Equity Forward with Dividend Reinvestment

We developed a pricing model to calculate unwinding values of equity forward with dividend reinvestment.

Suppose the initial trade date is set to time t_0, the maturity of the forward is T. If there is a delay of δ_0 for the forward contract to be effective, and a delay of δ_T for the contract to be settled, the forward price is calculated by

$$K = \left(S_{t_0} - \sum_{i=1}^{n} D_i \exp\left(-r_{t_0,t_i}\left(t_i - t_0\right)\right)\right) \cdot \exp\left(r_{t_0+\delta_0,T+\delta_T}\left(T + \delta_T - t_0 - \delta_0\right)\right) \qquad (1)$$

where S_{t_0} is the initial stock price at time t_0, D_i is the amount of the i^{th} dividend whose ex-date lies between the trade date and the maturity date, t_i is the payment time of the i^{th} dividend, and $r_{u,v}$ is the forward interest rate between time u and v. At time t, if there is a delay of δ_t for the contract to be unwound, the unwinding value of the forward is calculated by

$$V_t = S_t - \sum_{i} D_i \exp\left(-r_{t,t_i}\left(t_i - t\right)\right) - K \exp\left(-r_{t+\delta_t,T+\delta_T}\left(T + \delta_T - t - \delta_t\right)\right) \qquad (2)$$

where S_t is the stock price at time t. Detailed discussion is found in [1].

The term ‘value’ of forward contract refers the unwinding value of the forward hereinafter.

In general, owner of a forward contract does not receive dividends paid before the maturity of the contract. For simplicity in illustration, let us assume that there is only one dividend payment with amount D and payment date t_d. If the contract specifies that dividends will be reinvested and paid at maturity (*DIV_REINVEST_IND* set to YES), the dividend is invested by buying D/S_{t_d} shares of the underlying stock on the dividend payment date. Thus, forward contract owner gets D/S_{t_d} extra shares of stock at the maturity. Thus, with zero settle date lag, the contribution of this dividend at time t should be

$$E\left[\frac{D}{S_{t_d}} \cdot S_T\right] \cdot \exp\left(-r_{t,T} \cdot T\right) = D \cdot \exp\left(r_{t_d,T}\left(T - t_d\right)\right) \cdot \exp\left(-r_{t,T} T\right) \tag{3}$$

In a non-quanto forward, expression (3) becomes $D\exp\left(-r_{t,t_d}\left(t_d - t\right)\right)$. This is simply the present value of the dividend and, thus, the forward contract value becomes

$$\begin{aligned} V_t &= S_t - D \cdot \exp\left(-r_{t,t_d}\left(t_d - t\right)\right) + D \cdot \exp\left(-r_{t,t_d}\left(t_d - t\right)\right) - K \cdot \exp\left(-r_{t,T}\left(T - t\right)\right) \\ &= S_t - K \cdot \exp\left(-r_{t,T}\left(T - t\right)\right) \end{aligned} \tag{4}$$

For the quanto case, suppose the stock is in Canadian dollars and the forward contract is quantoed into US Dollars. The forward contract value with the reinvestment adjustment is

$$\begin{aligned} V_t = S_t &\cdot \exp\left(\left(r_{t,T}^{cad} - \rho\sigma_{ex}\sigma_s\right)\left(T - t\right)\right) \cdot \exp\left(-r_{t,T}^{usd}\left(T - t\right)\right) \\ &- D \cdot \exp\left(r_{t_d,T}^{cad}\left(T - t_d\right)\right) \cdot \exp\left(-r_{t,T}^{usd}\left(T - t\right)\right) \\ &+ D \cdot \exp\left(\left(r_{t_d,T}^{cad} - \rho\sigma_{ex}\sigma_s\right)\left(T - t_d\right)\right) \cdot \exp\left(-r_{t,T}^{usd}\left(T - t\right)\right) - K \cdot \exp\left(-r_{t,T}^{usd}\left(T - t\right)\right) \end{aligned} \tag{5}$$

where σ_s is the underlying stock volatility, σ_{ex} is the FX rate volatility and ρ is the correlation coefficient between the stock price and FX rate quoted in the units of USD per one unit of CAD. Currently, GAT model uses constant volatilities.

The equations (4) and (5) can be generalized with multiple dividends and non-zero settle date lag as the following. For non-quanto case, let *Pay* be the contract payoff currency (= underlying currency) and *Div* the dividend currency, then we have

$$V_t = S_t - PV(Div) + REI(Div) - K \cdot \exp\left(-r^{Pay}_{t+\delta_t, T+\delta_T}\left(T + \delta_T - t - \delta_t\right)\right) \tag{6}$$

Where

$$PV(Div) = \sum_{i=1}^{n} D_i \cdot X_t \cdot \exp\left(-r^{Div}_{t,t_i}(t_i - t)\right)$$

$$REI(Div) = \sum_{i=1}^{n} D_i \cdot X_0 \cdot \exp\left(-r^{Pay}_{t+\delta_t, t_i}\left(t_i - t - \delta_t\right)\right)$$

X_t is the current exchange rate and X_0 is the pre-specified exchange rate, both in units of *UL* currency (= payoff currency) per one unit of *Div* currency (see next section on *Dividend Currency* for details).

Note $X_t = X_0 = 1$ if *UL* currency = *Div* currency. Here, *PV(Div)* is the present value of the dividends whose ex-dates lie between the valuation date *t* and the maturity date. *REI(Div)* is the present value of the dividend reinvestment whose payment dates lie between the valuation date and the maturity date. The dividend that will go ex before maturity and be payable after maturity will not be invested. Rather, the cash (dividend) amount will be paid to the holder directly and, thus, only the discounted value at maturity *T* should be considered. It should be noted that this dividend amount should be discounted using the forward interest rate of the payoff currency.

For quanto case, let *UL* be the underlying stock currency and we have

$$V_t^Q = S_t^Q - PV^Q(Div) + REI^Q(Div) - K \cdot \exp\left(-r_{t+\delta_t, T+\delta_T}^{Pay}\left(T + \delta_T - t - \delta_t\right)\right) \qquad (7)$$

Where

$$S_t^Q = S_t \cdot \exp\left(\left(r_{t+\delta_t, T+\delta_T}^{UL} - \rho\sigma_{ex}\sigma_s - r_{t+\delta_t, T+\delta_T}^{Pay}\right) \cdot \left(T + \delta_T - t - \delta_t\right)\right)$$

$$PV^Q(Div) = \sum_{i=1}^{n} D_i \cdot X_t \cdot \exp\left(-r_{t,t_i}^{Div}(t_i - t) + \left(r_{t+\delta_t, T+\delta_T}^{UL} - r_{t+\delta_t, T+\delta_T}^{Pay}\right) \cdot \left(T + \delta_T - t - \delta_t\right)\right)$$

$$REI^Q(Div) = \sum_{i=1}^{n} D_i \cdot X_0 \cdot \exp\left(\left(r_{t_i, T+\delta_T}^{UL} - \rho\sigma_s\sigma_{ex}\right) \cdot \left(T + \delta_T - t_i\right) - r_{t+\delta_t, T+\delta_T}^{Pay}\left(T + \delta_T - t - \delta_t\right)\right)$$

where ρ is the correlation coefficient between the stock price and FX rate quoted in the units of *Pay* currency per one unit of *UL* currency. X_t and X_0 are quoted both in units of *UL* currency per one unit of *Div* currency. Note in *PV(Div)*, the dividend payments are considered as deterministic and there is no drift adjustment. On the other hand, *REI(Div)* considers stock purchase with dividend payments, there is drift adjustment in forward value. If there are dividend payments prior to the valuation date, the number of shares is increased by purchasing underlying stock on each dividend payment date. We assume there are *m* dividend payments prior to valuation date. The number of share at the beginning of the contract is one and on valuation date it becomes

$$N = \prod_{i=1}^{m}\left(1 + \frac{D_i}{S_{t_i}}\right)$$

where S_{t_i}, $i = 1, \cdots, m$ are historical stock prices and $t_i < t$. The equations (6) and (7) assume one share of underlying stock and they are generalized with *N* by simply multiplying the values by *N*. For non-quanto case,

$$V_t = N \cdot \left[S_t - PV(Div) + REI(Div) - K \cdot \exp\left(- r_{t+\delta_t, T+\delta_T} \left(T + \delta_T - t - \delta_t \right)\right)\right] \quad (6')$$

For quanto case,

$$V_t^Q = N \cdot \left[S_t^Q - PV^Q(Div) + REI^Q(Div) - K \cdot \exp\left(- r^{Pay}_{t+\delta_t, T+\delta_T} \left(T + \delta_T - t - \delta_t \right)\right)\right] \quad (7')$$

Currently, GAT model does not provide a check on dividends that should be included for valuing contracts in equity forwards when DIV_REINVEST_IND is set to YES. Including dividend, that should not be considered, result in incorrect GAT pricing in some cases. Middle office should ensure correct dividends for each contract.

This feature allows the dividends and the underlying stock to be denominated in two different currencies (*DIVIDEND_CURRENCY* is different from *UNDERLYING_CURRENCY)*. For example, *BMO.N* is traded in the New York Stock Exchange, but the dividend is still paid in Canadian Dollars. Suppose we have a forward contract on *BMO.N* with payoff currency in US dollars. The dividends are converted into US Dollars at a pre-specified exchange rate before reinvested into shares of *BMO.N* at the dividend payment date.

The value of the forward contract is given by

$$V_t = S_t - K \cdot \exp\left(- r^{Pay}_{t+\delta_t, T+\delta_T} \left(T + \delta_T - t - \delta_t \right)\right) - \sum_{i=1}^{n} D_i \cdot X_t \exp\left(- r^{Div}_{t,t_i} \cdot (t_i - t)\right)$$
$$+ \sum_{i=1}^{n} D_i \cdot X_0 \exp\left(- r^{Pay}_{t+\delta_t, t_i} \cdot (t_i - t - \delta_t)\right)$$

where $r^{Pay}_{t_1,t_2}$ is the forward rate in payoff currency (= underlying stock currency) between t_1 and t_2. Similarly, $r^{Div}_{t_1,t_2}$ is the forward rate in dividend currency between t_1 and t_2. X_t is the current exchange rate and X_0 is the pre-specified exchange rate as described above. Both exchange rates are quoted in the units of Payoff currency per one unit of Dividend currency.

When one enters a forward contract at time t_0 that matures at T, he does not have to borrow funds to finance the stock purchase until day $t_0+\delta_0$ and the loan will be finally settled at day $T_0+\delta_T$. This time delay is caused by the settle date lag for stock purchases and sales. Thus, the strike of the forward contract should be $K = S_{t_0} \cdot \exp\left(r_{t_0+\delta_0, T+\delta_T}\left(T+\delta_T - t_0 - \delta_0\right)\right)$.

From day t_0 to day $t_0+\delta_0$, if the stock price does not move, the trader would like to see a zero *P&L* since he has not borrowed any money. From day $t_0+\delta_0+1$ to day $T+\delta_T$, however, he should see interest borrow cost accrue every day. To accomplish this, the following *MTM* strategy is proposed. Let t be the valuation date. For simplicity of illustration, dividends are not considered in this section.

For $t_0 \le t \le t_0+\delta_0$,

$$
\begin{aligned}
V_t &= S_t - K \cdot \exp\left(-r_{t_0+\delta_0, T+\delta_T} \cdot \left(T+\delta_T - t_0 - \delta_0\right)\right) \\
&= S_t - S_0 \cdot \exp\left(r_{t_0+\delta_0, T+\delta_T} \cdot \left(T+\delta_T - t_0 - \delta_0\right)\right) \cdot \exp\left(-r_{t_0+\delta_0, T+\delta_T} \cdot \left(T+\delta_T - t_0 - \delta_0\right)\right) \\
&= S_t - S_0 .
\end{aligned}
$$

Thus, if stock price does not move from day t_0 to day $t_0+\delta_0$, i.e., $S_t = S_0$, there will be zero *P&L* at the end of every day.

For $t_0+\delta_0+1 \le t \le T$,

$$
\begin{aligned}
V_t &= S_t - K \cdot \exp\left(-r_{t, T+\delta_T}\left(T+\delta_T - t\right)\right) \\
&= S_t - S_0 \cdot \exp\left(r_{t_0+\delta_0, T+\delta_T}\left(T+\delta_T - t_0 - \delta_0\right)\right) \cdot \exp\left(-r_{t, T+\delta_T}\left(T+\delta_T - t\right)\right) \\
&= S_t - S_0 \cdot \exp(r_{t_0+\delta_0, t}(t - t_0 - \delta_0))
\end{aligned}
$$

Thus, if the stock price never moves, $S_t = S_0$, the *MTM* should be the accrued interest cost. Note, in this case, $t+\delta_t$ is also set to be t for *PV(Div)* and *REI(Div)* in equations (6) and (7).

Since the stock is sold on day *T*, its value should stay the same as S_T from Day T to $T+\delta_T$. Thus, the *MTM* of the forward should be

$$
\begin{aligned}
V_t &= S_T - K \cdot \exp(-r_{t,T+\delta_T}(T+\delta_T - t)) \\
&= S_T - S_0 \cdot \exp\left(r_{t_0+\delta_0,T+\delta_T}\left(T+\delta_T - t_0 - \delta_0\right)\right) \cdot \exp\left(-r_{t,T+\delta_T}\left(T+\delta_T - t\right)\right) \\
&= S_T - S_0 \cdot \exp(r_{t_0+\delta_0,t}(t - t_0 - \delta_0))
\end{aligned}
$$

Thus, on the last day $T+\delta_T$, $V_{T+\delta_T} = S_T - S_0 \exp\left(r_{t_0+\delta_0,T+\delta_T}\left(T+\delta_T - t_0 - \delta_0\right)\right) = S_T - K$. Note, in this case, $t+\delta_t$ is also set to be t and $S_t = S_T$ for *PV(Div)* and *REI(Div)* in equations (6) and (7).

References:

https://finpricing.com/product.html