded simply connected domain with the sufficiently smooth boundary and satisfy (2.5), then for any harmonic polynomial \(P_n\) in the form of (3.1), there exists a linear combination \(\mathcal{P}_N := \mathcal{P}_N(P_n, r, 0)\) in \(V_N\) with a source radius \(R \neq 1\), such that the following inequality

\[
\|P_n - \mathcal{P}_N\|_{q, \Omega} \leq C N^q \left( \frac{R}{r_{\max}} \right)^{2n-N} \left( \frac{r_{\max}}{r_{\min}} \right)^{n+\frac{1}{2}} n^2 \|P_n\|_{0, \Omega},
\]

(3.2)

holds provided \(q \geq 0\) and

\[
2^{q+1} \left( \frac{R}{r_{\max}} \right)^{-2N} \leq 1,
\]

(3.3)

where \(C\) is a constant independent of \(n\) and \(N\).

**Remark 3.3.** In [23], there is no a constant term in harmonic polynomial \(P_n\). Theorem 4.1 in [32] provided a very similar result but removed a factor \(\sqrt{n}\) which seems wrong and there is a mistake in its proof. In Appendix A, we provide a modified proof.

**Remark 3.4.** For an arbitrary geometry, we can choose \(R\) such that \(R > r_{\max}\) then for sufficiently large \(N\), condition (3.3) is satisfied easily.

For \(\Delta v = 0\), there exist the following trace theorem and the stability estimate in [38,39],

\[
\|v\|_{k,\partial \Omega} \leq C \|v\|_{k+\frac{3}{2},\Omega}, \quad \|v\|_{k+\frac{3}{2},\Omega} \leq C \|v\|_{k,\partial \Omega}.
\]

(3.4)

Throughout this paper \(C\) will denote a generic constant not necessarily the same everywhere.

For the solution \(u \in H^p(\Omega)\) of problem (2.1)-(2.3), by Lemma 3.1, there exists a harmonic polynomial \(P_n\) in the form of (3.1) such that

\[
\|R_n\|_{k,\Omega} = \|u - P_n\|_{k,\Omega} \leq C n^{k-p} \|u\|_{p,\Omega}.
\]

(3.5)

Denote by \(\mathcal{P}_N\) the approximation of \(P_n\) in \(V_N\) satisfying (3.2), then we can obtain an upper bound for \(I(\mathcal{P}_N)\) in the following lemma.

**Lemma 3.5.** Suppose that the conditions in Lemma 3.2 hold and \(u \in H^p(\Omega)\) for \(p \geq 2\) is the solution of problem (2.1)-(2.3). Denote by \(P_n\) the harmonic approximation of \(u\) and by \(\mathcal{P}_N\) the approximation of \(P_n\) in Lemma 3.2. Let \(N\) satisfy

\[
\left( \frac{R}{r_{\max}} \right)^{2n-N} \left( \frac{r_{\max}}{r_{\min}} \right)^{n+\frac{1}{2}} n^2 \leq \frac{1}{N^p},
\]

(3.6)

then we have

\[
I(\mathcal{P}_N) \leq C \left[ \frac{1 + \omega}{N^{p-2}} + \alpha \right] \|u\|_{p,\Omega}^2 + (1 + \omega) \delta^2.
\]

(3.7)

where \(C\) is a constant independent of \(\alpha, \delta, n\) and \(N\).

**Proof.** If \(N\) satisfies (3.6), note that \(p \geq 2\), when \(n \geq 2\) we have

\[
2^q \left( \frac{R}{r_{\max}} \right)^{-2N} \leq 16 \left( \frac{r_{\max}}{R} \right)^{2n} \left( \frac{r_{\min}}{r_{\max}} \right)^{n+1} \frac{1}{n^{2p}} \leq 1,
\]

Here, condition (3.3) holds for \(0 < q < \frac{1}{2}\). We may choose

\[
N \approx 2n + \frac{(n + \frac{1}{2}) \ln \left( \frac{r_{\max}}{r_{\min}} \right) + p \ln n}{\log \left( \frac{r_{\min}}{r_{\max}} \right)} \leq C n,
\]

where \(C = 2 + \frac{2 \ln \left( \frac{r_{\max}}{r_{\min}} \right)}{\ln n} > 2\) is a constant independent of \(n\) and \(N\), i.e. we have \(\frac{1}{N} \leq \frac{C}{n}\).

By (2.7), we have

\[
I(\mathcal{P}_N) = \|\mathcal{P}_N - f\|_{0, \Omega}^2 + \omega \|\frac{\partial \mathcal{P}_N}{\partial v}\|_{0, \Omega}^2 - g\|\mathcal{P}_N\|_{2, \Omega}^2 =: T_1 + T_2 + T_3.
\]
Next, from the embedding inequalities and the trace theorem in Sobolev space we give estimates for $T_1$, $T_2$, $T_3$ respectively.

\[
T_1 = \| \psi_N - f \|^2_{0, T} \leq 2 \| \psi_N - f \|^2_{0, T} + 2 \delta^2 \leq 2 \left( \| u - \psi_N \|^2_{0, \Omega} + \| R_N \|^2_{0, \Omega} + \delta^2 \right) \leq 4 \left( \| u - \psi_N \|^2_{0, \Omega} + \| R_N \|^2_{0, \Omega} + \delta^2 \right).
\]

By (3.5), (3.6) and Lemma 3.2 in which $q = 0$, we have

\[
T_1 \leq C \left\{ \frac{1}{N^{p-1}} \| P_n \|^2_{0, \Omega} + \frac{1}{N^{p-1}} \| u \|^2_{p, \Omega} + \delta^2 \right\}.
\]

Moreover, since $u \approx P_n$, we have

\[
\| P_n \|^2_{0, \Omega} \leq C \| u \|^2_{p, \Omega} \leq C \| u \|^2_{p, \Omega}.
\]

Therefore, we have

\[
T_1 \leq C \left\{ \frac{\| u \|^2_{p, \Omega}}{N^{p-1}} + \delta^2 \right\} \leq C \left\{ \frac{\| u \|^2_{p, \Omega}}{N^{p-1}} + \delta^2 \right\}. \tag{3.8}
\]

From the trace theorem and the stability estimate (3.4), there exists an inequality $\| \frac{\partial u}{\partial 
abla} \|^2_{0, \Omega} \leq C \| u \|^2_{0, \Omega} \leq C \| u \|^2_{1, \Omega}$, then by Lemmas 3.1 and 3.2 in which $q = 1$, we have

\[
T_2 = \omega \left\| \frac{\partial \psi_N}{\partial \nabla} - g \right\|^2_{0, T} \leq 2 \omega \left\| \frac{\partial \psi_N}{\partial \nabla} - g \right\|^2_{0, T} + \delta^2 \leq 2 \omega \left\| \frac{\partial u}{\partial \nabla} - \psi_N \right\|^2_{0, \Omega} + \delta^2 \leq C \omega \left\{ \| u - \psi_N \|^2_{0, \Omega} + \delta^2 \right\} \leq C \omega \left\{ \frac{1}{N^{p-1}} \| P_n \|^2_{0, \Omega} + \frac{1}{N^{p-1}} \| u \|^2_{p, \Omega} + \delta^2 \right\} \leq C \omega \left\{ \frac{1}{N^{p-1}} \| u \|^2_{p, \Omega} + \delta^2 \right\}. \tag{3.9}
\]

By Lemma 3.2 in which $q = \frac{5}{3}$, noting that $p \geq 2$, we have

\[
T_3 = \alpha \| \psi_N \|^2_{2, \Omega} \leq 2 \alpha \left\{ \| \psi_N - P_n \|^2_{2, \Omega} + \| P_n \|^2_{2, \Omega} \right\} \leq C \alpha \left\{ \| \psi_N - P_n \|^2_{2, \Omega} + \| u \|^2_{p, \Omega} \right\} \leq C \alpha \left\{ \frac{1}{N^{p-1}} \| u \|^2_{p, \Omega} + \| u \|^2_{p, \Omega} \right\} \tag{3.10}
\]

Combining (3.8)–(3.10) gives

\[
I(\psi_N) \leq C \left[ \left( \frac{1 + \alpha}{N^{p-1}} + \alpha \right) \| u \|^2_{p, \Omega} + (1 + \omega) \delta^2 \right].
\]

This completes the proof of Lemma 3.5. \hfill \square

Now, we give our main result.

**Theorem 3.6.** Let $\Omega$ satisfy (2.5) and $u \in H^p(\Omega)$ be the solution of (2.1)–(2.3). For a given $E > 0$, assume $u$ satisfies an a-priori bound

\[
\| u \|^2_{p, \Omega} \leq E, \quad \text{for } p \geq 2.
\]

Let $u^\delta_{\alpha, \beta}$ be the regularized solution given by (2.9), then we have an error estimate in $L^2$ norm

\[
\| u^\delta_{\alpha, \beta} - u \|^2_{0, \Omega} \leq C(M + \epsilon) \frac{1}{(\log (\frac{M + 1}{\epsilon}))^\mu}, \tag{3.11}
\]

where

\[
\epsilon = C \left\{ M^2 \left( (1 + \omega) \frac{\delta^4}{\beta} + E^2 (1 + \omega) \frac{1}{N^p} + E^2 \alpha^4 \right) + \left( (1 + \omega) \frac{\delta^4}{\beta} + E (1 + \omega) \frac{1}{N^p} + E \alpha^4 \right) \right\},
\]

\[
M = C \left\{ \left( 1 + \omega \right) \frac{\delta^4}{\sqrt{2N^p}} + 2 \right\} E + (1 + \omega)^2 \frac{\delta^4}{\sqrt{2}}
\]

and $\mu \in (0, 1)$, $C$ are constants independent of $\alpha$, $\delta$ and $N$. Furthermore, if choosing the regularization parameter $\alpha = \delta^2$ and taking $N$ such that $N \geq \frac{1}{\mu \delta^2}$, then we have a convergence rate for sufficiently small $\delta$

\[
\| u^\delta_{\alpha, \beta} - u \|^2_{0, \Omega} \leq C \left( \log \left( \frac{M + 1}{\epsilon} \right) \right)^\mu, \tag{3.12}
\]

where $C$, $\mu$ are constants independent of $\delta$. 


Proof. By the definition of \( L(v) \) in (2.9), we have \( L(u_N^0) \geq \alpha u_N^0 \|^2 \), further, \( u_N^0 \|^2 \leq \frac{1}{\sqrt{2\alpha}} L(\Phi_N) \). Note that \( L(u_N^0) \leq L(\Phi_N) \), then we have
\[
\|u_N^0 - u\|^2 \leq \|u_N^0\|^2 + \|u\|^2 \leq C \left\{ \frac{1}{\alpha} \sqrt{L(\Phi_N)} + \|u\|_{p,0} \right\} \leq \frac{1}{\sqrt{2\alpha}} \left\{ \sqrt{L(\Phi_N)} + \|u\|_{p,0} \right\}.
\] (3.13)
Substitute the estimate of \( L(\Phi_N) \) in Lemma 3.5 into the inequality above, we have
\[
\|u_N^0 - u\|^2 \leq \frac{1}{\sqrt{2\alpha}} \left\{ \left( 1 + \frac{\omega}{N} \right) \|u\|_{p,0}^2 + (1 + \omega)\delta^2 + \|u\|_{p,0} \right\} \leq \frac{1}{\sqrt{2\alpha}} \left\{ (1 + \omega) \frac{\delta^2}{\sqrt{\frac{\omega}{N}}} + 2 \right\} E + (1 + \omega) \frac{\delta^2}{\sqrt{\frac{\omega}{N}}} =: M. \tag{3.14}
\]
Also note that the definition of \( u_N^0 \), we have
\[
\|u_N^0 - u\|_{1,0}^2 \leq \|u_N^0 - f\|^2 + \delta \leq \sqrt{L(\Phi_N)} + \delta \leq \sqrt{L(\Phi_N)} + \delta \tag{3.15}
\]
and
\[
\|
\frac{\partial u_N^0}{\partial v} - \frac{\partial u}{\partial v}\|^2_{0,0} \leq \|
\frac{\partial u_N^0}{\partial v} - \frac{\partial u}{\partial v}\|^2_{0,0} + \delta \leq \sqrt{\omega^{-1}L(\Phi_N)} + \delta \leq \sqrt{\omega^{-1}L(\Phi_N)} + \delta. \tag{3.16}
\]
From the Sobolev embedding theorem and trace theorem, we have
\[
\|u_N^0 - u\|_{1,0} \leq \|u_N^0 - u\|_{1,0} \leq C\|u_N^0 - u\|^2 \leq C\|u_N^0 - u\|^2 \leq CM. \tag{3.17}
\]
By using an interpolation inequality and (3.15) and (3.17), we obtain
\[
\|u_N^0 - u\|_{1,0} \leq C \left\{ \|u_N^0 - u\|_{1,0} \|u_N^0 - u\|_{1,0} \right\} \leq C \left\{ M^2 \left( L(\Phi_N) + \delta^2 \right) \right\} \tag{3.18}
\]
From (3.16), (3.18) and Lemma 3.5, we obtain
\[
\|u_N^0 - u\|_{1,0} \leq \left\{ \|u_N^0 - u\|_{1,0} \|u_N^0 - u\|_{1,0} \right\} \leq C \left\{ M^2 \left( L(\Phi_N) + \delta^2 \right) + \omega^2 \left( L(\Phi_N) + \delta^2 \right) \right\} \leq \left\{ M^2 \left( (1 + \omega) \frac{\delta^2}{\sqrt{\frac{\omega}{N}}} + \frac{\delta^2}{\sqrt{\frac{\omega}{N}}} \right) \right\} \leq \epsilon. \tag{3.19}
\]
Combining (3.14) and (3.19), by the global conditional stability for the Cauchy problem (Theorem 1.9 in [36]), the desired result (3.11) is proved.

Suppose that \( \delta > 0 \) is sufficiently small, let \( N \geq \frac{1}{\delta \sqrt{\omega}} \) and choose the regularization parameter \( \alpha = \delta^2 \), then
\[
M = C \left\{ (1 + \omega)^2 + 2 \right\} \|u_N^0\|^2 \leq K \tag{3.20}
\]
and
\[
\epsilon = C \left\{ (1 + \omega) \frac{\delta^2}{\sqrt{\frac{\omega}{N}}} + \frac{\delta^2}{\sqrt{\frac{\omega}{N}}} \right\} \leq C \sqrt{\delta}. \tag{3.21}
\]
Substitute (3.20) and (3.21) into (3.11), for sufficiently small \( \delta \), we have
\[
\|u_N^0 - u\|_{0,0} \leq C \left( \frac{\log \left( \frac{K + \delta \sqrt{\omega}}{\epsilon} \right) ^2}{\epsilon} \right), \tag{3.22}
\]
where \( C \) and \( \epsilon \) are constants independent of \( \delta \).

It is clear that the right hand side of (3.22) tends to zero as \( \delta \to 0 \) and \( N \) goes to infinity as \( \delta \to 0 \), hence the error estimate (3.11) is convergent under the choice of parameters \( \alpha \). \( \square \)

Remark 3.7. In [32], the weight \( \omega \) is chosen as \( \frac{1}{\delta} \). However in our case, from (3.21), we can see that such a choice is bad due to the appearance of \( 1/\omega \) in \( \epsilon \). We choose a fixed constant \( \omega = 1 \) in our computations if without a special statement.

Remark 3.8. Note that under the choices of parameters \( \alpha \) and \( N \) in Theorem 3.6, from (3.14), we know \( \|u_N^0 - u\|_{1,0} \leq K \). By an interpolation theorem in Sobolev space, we have \( \|u_N^0 - u\|_{1,0} \leq \frac{\epsilon}{(\log \left( \frac{K + \delta \sqrt{\omega}}{\epsilon} \right) )^2} \). From the trace theorem in Sobolev space, we know \( \|u_N^0 - u\|_{0,0} \leq \frac{\epsilon}{(\log \left( \frac{K + \delta \sqrt{\omega}}{\epsilon} \right) )^2} \). Thus in our numerical experiments, we only focus on the accuracy of numerical solution on boundary \( \gamma \).
4. Numerical implementation

In this section, we propose a numerical implementation method for obtaining the regularized solution \( u^R_{x,N} \) and test four examples for supporting our analysis. Some numerical experiments by a collocation MFS in \([18]\) are provided to compare with the new method.

Let \( u^R_{x,N} = \sum_{j=1}^{N} c_j \phi_j \), inserting into the functional (2.7), we have

\[
F(c) = I(u^R_{x,N}) = \left\| \sum_{j=1}^{N} c_j \phi_j - f \right\|_{0,\Gamma}^2 + \omega \left\| \sum_{j=1}^{N} \frac{\partial c_j \phi_j}{\partial y} - g \right\|_{0,\Omega}^2 + \alpha \left\| \sum_{j=1}^{N} c_j \phi_j \right\|_{2,\Omega}^2.
\]

(4.1)

From \( \frac{df}{dx} = 0 \), \( k = 1, 2, \ldots, N \), we know \( c = (c_1, c_2, \ldots, c_N)^T \) satisfy the following linear equations

\[
Ac = b,
\]

(4.2)

where \( A = (a_{k,j})_{N \times N} \) with \( a_{k,j} = (\phi_k, \phi_j)_{0,\Gamma} + \omega \left( \frac{\partial \phi_k}{\partial x}, \frac{\partial \phi_j}{\partial x} \right)_{0,\Gamma} + \alpha (\phi_k, \phi_j)_{2,\Omega} \) and \( b = (b_k)_{N \times 1} \) with \( b_k = (\phi_k, f^\alpha)_{0,\Gamma} + \omega (\phi_k, g^\alpha)_{0,\Gamma} \) in which \((\cdot, \cdot)_{0,\Gamma}, (\cdot, \cdot)_{2,\Omega}\) are the corresponding inner products in the Sobolev spaces.

For simplicity, we use a star-like curve as a boundary, i.e. suppose

\[
\partial \Omega = \{ (x, y) | x = r(\theta) \cos(\theta), \; y = r(\theta) \sin(\theta), 0 \leq \theta \leq 2\pi \}
\]

and let

\[
\Gamma = \partial \Omega \cap \{ \pi < \theta < 2\pi \},
\]

\( \gamma = \partial \Omega \cap \{ 0 \leq \theta \leq \pi \} \).

The inner products can be calculated from the following schemes

\[
(\phi_k, \phi_j)_{0,\Gamma} = \int_{0}^{2\pi} \phi_k(\theta) \phi_j(\theta) \sqrt{r(\theta)^2 + (r'(\theta))^2} d\theta = \sum_{l=1}^{m} \phi_k(\theta_l) \phi_j(\theta_l) \sqrt{r(\theta_l)^2 + (r'(\theta_l))^2} \frac{h_m}{2},
\]

(4.3)

where \( h_m = \pi/m \), \( \theta_l = \pi + l \cdot h_m \) and in the following computations we always use \( m = 100 \) if no special instructions.

\[
(\phi_k, \phi_j)_{2,\Omega} = \int_{\Omega} \left\{ (\phi_k \phi_l + (\phi_k)_{x}(\phi_l)_{x} + (\phi_k)_{y}(\phi_l)_{y} + (\phi_k)_{xx}(\phi_l)_{xx} + (\phi_k)_{xy}(\phi_l)_{xy} + (\phi_k)_{yy}(\phi_l)_{yy} \right\} dx dy
\]

\[
= \int_{0}^{2\pi} \int_{0}^{r(\theta)} H_{ij}(r, \theta) r dr d\theta,
\]

(4.4)

where \( H_{ij}(r, \theta) \) are the integrands with \( r \) and \( \theta \) as polar coordinates. The function \( \text{quad2d} \) in Matlab is used for calculating these integrations.

In our computations, the noisy data are generated by

\[
f^\alpha(\theta_l) = f(\theta_l)(1 + \varepsilon \text{rand}(l)), \quad l = 1, 2, \ldots, m
\]

and

![Fig. 2. Solution domain for Examples 1–3.](image-url)
\[ g^e(\theta_i) = g(\theta_i)(1 + \varepsilon \text{rand}(l)), \quad l = 1, 2, \ldots, m, \]

where \( f(\theta_i), g(\theta_i) \) are the exact Cauchy data at points \((r(\theta_i), \theta_i)\) and \(\varepsilon\) is a relative noise level; the function \(\text{rand}(l)\) are random numbers uniformly distributed in \([-1, 1]\).

To show the numerical accuracy, we compute the relative root mean square error for the approximate solution \(u_{\varepsilon,N}^e\), denoted by

\[ e_\varepsilon(u) = \frac{\sqrt{\sum_{i=1}^{M} [u_{\varepsilon,N}^e(\theta_i) - u(\theta_i)]^2}}{\sqrt{\sum_{i=1}^{M} u(\theta_i)^2}}, \quad (4.5) \]

where \(\theta_i\) are test points on \(\gamma\) and \(M\) is the total number of test points. In all the numerical examples, we always fix \(M = 300\).

In the following Examples 1–3, we use an apple-shaped curve

\[ r(\theta) = 0.7 \frac{0.53 + 0.4 \cos(\theta + \pi/4) + 0.1 \sin(2(\theta + \pi/4))}{1 + 0.8 \cos(\theta + \pi/4)} \]

as the boundary of solution domain, see Fig. 2.

Fig. 3. Example 1. Exact solution and approximate solutions with \(\varepsilon = 0, 0.001, 0.01\) by the variational MFS.

Fig. 4. Example 1. Exact solution and approximate solutions with \(\varepsilon = 0, 0.001, 0.01\) by the collocation MFS.
**Example 1.** Let \( u = x^2 - y^2 + e^x \sin(y) \). It is a harmonic function in \( \mathbb{R}^2 \). The Cauchy data \( f \) and \( g \) on \( \Gamma \) can be calculated directly.

In the computation of **Example 1**, we take the source radius \( R = 8 \), the number of source points \( N = 30 \).

Numerical results with various noise levels \( \varepsilon = 0, 0.001, 0.01 \) are shown in Fig. 3 in which the regularization parameters are \( \alpha = 1e-13, 1e-11, 1e-9 \), respectively. The approximate solutions are in very good agreement with the exact one for both of the exact and the noisy Cauchy data. Numerical results given by a collocation MFS are provided in Fig. 4. We can see that for a little small noise level (\( \leq 1\% \)), both methods can provide satisfactorily accurate results.

To verify the robustness of our proposed method, we present numerical results with higher noise levels \( \varepsilon = 0.05, 0.15, 0.25 \) in Fig. 5 in which the regularization parameters are \( \alpha = 1e-7, 1e-6, 1e-6 \), respectively. The approximate solutions match the exact one also quite well even for such large noise levels. Numerical results given by a collocation MFS are shown in Fig. 6. From Figs. 5 and 6, it can be seen that our proposed variational type MFS is more stable than the collocation MFS.

The weight \( \omega \) in the functional \( I(v) \) is an adjust factor, usually it is taken \( \omega = 1 \) which means that the defects to Dirichlet data and Neumann data are equally important. To show the influence of \( \omega \), we display the relative errors for numerical solutions with various weights \( \omega \) for **Example 1** in Fig. 7. It can be seen that the \( \omega \) has a wide range for getting a good accuracy. In Fig. 8, we present the recovered results with various \( \omega \) for **Example 1**, we can see again that the numerical solutions are not so sensitive to \( \omega \). Thus in the following tests, we always fix \( \omega = 1 \).

---

**Fig. 5.** Example 1. Exact solution and approximate solutions with \( \varepsilon = 0.05, 0.15, 0.25 \) by the variational MFS.

**Fig. 6.** Example 1. Exact solution and approximate solutions with \( \varepsilon = 0.05, 0.15, 0.25 \) by the collocation MFS.
We investigate the relationship between the relative error $e_r(u)$ and the source radius $R$. Numerical results are shown in Fig. 9. It can be observed that the numerical accuracy keep a reasonably stable level when the source radius $R$ is in the wide range 7–17. In the following computations, we always fix $R = 8$.

**Example 2.** Take a harmonic function in $\Omega$ with a singularity outside $\Omega$ nearby $c$, i.e., we take $u = \ln \sqrt{(x-x_p)^2 + (y-y_p)^2}$ where $(x_p, y_p) = (0, 1)$ is a singularity. The Cauchy data $f$ and $g$ on $\Gamma$ can be calculated directly. In Example 2, we take the number of source points $N = 30$.

Numerical results for various noise levels $\varepsilon = 0.05, 0.15, 0.25$ are shown in Fig. 10 in which we choose the regularization parameters $\lambda = 10^{\varepsilon}/C_0, 10^{\varepsilon}/C_0, 10^{\varepsilon}/C_0$ respectively. The proposed approach produces very accurate approximate solutions. The corresponding results by a collocation MFS is displayed in Fig. 11. By comparing Figs. 10 and 11, it can be seen that the variational MFS is somewhat better than the collocation MFS. For both approaches, we find that numerical accuracy deteriorates considerably as the singularity $(x_p, y_p)$ is close to the boundary $\gamma$ down to point (0,0.3).

Table 1 shows the root mean square errors $e_r(u)$ with various noise levels $\varepsilon$ for Examples 1, 2 by two kinds of MFS, which also provide a clear view of superiority of the new method over the collocation MFS on stability to large noises. We note that the collocation MFS is very efficient to the exact Cauchy data, but sensitive to noises in Cauchy data.
Fig. 9. Example 1. Relative error $\varepsilon_r(u)$ versus $R$ for a fixed $\varepsilon = 0.005$.

Fig. 10. Example 2. Exact solution and approximate solutions with $\varepsilon = 0.05, 0.15, 0.25$ by the variational MFS.

Fig. 11. Example 2. Exact solution and approximate solutions with $\varepsilon = 0.05, 0.15, 0.25$ by the collocation MFS.
Example 3. For this example, no analytic solution is given and we should solve a direct problem at first such that
\[ u_j = \frac{C_0}{8} x^2 + \frac{1}{30} x + 2 y \] and \( g(x, y) = -1 \). The Dirichlet data \( f = u_{|r} \) is obtained by a method of integral equation. For solving Example 3, we take the number of source points \( N = 30 \).
Numerical results for various noise levels $\epsilon = 0.05, 0.15, 0.25$ are shown in Fig. 12 in which the relative errors for approximate solutions are $\varepsilon(u) = 0.1448, 0.1454, 0.1353$ with the regularization parameters $\alpha = 1e-5, 1e-5, 1e-4$, respectively. We can see that the approximate solutions are still reasonably accurate and stable. For this example and Example 4, the collocation MFS in [18] is fail to give acceptable numerical reconstructions. The reason may be that the Dirichlet data given by solving the direct problem contain a little large computational error and the solution are not smooth in Example 4.

Example 4. We consider a non-smooth solution but with an analytic boundary $\partial \Omega = \{(r, \theta) \mid r = 1, 0 \leq \theta \leq 2\pi\}$. The Dirichlet data $u|\Gamma$ is given by solving the direct problem

$$
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega, \\
 u &= \varphi \quad \text{on } \gamma, \\
 \frac{\partial u}{\partial n} &= g \quad \text{on } \Gamma
\end{align*}
$$

with

$$
\varphi = \begin{cases}
6\theta/\pi, & [0, \pi/2], \\
-6\theta/\pi + 6, & \left(\frac{\pi}{2}, \pi\right]
\end{cases}
$$

and $g = -1$. In the computation of Example 4, the parameters we used are $N = 30, m = 300$.

Numerical results for various noise levels $\epsilon = 0.05, 0.15, 0.25$ are shown in Fig. 13. The relative errors for approximate solutions are $\varepsilon(u) = 0.0692, 0.0847, 0.0963$ with the regularization parameters $\alpha = 1e-9, 1e-7, 1e-5$ respectively. It is seen that the approximate solutions are in good agreement with the exact one except a region near the cusped point. Our proposed method still gives reasonable results although the solution is not in $H^2(\Omega)$.

5. Conclusions

In this paper, we propose a variational-type regularized method of fundamental solutions for solving the Cauchy problem of Laplace’s equation in a simply connected bounded domain with noisy Cauchy data. Under the suitable choice of the regularization parameter, we obtain a convergence result for the regularized solution. Numerical experiments are investigated and show the robustness of our proposed method even for high level of noise in Cauchy data. Compared with the collocation MFS, we observe that the variational method in this paper is much more effective and stable. For non-smooth solutions, our proposed new approach can be used to obtain reasonable approximations, and meanwhile the collocation MFS is fail to use. However, in our computations, we choose the regularization parameters by trial and error. The a-priori choice rule provided in this paper is fail to give accurate numerical results. The good choice rule for the regularization parameter is still an open problem in the proposed method.

Acknowledgments

This paper was supported by the NSF of China (10971089, 11171136) and the Fundamental Research Funds for the Central Universities (lzujbky-2012-k25). The part of work was done when the first author visited Texas A&M University supported by the CSC of China.

Appendix A

Proof. Suppose

$$
P_n(\rho, \theta) = \frac{d_n}{2} + \sum_{m=1}^{n} (a_m \rho^m \cos m\theta + b_m \rho^m \sin m\theta).
$$

Using the notations in $V_N$ and some ones in [32], denote

$$
\Sigma_N(1; \rho, \theta) = \frac{h}{2\pi \ln \rho} \sum_{k=1}^{N} \phi_k(\rho, \theta), \quad h = \frac{2\pi}{N},
$$

$$
\Sigma_N(\rho^m \cos m\theta; \rho, \theta) = \frac{\rho^m h}{\pi} \sum_{k=1}^{N} \cos(mkh) \phi_k(\rho, \theta),
$$

$$
\Sigma_N(\rho^m \sin m\theta; \rho, \theta) = \frac{\rho^m h}{\pi} \sum_{k=1}^{N} \sin(mkh) \phi_k(\rho, \theta)
$$

and define.
\[ \Psi_N(P_n; \rho, \theta) = \frac{a_0}{2} \Sigma_N(1; \rho, \theta) + \sum_{m=1}^n a_m \Sigma_N(\rho^m \cos m\theta; \rho, \theta) + b_m \Sigma_N(\rho^m \sin m\theta; \rho, \theta). \]  

(5.5)

Using the proof in [32] with condition (3.3), we have

\[ \|E_m\|_{q, C_{r_{\max}}} := \|\rho^m \cos m\theta - \Sigma_N(\rho^m \cos m\theta; \rho, \theta)\|_{q, C_{r_{\max}}} \leq Cr_{\max}^2 N^q \left( \frac{R}{r_{\max}} \right)^{-n} R^n, \]

where \( C_{r_{\max}} \) is the circle with radius \( r_{\max} \).

By a very similar method, we can prove

\[ \|E_S\|_{q, C_{r_{\max}}} := \|\rho^m \sin m\theta - \Sigma_N(\rho^m \sin m\theta; \rho, \theta)\|_{q, C_{r_{\max}}} \leq Cr_{\max}^2 N^q \left( \frac{R}{r_{\max}} \right)^{-n} R^n \]

and

\[ \|E_0\|_{q, C_{r_{\max}}} := \|1 - \Sigma_N(1; \rho, \theta)\|_{q, C_{r_{\max}}} \leq Cr_{\max}^2 N^q \left( \frac{R}{r_{\max}} \right)^{-n} \left| \frac{1}{\ln R} \right|. \]

In [32], there is a mistake for \( \|E_S\|_{q, C_{r_{\max}}} = \|E_0\|_{q, C_{r_{\max}}} = 0 \).

Note that \( \frac{r_{\max}}{R} > 1 \) and \( \frac{r_{\max}}{r_{\min}} > 1 \), we have

\[ \|P_n - \Psi_N\|_{q, C_{r_{\max}}} \leq \frac{a_0}{2} \left\| E_0 \right\|_{q, C_{r_{\max}}} + \sum_{m=1}^n \left( |a_m| \|E_m\|_{q, C_{r_{\max}}} + |b_m| \|E_S\|_{q, C_{r_{\max}}} \right) \]

\[ \leq Cr_{\max}^2 N^q \left( \frac{R}{r_{\max}} \right)^{-n} \max \left\{ \frac{1}{\ln R}, 1 \right\} \left( \frac{a_0}{2} + \sum_{m=1}^n (|a_m| + |b_m|) \left( \frac{R}{r_{\max}} \right)^m R^n \right) \]

\[ \leq C(R) r_{\max}^2 N^q \left( \frac{R}{r_{\min}} \right)^{-n} \left( \frac{a_0}{2} + \sum_{m=1}^n r_{\min}^m (|a_m| + |b_m|) \right) \]

\[ \leq C(R) r_{\max}^2 N^q \left( \frac{R}{r_{\min}} \right)^{-n} \left( \frac{r_{\max}}{r_{\min}} \right)^n \left( \frac{R}{r_{\min}} \right)^n \left( \frac{R}{r_{\min}} \right)^{2n+1} \left( \frac{a_0}{2} + \sum_{m=1}^n r_{\min}^m (|a_m|^2 + |b_m|^2) \right) \]

\[ \leq C(R) N^q \left( \frac{R}{r_{\max}} \right)^{2n-N} \left( \frac{r_{\max}}{r_{\min}} \right)^{n+1} n!^2 \|P_n\|_{0, C_{r_{\min}}}, \]

where \( C_{r_{\min}} \) is the circle with radius \( r_{\min} \).

Note that \( P_n \) and \( \Psi_N \) are harmonic functions on \( \Omega_{r_{\max}} \), by the trace theorem and the stability estimate (3.4), we have

\[ \|P_n - \Psi_N\|_{q, 0, \Omega} \leq C \|P_n - \Psi_N\|_{q, 1/2, \Omega} \leq C \|P_n - \Psi_N\|_{q, 1/2, r_{\max}} \leq C \|P_n - \Psi_N\|_{q, C_{r_{\max}}}. \]

Similarly, we know \( \|P_n - \Psi_N\|_{q, C_{r_{\max}}} \leq \|P_n - \Psi_N\|_{q, 1/2, 0} \). Then the proof is finished by simple substitutions. \( \square \)

References


