Defaultable Derivative Pricing Model

Tim Xiao

ABSTRACT

This article presents a comprehensive framework for valuing financial instruments subject to credit risk. In particular, we focus on the impact of default dependence on asset pricing, as correlated default risk is one of the most pervasive threats in financial markets. We bring the concept of *comvariance* into the area of credit risk modeling to capture the statistical relationship among three or more random variables. Furthermore, we define a new statistics, *comrelation*, as a scaled version of comvariance. Accounting for default correlations and comrelations becomes important in determining CDS premia, especially during the credit crisis. We find that the default comvariance/comrelation has substantial effects on the asset pricing and risk management, which have never been documented.

Key Words: asset pricing; credit risk modeling; credit risk; collateralization; comvariance; comrelation; correlation.

1 Introduction

A broad range of financial instruments bear credit risk. Credit risk may be unilateral, bilateral, or multilateral. Some instruments such as, loans, bonds, etc, by nature contain only unilateral credit risk because only the default risk of one party appears to be relevant, whereas some other instruments, such as, over the counter (OTC) derivatives, securities financing transactions (SFT), and credit derivatives, bear bilateral or multilateral credit risk because two or more parties are susceptible to default risk.

There are two primary types of models that attempt to describe default processes in the literature: structural models and reduced-form (or intensity) models. Many practitioners in the credit trading arena have tended to gravitate toward the reduced-from models given their mathematical tractability. They can be made consistent with the risk-neutral probabilities of default backed out from corporate bond prices or credit default swap (CDS*)* spreads/premia.

Central to the reduced-form models is the assumption that multiple defaults are independent conditional on the state of the economy. In reality, however, the default of one party might affect the default probabilities of other parties. Collin-Dufresne et al. (2003) and Zhang and Jorion (2007) find that a major credit event at one firm is associated with significant increases in the credit spreads of other firms. Giesecke (2004), Das et al. (2006), and Lando and Nielsen (2010) find that a defaulting firm can weaken the firms in its network of business links. These findings have important implications for the management of credit risk portfolios, where default relationships need to be explicitly modeled.

The main drawback of the conditionally independent assumption or the reduced-form models is that the range of default correlations that can be achieved is typically too low when compared with empirical default correlations (see Das et al. (2007)). The responses to correct this weakness can be generally classified into two categories: endogenous default relationship approaches and exogenous default relationship approaches.

The endogenous approaches include the contagion (or infectious) models and frailty models. The frailty models (see Duffie et al. (2009), Koopman et al. (2011), etc) describe default clustering based on some unobservable explanatory variables. In variations of contagion or infectious type models (see Davis and Lo (2001), Jarrow and Yu (2001), etc.), the assumption of conditional independence is relaxed and default intensities are made to depend on default events of other entities. Contagion and frailty models fill an important gap but at the cost of analytic tractability. They can be especially difficult to implement for large portfolios.

The exogenous approaches (see Li (2000), Laurent and Gregory (2005), Hull and White (2004), Brigo et al. (2011), etc) attempt to link marginal default probability distributions to the joint default probability distribution through some external functions. Due to their simplicity in use, the exogenous approaches become very popular in practice.

Collateralization is one of the most important and widespread credit risk mitigation techniques used in derivatives transactions. According the ISDA (2012), 71% of all OTC derivatives transactions are subject to collateral agreements. The use of collateral in the financial markets has increased sharply over the past decade, yet the research on collateralized valuation is relatively sparse. Previous studies seem to turn away from direct and detailed modeling of collateralization (see Fuijii and Takahahsi (2012)). For example, Johannes and Sundaresan (2007), and Fuijii and Takahahsi (2012) characterize collateralization via a cost-of-collateral instantaneous rate (or stochastic dividend or convenience yield). Piterbarg (2010) regards collateral as a regular asset in a portfolio and uses the replication approach to price collateralized contracts.

2 Pricing Financial Instruments Subject to Bilateral Credit Risk

In the reduced-form approach, the stopping (or default) time τ_i of firm *i* is modeled as a Cox arrival process (also known as a doubly stochastic Poisson process) whose first jump occurs at default and is defined by,

$$\tau_i = \inf\left\{t : \int_0^t h_i(s, Z_s)ds \geq H_i\right\} \tag{1}$$

where $h_i(t)$ or $h_i(t, Z_t)$ denotes the stochastic hazard rate or arrival intensity dependent on an exogenous common state Z_t, and H_i is a unit exponential random variable independent of Z_t.

It is well-known that the survival probability from time t to s in this framework is defined by

$$p_i(t,s) := P_i(\tau > s \mid \tau > t, Z_t) = \exp\left(-\int_t^s h_i(u)du\right) \tag{2a}$$

The default probability for the period (t, s) in this framework is given by

$$q_i(t,s) := P_i(\tau \le s \mid \tau > t, Z_t) = 1 - p_i(t,s) = 1 - \exp\left(-\int_t^s h_i(u)du\right) \tag{2b}$$

There is ample evidence that corporate defaults are correlated. The default of a firm's counterparty might affect its own default probability. Thus, default correlation/dependence arises due to the counterparty relations.

Two counterparties are denoted as A and B. The binomial default rule considers only two possible states: default or survival. Therefore, the default indicator Y_j for party j (j=A, B) follows a Bernoulli distribution, which takes value 1 with default probability q_j, and value 0 with survival probability p_j, i.e., $P\{Y_j = 0\} = p_j$ and $P\{Y_j = 1\} = q_j$. The marginal default distributions can be determined by the reduced-form models. The joint distributions of a multivariate Bernoulli variable can be easily obtained via the marginal distributions by introducing extra correlations.

Consider a pair of random variables (Y_A, Y_B) that has a bivariate Bernoulli distribution. The joint probability representations are given by

$$p_{00} := P(Y_A = 0, Y_B = 0) = p_A p_B + \sigma_{AB} \tag{3a}$$

$$p_{01} := P(Y_A = 0, Y_B = 1) = p_A q_B - \sigma_{AB} \tag{3b}$$

$$p_{10} := P(Y_A = 1, Y_B = 0) = q_A p_B - \sigma_{AB} \tag{3c}$$

$$p_{11} := P(Y_A = 1, Y_B = 1) = q_A q_B + \sigma_{AB} \tag{3d}$$

where $E(Y_j) = q_j$, $\sigma_j^2 = p_j q_j$, and $\sigma_{AB} := E[(Y_A - q_A)(Y_B - q_B)] = \rho_{AB}\sigma_A\sigma_B = \rho_{AB}\sqrt{q_A p_A q_B p_B}$ where ρ_{AB} denotes the default correlation coefficient, and σ_{AB} denotes the default covariance.

A critical ingredient of the pricing of a bilateral defaultable instrument is the default settlement rules. There are two rules in the market. The *one-way payment rule* was specified by the early International

Swap Dealers Association (ISDA) master agreement. The non-defaulting party is not obligated to compensate the defaulting party if the remaining market value of the instrument is positive for the defaulting party. The *two-way payment rule* is based on current ISDA documentation. The non-defaulting party will pay the full market value of the instrument to the defaulting party if the contract has positive value to the defaulting party.

1.1 Risky valuation without collateralization

Consider a defaultable instrument that promises to pay a X_T from party *B* to party *A* at maturity date *T*, and nothing before date *T*. The payoff X_T may be positive or negative, i.e. the instrument may be either an asset or a liability to each party. All calculations are from the perspective of party *A*.

We divide the time period (*t*, *T*) into *n* very small time intervals (Δt) and use the approximation $\exp(y) \approx 1 + y$ provided that *y* is very small. The survival and the default probabilities for the period (*t*, $t + \Delta t$) are given by

$$\hat{p}(t) := p(t, t+\Delta t) = \exp(-h(t)\Delta t) \approx 1 - h(t)\Delta t \tag{4a}$$

$$\hat{q}(t) := q(t, t+\Delta t) = 1 - \exp(-h(t)\Delta t) \approx h(t)\Delta t \tag{4b}$$

Suppose that the value of the instrument at time $t + \Delta t$ is $V(t + \Delta t)$ that can be an asset or a liability. There are a total of four ($2^2 = 4$) possible states shown in Table 1.

The risky value of the instrument at time *t* is the discounted expectation of all the payoffs and is given by

$$\begin{aligned} V(t) = E\{\exp(-r(t)\Delta t)[&1_{V(t+\Delta t)\geq 0}\langle p_{00}(t) + \varphi_B(t)p_{01}(t) + \overline{\varphi}_B(t)p_{10}(t) + \varphi_{AB}(t)p_{11}(t)\rangle V(t+\Delta t)|\mathcal{F}_t \\ &+ 1_{V(t+\Delta t)<0}\langle p_{00}(t) + \overline{\varphi}_A(t)p_{01}(t) + \varphi_A(t)p_{10}(t) + \varphi_{AB}(t)p_{11}(t)\rangle V(t+\Delta t)|\mathcal{F}_t]\} \\ \approx E\{\exp[-(r(t) + 1_{V(t+\Delta t)\geq 0}&l_B(t) + 1_{V(t+\Delta t)<0}l_A(t))\Delta t]V(t+\Delta t)|\mathcal{F}_t\} = E\{\exp(-g(t)\Delta t)V(t+\Delta t)|\mathcal{F}_t\} \end{aligned} \tag{5a}$$

where

$$g(t) = r(t) + 1_{V(t+\Delta t)\geq 0}l_B(t) + 1_{V(t+\Delta t)<0}l_A(t) \tag{5b}$$

$$l_B(t) = (1-\varphi_B(t))h_B(t) + (1-\overline{\varphi}_B(t))h_A(t) - (1-\varphi_B(t) - \overline{\varphi}_B(t) + \varphi_{AB}(t))\rho_{AB}(t)\sqrt{h_A(t)h_B(t)} \tag{5c}$$

$$l_A(t) = (1-\varphi_A(t))h_A(t) + (1-\overline{\varphi}_A(t))h_B(t) - (1-\varphi_A(t) - \overline{\varphi}_A(t) + \varphi_{AB}(t))\rho_{AB}(t)\sqrt{h_A(t)h_B(t)} \tag{5d}$$

where 1_Y is an indicator function that is equal to one if Y is true and zero otherwise, $E\{\bullet|\mathcal{F}_t\}$ is the expectation conditional on the $\mathcal{F}_t$, $r(t)$ is the risk-free short rate, and φ_i is the recovery rate.

The pricing equation above keeps terms of order Δt. All higher order terms of Δt are omitted. Similarly, we have

$$V(t+\Delta t) = E\{\exp(-g(t+\Delta t)\Delta t)V(t+2\Delta t)|\mathcal{F}_{t+\Delta t}\} \tag{6}$$

Note that $\exp(-g(t)\Delta t)$ is $\mathcal{F}_{t+\Delta t}$ -measurable. By definition, an $\mathcal{F}_{t+\Delta t}$ -measurable random variable is a random variable whose value is known at time $t+\Delta t$. Based on the *taking out what is known* and *towe*r properties of conditional expectation, we have

$$\begin{aligned} V(t) &= E\{\exp(-g(t)\Delta t)V(t+\Delta t)|\mathcal{F}_t\} = E\{\exp(-g(t)\Delta t)E[\exp(-g(t+\Delta t)\Delta t)V(t+2\Delta t)|\mathcal{F}_{t+\Delta t}]|\mathcal{F}_t\} \\ &= E\left\{\exp\left(-\sum_{i=0}^{1} g(t+i\Delta t)\Delta t\right)V(t+2\Delta t)|\mathcal{F}_t\right\} \end{aligned} \tag{7}$$

By recursively deriving from t forward over T where $V(T) = X_T$ and taking the limit as Δt approaches zero, we obtain

$$V(t) = E\{G(t,T)X_T|\mathcal{F}_t\} = E\left\{\exp\left[-\int_t^T g(u)du\right]X_T|\mathcal{F}_t\right\} \tag{8}$$

We may think of $G(t,T)$ as the bilateral risk-adjusted discount factor and $g(u)$ as the bilateral risk-adjusted short rate. Equation (8) has a general form that applies in a particular situation where we assume that parties *A* and *B* have independent default risks, i.e. $\rho_{AB} = 0$ and $\varphi_{AB} = 0$. Thus, we have:

$$V(t) = E\{\overline{G}(t,T)X_T|\mathcal{F}_t\} = E\left\{\exp\left[-\int_t^T \overline{g}(u)du\right]X_T|\mathcal{F}_t\right\} \tag{9a}$$

where

$$\overline{g}(u) = r(u) + 1_{V(u)\geq 0}\bar{l}_B(u) + 1_{V(u)<0}\bar{l}_A(u) \tag{9b}$$

$$\bar{l}_B(u) = (1-\varphi_B(u))h_B(u) + (1-\overline{\varphi}_B(u))h_A(u) \tag{9c}$$

$$\bar{l}_A(u) = (1-\varphi_A(u))h_A(u) + (1-\overline{\varphi}_A(u))h_B(u) \tag{9d}$$

Equation (9) is the same as equation (2.5') in Duffie and Huang (1996).

In theory, a default may happen at any time, i.e., a risky contract is continuously defaultable. This Continuous Time Risky Valuation Model is accurate but sometimes complex and expensive. For simplicity, people sometimes prefer the Discrete Time Risky Valuation Model that assumes that a default may only happen at some discrete times. A natural selection is to assume that a default may occur only on the payment dates. Fortunately, the level of accuracy for this discrete approximation is well inside the typical bid-ask spread for most applications (see O'Kane and Turnbull (2003)). From now on, we will focus on the discrete setting only, but many of the points we make are equally applicable to the continuous setting.

If we assume that a default may occur only on the payment date, the risky value of the instrument in a discrete-time setting is given by

$$\begin{aligned} V(t) = E\{D(t,T)[1_{X_T \geq 0}\langle p_{00}(t,T) + \varphi_B(T) p_{01}(t,T) + \bar{\varphi}_B(T) p_{10}(t,T) + \varphi_{AB}(T) p_{11}(t,T)\rangle X_T | \mathcal{F}_t \\ + 1_{X_T < 0}\langle p_{00}(t,T) + \bar{\varphi}_A(T) p_{01}(t,T) + \varphi_A(T) p_{10}(t,T) + \varphi_{AB}(T) p_{11}(t,T)\rangle X_T | \mathcal{F}_t]\} \\ = E[D(t,T)(1_{X_T \geq 0} k_B(t,T) + 1_{X_T < 0} k_A(t,T))X_T | \mathcal{F}_t] = E(K(t,T) X_T | \mathcal{F}_t) \end{aligned} \quad (10a)$$

where

$$\begin{aligned} k_B(t,T) = p_B(t,T)p_A(t,T) + \varphi_B(T)q_B(t,T)p_A(t,T) + \bar{\varphi}_B(T)p_B(t,T)q_A(t,T) \\ + \varphi_{AB}(T)q_B(t,T)q_A(t,T) + \sigma_{AB}(t,T)(1 - \varphi_B(T) - \bar{\varphi}_B(T) + \varphi_{AB}(T)) \end{aligned} \quad (10b)$$

$$\begin{aligned} k_A(t,T) = p_B(t,T)p_A(t,T) + \varphi_A(T)q_A(t,T)p_B(t,T) + \bar{\varphi}_A(T)p_A(t,T)q_B(t,T) \\ + \varphi_{AB}(T)q_B(t,T)q_A(t,T) + \sigma_{AB}(t,T)(1 - \varphi_A(T) - \bar{\varphi}_A(T) + \varphi_{AB}(T)) \end{aligned} \quad (10c)$$

where $D(t,\tau)$ denotes the stochastic risk-free discount factor at t for the maturity T given by

$$D(t,T) = \exp\left[-\int_t^T r(u)du\right] \quad (10d)$$

We may think of $K(t,T)$ as the risk-adjusted discount factor, and $k_A(t,T)$ and $k_B(t,T)$ as the adjustment factors. Equation (10) tells us that the bilateral risky price of a single-payment instrument can be expressed as the present value of the payoff discounted by a risk-adjusted discount factor that has a switching-type dependence on the sign of the payoff.

Equation (10) can be easily extended from one-period to multiple-periods. Suppose that a defaultable instrument has *m* cash flows. Let the *m* cash flows be represented as $X_1, \ldots, X_m$ with payment dates $T_1, \ldots, T_m$. Each cash flow may be positive or negative. We have the following proposition.

Proposition 1: *The risky value of the multiple-payment instrument is given by*

$$V(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} K(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \tag{11a}$$

where $t = T_0$ *and*

$$K(T_j, T_{j+1}) = D(T_j, T_{j+1})\left(1_{(X_{j+1}+V(T_{j+1}))\geq 0} k_B(T_j, T_{j+1}) + 1_{(X_{j+1}+V(T_{j+1}))<0} k_A(T_j, T_{j+1})\right) \tag{11b}$$

where $k_A(T_j, T_{j+1})$ *and* $k_B(T_j, T_{j+1})$ *are defined in Equation (10).*

Proof: See the Appendix.

From Proposition 1, we can see that the intermediate values are vital to determine the final price. For a payment interval, the current risky value has a dependence on the future risky value. Only on the final payment date T_m, the value of the instrument and the maximum amount of information needed to determine the risk-adjusted discount factor are revealed.

1.2 Risky valuation with collateralization

Collateralization is the most important and widely used technique in practice to mitigate credit risk. The posting of collateral is regulated by the Credit Support Annex (CSA) that specifies a variety of terms including the threshold, the independent amount, and the minimum transfer amount (MTA), etc. The threshold is the unsecured credit exposure that a party is willing to bear. The minimum transfer amount is the smallest amount of collateral that can be transferred. The independent amount plays the same role as the initial margin (or haircuts).

The collateral amount posted at time *t* is given by

$$C(t) = \begin{cases} V(t) - H(t) & \text{if } V(t) > H(t) \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

where $H(t)$ is the collateral threshold. In particular, $H(t)=0$ corresponds to full-collateralization[1]; $H>0$ represents partial/under-collateralization; and $H<0$ is associated with over-collateralization. Full collateralization becomes increasingly popular at the transaction level. In this paper, we focus on full collateralization only, i.e., $C(t)=V(t)$.

The main role of collateral should be viewed as an improved recovery in the event of a counterparty default. According to Bankruptcy law, if there has been no default, the collateral is returned to the collateral giver by the collateral taker. If a default occurs, the collateral taker possesses the collateral. In other words, collateral does not affect the survival payment; instead, it takes effect on the default payment only.

The value of the collateralized instrument at time *t* is the discounted expectation of all the payoffs and is given by

$$\begin{aligned} V(t) &= E\left[D(t,u)\left(p_{00}(t,u)V(u)+p_{01}(t,u)C(u)+p_{10}(t,u)C(u)+p_{11}(t,u)C(u)\right)\middle|\mathcal{F}_t\right] \\ &= E\left[D(t,u)p_{00}(t,u)V(u)\,|\,\mathcal{F}_t\right]+\left[1-E\left(p_{00}(t,u)\,|\,\mathcal{F}_t\right)\right]V(t) \end{aligned} \tag{13a}$$

or

$$V(t)=E\left[D(t,s)p_{00}(t,s)V(s)\,|\,\mathcal{F}_t\right]/\,E\left(p_{00}(t,s)\,|\,\mathcal{F}_t\right) \tag{13b}$$

If we assume that default probabilities are uncorrelated with interest rates and payoffs[2], we have

$$V(t)=E\left[D(t,s)V(s)\,|\,\mathcal{F}_t\right] \tag{14}$$

Equation (14) is the formula for the risk-free valuation. Thus, we have the following proposition.

Proposition 2*: If a bilateral risky instrument is fully collateralized, the risky value of the instrument is equal to the risk-free value, as shown in equation (14).*

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[1] There are three types of collateralization: Full-collateralization is a process where the posting of collateral is equal to the current MTM value. Partial/under-collateralization is a process where the posting of collateral is less than the current MTM value. Over-collateralization is a process where the posting of collateral is greater than the current MTM value.

[2] Moody's Investor's Service (2000) presents statistics that suggest that the correlations between interest rates, default probabilities, and recovery rates are very small and provides a reasonable comfort level for the uncorrelated assumption.

Since an *IRS* is a typical bilateral risky contract, Proposition 2 squares with the results of Johannes and Sundaresan (2007), and is also consistent with the current market practice in which market participants commonly assume fully collateralized swaps are risk-free and it is common to build models of swap rates assuming that swaps are free of counterparty risk.

1.3 Numerical results

We first assume that i) counterparties *A* and *B* have independent default risks; ii) the hazard rates are deterministic; and iii) both parties have a constant recovery of 60%. We use the LMM to evolve the interest rates and then price the risky IRS according to Proposition 1. The risky swap rates are computed and shown in Table 3.

From Table 3, we derive the following conclusions: First, a fixed-rate payer with lower credit quality (higher credit risk) pays a higher fixed rate. Second, a credit spread of about 100 basis points translates into a swap spread of about 1.3 basis points. Finally, the credit impact on swap rates is approximately linear within the range of normally encountered credit quality. This confirms the findings of Duffie and Huang (1996). Intuitively, a risk-free floating-rate payer demands a higher fixed rate if the fixed-rate payer has a lower credit score.

We next present some new results. Assume that party *A* has an 'A+300bps' credit quality, i.e., a '300 basis points' parallel shift in the A-rated credit spreads, and party *B* has an 'A' credit quality. The risky swap rate with asymmetric credit qualities is calculated as 0.03436.

Assume $\varphi_{AB} = 0.5$. The effect of the default correlation ρ_{AB} on swap rate is shown in Figure 1. We can draw the following conclusions from the results: First, the counterparty default correlation and the swap rates have a negative relation, i.e., a negative sensitivity of swap rates to changes in counterparty default correlation is obtained. Second, the graph suggests an almost linear relationship between the swap rates and the default correlation. Finally, the impact of the default correlation is modest (e.g., in the range of [-2, 2] basis points).

Bilateral credit risk modeling is probably the simplest example involving default dependency, but it shows several essential features for modeling correlated credit risk, which will help the reader better understand the increasingly complex cases in the following section.

2 Pricing Financial Instruments Subject to Multilateral Credit Risk

The interest in the financial industry for the modeling and pricing of multilateral defaultable instruments arises mainly in two respects: in the management of credit risk at a portfolio level and in the valuation of credit derivatives. Central to the pricing and risk management of credit derivatives and credit risk portfolios is the issue of default relationships.

Let us discuss the three-party case first. A CDS is a good example of a trilateral defaultable instrument where the three parties are counterparties *A*, *B* and reference entity *C*. In a standard CDS contract one party purchases credit protection from another party, to cover the loss of the face value of a reference entity following a credit event. The protection buyer makes periodic payments to the seller until the maturity date or until a credit event occurs. A credit event usually requires a final accrual payment by the buyer and a loss protection payment by the protection seller. The protection payment is equal to the difference between par and the price of the cheapest to deliver (CTD) asset of the reference entity on the face value of the protection.

The default indicator Y_j for firm *j* (*j* = *A* or *B* or *C*) follows a Bernoulli distribution, which takes value 1 with default probability q_j, and value 0 with survival probability p_j. The joint probability representations of a trivariate Bernoulli distribution (see Teugels (1990)) are given by

$$p_{000} := P(Y_A = 0, Y_B = 0, Y_C = 0) = p_A p_B p_C + p_C \sigma_{AB} + p_B \sigma_{AC} + p_A \sigma_{BC} - \theta_{ABC} \tag{15a}$$

$$p_{100} := P(Y_A = 1, Y_B = 0, Y_C = 0) = q_A p_B p_C - p_C \sigma_{AB} - p_B \sigma_{AC} + q_A \sigma_{BC} + \theta_{ABC} \tag{15b}$$

$$p_{010} := P(Y_A = 0, Y_B = 1, Y_C = 0) = p_A q_B p_C - p_C \sigma_{AB} + q_B \sigma_{AC} - p_A \sigma_{BC} + \theta_{ABC} \tag{15c}$$

$$p_{001} := P(Y_A = 0, Y_B = 0, Y_C = 1) = p_A p_B q_C + q_C \sigma_{AB} - p_B \sigma_{AC} - p_A \sigma_{BC} + \theta_{ABC} \tag{15d}$$

$$p_{110} := P(Y_A = 1, Y_B = 1, Y_C = 0) = q_A q_B p_C + p_C \sigma_{AB} - q_B \sigma_{AC} - q_A \sigma_{BC} - \theta_{ABC} \tag{15e}$$

$$p_{101} := P(Y_A = 1, Y_B = 0, Y_C = 1) = q_A p_B q_C - q_C \sigma_{AB} + p_B \sigma_{AC} - q_A \sigma_{BC} - \theta_{ABC} \tag{15f}$$

$$p_{011} := P(Y_A = 0, Y_B = 1, Y_C = 1) = p_A q_B q_C - q_C \sigma_{AB} - q_B \sigma_{AC} + p_A \sigma_{BC} - \theta_{ABC} \tag{15g}$$

$$p_{111} := P(Y_A = 1, Y_B = 1, Y_C = 1) = q_A q_B q_C + q_C \sigma_{AB} + q_B \sigma_{AC} + q_A \sigma_{BC} + \theta_{ABC} \tag{15h}$$

where

$$\theta_{ABC} := E\left((Y_A - q_A)(Y_B - q_B)(Y_C - q_C)\right) \tag{15i}$$

Equation (15) tells us that the joint probability distribution of three defaultable parties depends not only on the bivariate statistical relationships of all pair-wise combinations (e.g., σ_{ij}) but also on the trivariate statistical relationship (e.g., θ_{ABC}). θ_{ABC} was first defined by Deardorff (1982) as *comvariance*, who use it to correlate three random variables that are the value of commodity net imports/exports, factor intensity, and factor abundance in international trading.

We introduce the concept of *comvariance* into credit risk modeling arena to exploit any statistical relationship among multiple random variables. Furthermore, we define a new statistic, *comrelation,* as a scaled version of comvariance (just like correlation is a scaled version of covariance) as follows:

Definition 1*: For three random variables* X_A, X_B, *and* X_C, *let* μ_A, μ_B, *and* μ_C *denote the means of* X_A, X_B, *and* X_C. *The comrelation of* X_A, X_B, *and* X_C *is defined by*

$$\zeta_{ABC} = \frac{E\left[(X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)\right]}{\sqrt[3]{E|X_A - \mu_A|^3 \times E|X_B - \mu_B|^3 \times E|X_C - \mu_C|^3}} \tag{16}$$

According to the Holder inequality, we have

$$\left|E\left((X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)\right)\right| \le E\left|(X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)\right| \le \sqrt[3]{E|X_A - \mu_A|^3 \times E|X_B - \mu_B|^3 \times E|X_C - \mu_C|^3} \tag{17}$$

Obviously, the comrelation is in the range of [-1, 1]. Given the comrelation, Equation (15i) can be rewritten as

$$\begin{aligned} \theta_{ABC} &:= E\left((Y_A - q_A)(Y_B - q_B)(X_C - q_C)\right) = \zeta_{ABC} \sqrt[3]{E|Y_A - q_A|^3 \times E|Y_B - q_B|^3 \times E|Y_C - q_C|^3} \\ &= \zeta_{ABC} \sqrt[3]{\left|p_A q_A (p_A^2 + q_A^2) p_B q_B (p_B^2 + q_B^2) p_C q_C (p_C^2 + q_C^2)\right|} \end{aligned} \tag{18}$$

where $E(Y_j) = q_j$ and $E|Y_j - q_j|^3 = p_j q_j (p_j^2 + q_j^2)$, *j=A, B, or C.*

If we have a series of *n* measurements of X_A, X_B, *and* X_C written as x_{Ai}, x_{Bi} and x_{Ci} where *i* = 1,2,…,*n*, the sample *comrelation coefficient* can be obtained as:

$$\zeta_{ABC} = \frac{\sum_{i=1}^{n}(x_{Ai} - \mu_A)(x_{Bi} - \mu_B)(x_{Ci} - \mu_C)}{\sqrt[3]{\sum_{i=1}^{n}|x_{Ai} - \mu_A|^3 \times \sum_{i=1}^{n}|x_{Bi} - \mu_B|^3 \times \sum_{i=1}^{n}|x_{Ci} - \mu_C|^3}} \tag{19}$$

More generally, we define the *comrelation* in the context of *n* random variables as

Definition 2*: For n random variables* X_1, X_2*,…,* X_n*, let* μ_i *denote the mean of* X_i *where i=1,..,n. The comrelation of* X_1, X_2*,…,* X_n *is defined as*

$$\zeta_{12...n} = \frac{E[(X_1 - \mu_1)(X_2 - \mu_2)\cdots(X_n - \mu_n)]}{\sqrt[n]{E|X_1 - \mu_1|^n \times E|X_2 - \mu_2|^n \cdots \times E|X_n - \mu_n|^n}} \tag{20}$$

The correlation is just a specific case of the comrelation where n = 2. Again, the comrelation $\zeta_{12...n}$ is in the range of [-1, 1] according to the Holder inequality.

2.1 Risky valuation without collateralization

Let valuation date be *t*. Suppose that a CDS has *m* scheduled payments. Let each payment be represented as $X_i = -sN\delta(T_{i-1}, T_i)$ with payment dates $T_1, \ldots, T_m$ where *i=1,,,,m,* $\delta(T_{i-1}, T_i)$ denotes the accrual factor for period (T_{i-1}, T_i), *N* denotes the notional/principal, and *s* denotes the CDS premium. Party *A* pays the premium/fee to party *B* if reference entity *C* does not default. In return, party *B* agrees to pay the protection amount to party *A* if reference entity *C* defaults before the maturity. We have the following proposition.

Proposition 3: *The value of the multiple-payment CDS is given by*

$$V(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} O(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] + \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-2} O(T_j, T_{j+1})\right) \Omega(T_{i-1}, T_i) R(T_{i-1}, T_i) \middle| \mathcal{F}_t\right] \tag{21a}$$

where $t = T_0$ *and*

$$O(T_j, T_{j+1}) = 1_{(V(T_{j+1}) + X_{j+1}) \geq 0}\, \phi_B(T_j, T_{j+1}) + 1_{(V(T_{j+1}) + X_{j+1}) < 0}\, \phi_A(T_j, T_{j+1}) \tag{21b}$$

$$\begin{aligned}\phi_A(T_j,T_{j+1}) = \{& p_A(T_j,T_{j+1})p_B(T_j,T_{j+1})p_C(T_j,T_{j+1}) + q_A(T_j,T_{j+1})p_B(T_j,T_{j+1})p_C(T_j,T_{j+1})\varphi_A(T_{j+1}) \\ &+ p_A(T_j,T_{j+1})q_B(T_j,T_{j+1})p_C(T_j,T_{j+1})\overline{\varphi}_A(T_{j+1}) + q_A(T_j,T_{j+1})q_B(T_j,T_{j+1})p_C(T_j,T_{j+1})\varphi_{AB}(T_{j+1}) \\ &+ p_C(T_j,T_{j+1})\sigma_{AB}(T_j,T_{j+1})\left(1-\varphi_A(T_{j+1}) - \overline{\varphi}_A(T_{j+1}) + \varphi_{AB}(T_{j+1})\right) \\ &+ \sigma_{AC}(T_j,T_{j+1})\left[p_B(T_j,T_{j+1})\left(1-\varphi_A(T_{j+1})\right) + q_B(T_j,T_{j+1})\left(\overline{\varphi}_A(T_{j+1}) - \varphi_{AB}(T_{j+1})\right)\right] \\ &+ \sigma_{BC}(T_j,T_{j+1})\left[p_A(T_j,T_{j+1})\left(1-\overline{\varphi}_A(T_{j+1})\right) + q_A(T_j,T_{j+1})\left(\varphi_A(T_{j+1}) - \varphi_{AB}(T_{j+1})\right)\right] \\ &+ \theta_{ABC}(T_j,T_{j+1})\left(-1+\overline{\varphi}_A(T_{j+1}) - \varphi_{AB}(T_{j+1}) + \varphi_A(T_{j+1})\right)\}D(T_j,T_{j+1})\end{aligned} \tag{21c}$$

$$\begin{aligned}\phi_B(T_j,T_{j+1}) = \{& p_A(T_j,T_{j+1})p_B(T_J,T_{j+1})p_C(T_j,T_{j+1}) + q_A(T_j,T_{j+1})p_B(T_j,T_{j+1})p_C(T_j,T_{j+1})\overline{\varphi}_B(T_{j+1}) \\ &+ p_A(T_j,T_{j+1})q_B(T_j,T_{j+1})p_C(T_j,T_{j+1})\varphi_B(T_{j+1}) + q_A(T_j,T_{j+1})q_B(T_j,T_{j+1})p_C(T_j,T_{j+1})\varphi_{AB}(T_{j+1}) \\ &+ p_C(T_j,T_{j+1})\sigma_{AB}(T_j,T_{j+1})\left(1-\varphi_B(T_{j+1}) - \overline{\varphi}_B(T_{j+1}) + \varphi_{AB}(T_{j+1})\right) \\ &+ \sigma_{AC}(T_j,T_{j+1})\left[p_B(T_j,T_{j+1})\left(1-\overline{\varphi}_B(T_{j+1})\right) + q_B(T_j,T_{j+1})\left(\varphi_B(T_{j+1}) - \varphi_{AB}(T_{j+1})\right)\right] \\ &+ \sigma_{BC}(T_j,T_{j+1})\left[p_A(T_j,T_{j+1})\left(1-\varphi_B(T_j)\right) + q_A(T_j,T_{j+1})\left(\overline{\varphi}_B(T_{j+1}) - \varphi_{AB}(T_{j+1})\right)\right] \\ &+ \theta(T_j,T_{j+1})\left(-1+\overline{\varphi}_B(T_{j+1}) - \varphi_{AB}(T_{j+1}) + \varphi_B(T_{j+1})\right)\}D(T_j,T_{j+1})\end{aligned} \tag{21d}$$

$$\begin{aligned}\Omega(T_j,T_{j+1}) = \{& p_A(T_j,T_{j+1})p_B(T_j,T_{j+1})q_C(T_j,T_{j+1}) + q_A(T_j,T_{j+1})p_B(T_j,T_{j+1})q_C(T_j,T_{j+1})\overline{\varphi}_B(T_{j+1}) \\ &+ p_A(T_j,T_{j+1})q_B(T_j,T_{j+1})q_C(T_j,T_{j+1})\varphi_B(T_{j+1}) + q_A(T_j,T_{j+1})q_B(T_j,T_{j+1})q_C(T_j,T_{j+1})\varphi_{AB}(T_{j+1}) \\ &+ q_C(T_j,T_{j+1})\sigma_{AB}(T_j,T_{j+1})\left(1-\varphi_B(T_{j+1}) - \overline{\varphi}_B(T_{j+1}) + \varphi_{AB}(T_{j+1})\right) \\ &- \sigma_{AC}(T_j,T_{j+1})\left[p_B(T_j,T_{j+1})\left(1-\overline{\varphi}_B(T_{j+1})\right) + q_B(T_j,T_{j+1})\left(\varphi_B(T_{j+1}) - \varphi_{AB}(T_{j+1})\right)\right] \\ &- \sigma_{BC}(T_j,T_{j+1})\left[p_A(T_j,T_{j+1})\left(1-\varphi_B(T_{j+1})\right) + q_A(T_j,T_{j+1})\left(\overline{\varphi}_B(T_{j+1}) - \varphi_{AB}(T_{j+1})\right)\right] \\ &+ \theta_{ABC}(T_j,T_{j+1})\left(1-\overline{\varphi}_B(T_{j+1}) + \varphi_{AB}(T_{j+1}) - \varphi_B(T_{j+1})\right)\}D(T_j,T_{j+1})\end{aligned} \tag{21e}$$

where $R(T_j,T_{j+1}) = \left(N\left(1-\varphi_C(T_{j+1})\right) - \alpha(T_j,T_{j+1})\right)$, $\alpha(T_j,T_{j+1}) = sN\delta(T_S,T)/2$, and $X_i = -sN\delta(T_j,T_{j+1})$.

Proof: See the Appendix.

We may think of $O(t,T)$ as the risk-adjusted discount factor for the premium and $\Omega(t,T)$ as the risk-adjusted discount factor for the default payment. Proposition 3 says that the pricing process of a multiple-payment instrument has a backward nature since there is no way of knowing which risk-adjusted discounting rate should be used without knowledge of the future value. Only on the maturity date, the value of an instrument and the decision strategy are clear. Therefore, the evaluation must be done in a backward fashion, working from the final payment date towards the present. This type of valuation process is referred to as backward induction.

Proposition 3 provides a general form for pricing a CDS. Applying it to a particular situation in which we assume that counterparties *A and B* are default-free, i.e., $p_j = 1$, $q_j = 0$, $\rho_{kl} = 0$, and $\varsigma_{ABC} = 0$, where *j=A or B* and *k, l=A, B, or C,* we derive the following corollary.

Corollary 1: *If counterparties A and B are default-free, the value of the multiple-payment CDS is given by*

$$\begin{aligned} V(t) &= \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} O(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] + \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-2} O(T_j, T_{j+1})\right) \Omega(T_{i-1}, T_i) R(T_{i-1}, T_i) \middle| \mathcal{F}_t\right] \\ &= \sum_{i=1}^{m} E\left[D(t, T_i) p_C(t, T_i) X_i \middle| \mathcal{F}_t\right] + \sum_{i=1}^{m} E\left[D(t, T_i) p_C(t, T_{i-1}) q_C(T_{i-1}, T_i) R(T_{i-1}, T_i) \middle| \mathcal{F}_t\right] \end{aligned} \tag{22}$$

where $O(T_{i-1}, T_i) = D(T_{i-1}, T_i) p_C(T_{i-1}, T_i)$; $\Omega(T_{i-1}, T_i) = D(T_{i-1}, T_i) q_C(T_{i-1}, T_i)$.

The proof of this corollary becomes straightforward according to Proposition 3 by setting ρ_{kl} =0, $\varphi_{AB} = 0$, $\varsigma_{ABC} = 0$, $p_j = 1$, $q_j = 0$, $p_C(t, T_i) = \prod_{g=0}^{i-1} p(T_g, T_{g+1})$, and $D(t, T_i) = \prod_{g=0}^{i-1} D(T_g, T_{g+1})$.

If we further assume that the discount factor and the default probability of the reference entity are uncorrelated and the recovery rate φ_C is constant, we have

Corollary 2: *Assume that i) counterparties A and B are default-free, ii) the discount factor and the default probability of the reference entity are uncorrelated; iii) the recovery rate* φ_C *is constant; the value of the multiple-payment CDS is given by*

$$V(t) = \sum_{i=1}^{m} P(t, T_i) \bar{p}_C(t, T_{i-1}) \bar{q}_C(T_{i-1}, T_i) \left(N(1 - \varphi_C) - \alpha(T_{i-1}, T_i)\right) - \sum_{i=1}^{m} P(t, T_i) \bar{p}_c(t, T_i) s N \delta(T_{i-1}, T_i) \tag{23}$$

where $P(t, T_i) = E[D(t, T_i) | \mathcal{F}_t]$ *denotes the bond price,* $\bar{p}_c(t, T_i) = E[p_c(t, T_i) | \mathcal{F}_t]$, $\bar{q}_c(t, T_i) = 1 - \bar{p}_c(t, T_i)$, $\bar{p}(t, T_{i-1}) \bar{q}(T_{i-1}, T_i) = \bar{p}(t, T_{i-1}) - \bar{p}(t, T_i)$.

This corollary is easily proved according to Corollary 1 by setting $E[XY | \mathcal{F}_t] = E[X | \mathcal{F}_t] E[Y | \mathcal{F}_t]$ when *X* and *Y* are uncorrelated. *Corollary 2 is the formula for pricing CDS in the market.*

Our methodology can be extended to the cases where the number of parties $n \geq 4$. A generating function for the (probability) joint distribution (see details in Teugels (1990)) of *n*-variate Bernoulli can be expressed as

$$p^{(n)} = \begin{bmatrix} p_n & -1 \\ q_n & 1 \end{bmatrix} \otimes \begin{bmatrix} p_{n-1} & -1 \\ q_{n-1} & 1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} p_1 & -1 \\ q_1 & 1 \end{bmatrix} \sigma^{(n)} \tag{24}$$

where $\otimes$ denotes the Kronecker product; $p^{(n)} = \{p_k^{(n)}\}$ and $\sigma^{(n)} = \{\sigma_k^{(n)}\}$ are vectors containing 2^n components: $p_k^{(n)} = p_{k_1, k_2, \ldots, k_n}$, $k = 1 + \sum_{i=1}^{n} k_i 2^{i-1}$, $k_i \in \{0,1\}$; $\sigma_k^{(n)} = \sigma_{k_1, k_2, \ldots, k_n} = E\left(\prod_{i=1}^{n} (Y_i - q_i)^{k_i}\right)$.

2.2 Risky valuation with collateralization

We assume that a CDS is fully collateralized, i.e., the posting of collateral is equal to the amount of the current *MTM* value: $C(t) = V(t)$. For a discrete one-period (*t*, *u*) economy, there are several possible states at time *u*: i) *A*, *B*, and *C* survive with probability p_{000}. The instrument value is equal to the market value $V(u)$; ii) *A* and *B* survive, but *C* defaults with probability p_{001}. The instrument value is the default payment $R(u)$; iii) For the remaining cases, either or both counterparties *A* and *B* default. The instrument value is the future value of the collateral $V(t)/D(t,u)$ (Here we consider the time value of money only). The value of the collateralized instrument at time *t* is the discounted expectation of all the payoffs and is given by

$$\begin{aligned} V(t) = E\{&D(t,u)[p_{000}(t,u)V(u) + p_{001}(t,u)R(u) \\ &+ (p_{100}(t,u) + p_{010}(t,u) + p_{110}(t,u) + p_{101}(t,u) + p_{011}(t,u) + p_{111}(t,u))V(t)/D(t,u)] | \mathcal{F}_t\} \\ = E\{&[D(t,u)(p_{000}(t,u)V(s) + p_{001}(t,u)R(u)) + (1 - p_{000}(t,u) - p_{001}(t,u))V(t)] | \mathcal{F}_t\} \end{aligned} \tag{25a}$$

or

$$\begin{aligned} E[(p_A(t,u)p_B(t,u) + \sigma_{AB}(t,u)) | \mathcal{F}_t]V(t) \\ = E\{[D(t,u)(p_C(t,u)V(u) + q_C(t,u)R(u))(p_A(t,u)p_B(t,u) + \sigma_{AB}(t,u)) \\ + D(t,u)(V(u) - R(u))(p_B(t,u)\sigma_{AC}(t,u) + p_A(t,u)\sigma_{BC}(t,u) - \theta_{ABC}(t,u))] | \mathcal{F}_t\} \end{aligned} \tag{25b}$$

If we assume that $(p_A(t,u)p_B(t,u) + \sigma_{AB}(t,u))$ and $D(t,u)(p_C(t,u)V(u) + q_C(t,u)R(u))$ are uncorrelated, we have

$$V(t) = V^F(t) + \xi_{ABC}(t,u)/\psi_{AB}(t,u) \tag{26a}$$

where

$$V^F(t) = E\{D(t,u)[p_C(t,u)V(u) + q_C(t,u)R(u)] | \mathcal{F}_t\} \tag{26b}$$

$$\psi_{AB}(t,u) = E\{[p_A(t,u)p_B(t,u) + \sigma_{AB}(t,u)] | \mathcal{F}_t\} \tag{26c}$$

$$\xi_{ABC}(t,u) = E\{[D(t,u)(p_B(t,u)\sigma_{AC}(t,u) + p_A(t,u)\sigma_{BC}(t,u) - \theta_{ABC}(t,u))(V(u) - R(u))] | \mathcal{F}_t\} \tag{26d}$$

The first term $V^F(t)$ in equation (26) is the counterparty-risk-free value of the CDS and the second term is the exposure left over under full collateralization, which can be substantial.

***Proposition 4**: If a CDS is fully collateralized, the risky value of the CDS is NOT equal to the counterparty-risk-free value, as shown in equation (26).*

Proposition 4 or equation (26) provides a theoretical explanation for the failure of full collateralization in the CDS market. It tells us that under full collateralization the risky value is in general not equal to the counterparty-risk-free value except in one of the following situations: i) the market value is equal to the default payment, i.e., $V(u) = R(u)$; ii) firms *A*, *B*, and *C* have independent credit risks, i.e., $\rho_{ij} = 0$ and $\varsigma_{ABC} = 0$; or iii) $p_B \sigma_{AC} + p_A \sigma_{BC} = \theta_{ABC}$.

2.3 Numerical results

Since the payoffs of a CDS are mainly determined by credit events, we need to characterize the evolution of the hazard rates. Here we choose the *Cox-Ingersoll-Ross* (CIR) model. The CIR process has been widely used in the literature of credit risk and is given by

$$dh_t = a(b - h_t)dt + \sigma\sqrt{h_t}\,dW_t \tag{27}$$

where a denotes the mean reversion speed, *b* denotes the long-term mean, and σ denotes the volatility.

The calibrated parameters are shown in table 4. We assume that interest rates are deterministic and select the regression-based Monte-Carlo simulation (see Longstaff and Schwartz (2001)) to perform risky valuation.

We first assume that counterparties *A*, *B*, and reference entity *C* have independent default risks, i.e., $\rho_{AB} = \rho_{AC} = \rho_{BC} = \varphi_{AB} = \zeta_{ABC} = 0$, and examine the following cases: i) *B* is risk-free and *A* is risky; and ii) *A* is risk-free and *B* is risky. We simulate the hazard rates using the CIR model and then determine the appropriate discount factors according to Proposition 3. Finally we calculate the prices via the regression-based Monte-Carlo method. The results are shown in Table 5 and 6.

From table 5 and 6, we find that a credit spread of about 100 basis points maps into a CDS premium of about 0.4 basis points for counterparty *A* and about -0.7 basis points for counterparty *B*. The credit impact on the CDS premia is approximately linear. As would be expected, i) the dealer's credit quality has a larger impact on CDS premia than the investor's credit quality; ii) the higher the investor's credit risk, the higher the premium that the dealer charges; iii) the higher the dealer's credit risk, the lower the premium that the dealer asks. Without considering default correlations and comrelations, we find that, in general, the impact of counterparty risk on CDS premia is relatively small. This is in line with the empirical findings of Arora, Gandhi, and Longstaff (2009).

Next, we study the sensitivity of CDS premia to changes in the joint credit quality of associated parties. Sensitivity analysis is a very popular way in finance to find out how the value and risk of an instrument/portfolio changes if risk factors change. One of the simplest and most common approaches involves changing one factor at a time to see what effect this produces on the output. We are going to examine the impacts of the default correlations ρ_{AB}, ρ_{AC}, ρ_{BC}, and the comrelation ζ_{ABC} separately. Assume that party A has an 'A+100bps' credit quality and party B has an 'A' credit quality. The 5-year risky CDS premium is calculated as 0.02703.

As the absolute value of the slope increases, so does the sensitivity. The results illustrate that ρ_{BC} has the largest effect on CDS premia. The second biggest one is ζ_{ABC}. The impacts of ρ_{AB} and ρ_{AC} are very small. In particular, the effect of the comrelation is substantial and has never been studies before. A natural intuition to have on CDS is that the party buying default protection should worry about the default correlations and comrelation.

3 Conclusion

This article presents a new valuation framework for pricing financial instruments subject to credit risk. In particular, we focus on modeling default relationships. Some well-known risky valuation models in the market can be viewed as special cases of this framework, when the default dependencies are ignored.

To capture the default relationships among more than two defaultable entities, we introduce a new statistic: *comrelation*, an analogue to correlation for multiple variables, to exploit any multivariate statistical relationship. Our research shows that accounting for default correlations and comrelations becomes important, especially under market stress. The existing valuation models in the credit derivatives market, which take into account only pair-wise default correlations, may underestimate credit risk and may be inappropriate.

We study the sensitivity of the price of a defaultable instrument to changes in the joint credit quality of the parties. For instance, our analysis shows that the effect of default dependence on CDS premia from large to small is the correlation between the protection seller and the reference entity, the comrelation, the correlation between the protection buyer and the reference entity, and the correlation between the protection buyer and the protection seller.

The model shows that a fully collateralized swap is risk-free, while a fully collateralized CDS is not equivalent to a risk-free one. Therefore, we conclude that collateralization designed to mitigate counterparty risk works well for financial instruments subject to bilateral credit risk, but fails for ones subject to multilateral credit risk.

Appendix

Proof of Proposition 1. Let $t = T_0$. On the first cash flow payment date T_1, let $V(T_1)$ denote the market value of the instrument excluding the current cash flow X_1. According to Equation (10), we have

$$V(t) = E\left[K(T_0, T_1)\left(X_1 + V(T_1)\right)\middle|\mathcal{F}_t\right] \tag{A1}$$

Similarly, we have

$$V(T_1) = E\left[K(T_1, T_2)\left(X_2 + V(T_2)\right)\middle|\mathcal{F}_{T_1}\right] \tag{A2}$$

Note that $K(T_0, T_1)$ is $\mathcal{F}_{T_1}$ -measurable. According to *taking out what is known* and *towe*r properties of conditional expectation, we have

$$\begin{aligned} V(t) &= E\left[K(T_0,T_1)\left(X_1+V(T_1)\right)\middle|\mathcal{F}_t\right] = E\left[K(T_0,T_1)X_1\middle|\mathcal{F}_t\right] \\ &\quad + E\left\{K(T_0,T_1)\left[E(K(T_1,T_2)X_2\middle|\mathcal{F}_{T_1}) + E(K(T_1,T_2)V(T_2)\middle|\mathcal{F}_{T_1})\right]\middle|\mathcal{F}_t\right\} \\ &= E\left[K(T_0,T_1)X_1\middle|\mathcal{F}_t\right] + E\left[\left(\prod_{j=0}^{1} K(T_j,T_{j+1})\right)X_2\middle|\mathcal{F}_t\right] + E\left[\left(\prod_{j=0}^{1} K(T_j,T_{j+1})\right)V(T_2)\middle|\mathcal{F}_t\right] \end{aligned} \tag{A3}$$

By recursively deriving from T_2 forward over T_m, where $V(T_m) = X_m$, we have

$$V(t)=\sum_{i=1}^{m}E\left[\left(\prod_{j=0}^{i-1}K(T_j,T_{j+1})\right)X_i\middle|\mathcal{F}_t\right] \tag{A4}$$

Proof of Proposition 3. Let $t=T_0$. On the first payment date T_1, let $V(T_1)$ denote the market value of the CDS excluding the current cash flow X_1. There are a total of eight ($2^3=8$) possible states shown in Table A1. The risky price is the discounted expectation of the payoffs and is given by

$$\begin{aligned}V(t)=E\Big\{\Big[&1_{(V(T_1)X_1\geq 0}\left\langle\left(p_{000}(T_0,T_1)+p_{100}(T_0,T_1)\overline{\varphi}_B(T_1)+p_{010}(T_0,T_1)\varphi_B(T_1)+p_{110}(T_0,T_1)\varphi_{AB}(T_1)\right)(V(T_1)+X_1)\middle|\mathcal{F}_t\right\rangle\\&+1_{(V(T_1)+X_1)<0}\left\langle\left(p_{000}(T_0,T_1)+p_{100}(T_0,T_1)\varphi_A(T_1)+p_{010}(T_0,T_1)\overline{\varphi}_A(T_1)+p_{110}(T_0,T_1)\varphi_{AB}(T_1)\right)(V(T_1)+X_1)\middle|\mathcal{F}_t\right\rangle\\&+\left(p_{001}(T_0,T_1)+p_{101}(T_0,T_1)\overline{\varphi}_B(T_1)+p_{011}(T_0,T_1)\varphi_B(T_1)+p_{111}(T_0,T_1)\varphi_{AB}(T_1)\right)R(T_s,T_1)\Big|\mathcal{F}_t\Big]D(t,T)\Big\}\\=E\Big\{&\left[O(T_0,T_1)(V(T_1)+X_1)+\Omega(T_0,T_1)R(T_s,T_1)\right]\Big|\mathcal{F}_t\Big\}\end{aligned} \tag{A5a}$$

where

$$O(T_0,T_1)=1_{(V(T_1)+X_1)\geq 0}\phi_B(T_0,T_1)+1_{(V(T_1)+X_1)<0}\phi_A(T_0,T_1) \tag{A5b}$$

$$\begin{aligned}\phi_A(T_0,T_1)=\{&p_A(T_0,T_1)p_B(T_0,T_1)p_C(T_0,T_1)+q_A(T_0,T_1)p_B(T_0,T_1)p_C(T_0,T_1)\varphi_A(T_1)\\&+p_A(T_0,T_1)q_B(T_0,T_1)p_C(T_0,T_1)\overline{\varphi}_A(T_1)+q_A(T_0,T_1)q_B(T_0,T_1)p_C(T_0,T_1)\varphi_{AB}(T_1)\\&+p_C(T_0,T_1)\sigma_{AB}(T_0,T_1)\left(1-\varphi_A(T_1)-\overline{\varphi}_A(T_1)+\varphi_{AB}(T_1)\right)\\&+\sigma_{AC}(T_0,T_1)\left[p_B(T_0,T_1)\left(1-\varphi_A(T_1)\right)+q_B(T_0,T_1)\left(\overline{\varphi}_A(T_1)-\varphi_{AB}(T_1)\right)\right]\\&+\sigma_{BC}(T_0,T_1)\left[p_A(T_0,T_1)\left(1-\overline{\varphi}_A(T_1)\right)+q_A(T_0,T_1)\left(\varphi_A(T_1)-\varphi_{AB}(T_1)\right)\right]\\&+\theta_{ABC}(T_0,T_1)\left(-1+\overline{\varphi}_A(T_1)-\varphi_{AB}(T_1)+\varphi_A(T_1)\right)\}D(T_0,T_1)\end{aligned} \tag{A5c}$$

$$\begin{aligned}\phi_B(T_0,T_1)=\{&p_A(T_0,T_1)p_B(T_0,T_1)p_C(T_0,T_1)+q_A(T_0,T_1)p_B(T_0,T_1)p_C(T_0,T_1)\overline{\varphi}_B(T_1)\\&+p_A(T_1,T_1)q_B(T_0,T_1)p_C(T_0,T_1)\varphi_B(T_1)+q_A(T_0,T_1)q_B(T_0,T_1)p_C(T_0,T_1)\varphi_{AB}(T_1)\\&+p_C(T_0,T_1)\sigma_{AB}(T_0,T_1)\left(1-\varphi_B(T_1)-\overline{\varphi}_B(T_1)+\varphi_{AB}(T_1)\right)\\&+\sigma_{AC}(T_0,T_1)\left[p_B(T_0,T_1)\left(1-\overline{\varphi}_B(T_1)\right)+q_B(T_0,T_1)\left(\varphi_B(T_1)-\varphi_{AB}(T_1)\right)\right]\\&+\sigma_{BC}(T_0,T_1)\left[p_A(T_0,T_1)\left(1-\varphi_B(T_1)\right)+q_A(T_0,T_1)\left(\overline{\varphi}_B(T_1)-\varphi_{AB}(T_1)\right)\right]\\&+\theta_{ABC}(T_0,T_1)\left(-1+\overline{\varphi}_B(T_1)-\varphi_{AB}(T_1)+\varphi_B(T_1)\right)\}D(T_0,T_1)\end{aligned} \tag{A5d}$$

$$\begin{aligned}\Omega(T_0,T)=\{&p_A(T_0,T)p_B(T_0,T)q_C(T_0,T)+q_A(T_0,T)p_B(T_0,T)q_C(T_0,T)\overline{\varphi}_B(T_1)\\&+p_A(T_0,T)q_B(T_0,T)q_C(T_0,T)\varphi_B(T_1)+q_A(T_0,T)q_B(T_0,T)q_C(T_0,T)\varphi_{AB}(T_1)\\&+q_C(T_0,T)\sigma_{AB}(T_0,T)\left(1-\varphi_B(T_1)-\overline{\varphi}_B(T_1)+\varphi_{AB}(T_1)\right)\\&-\sigma_{AC}(T_0,T)\left[p_B(T_0,T)\left(1-\overline{\varphi}_B(T_1)\right)+q_B(T_0,T)\left(\varphi_B(T_1)-\varphi_{AB}(T_1)\right)\right]\\&-\sigma_{BC}(T_0,T)\left[p_A(T_0,T)\left(1-\varphi_B(T_1)\right)+q_A(T_0,T)\left(\overline{\varphi}_B(T_1)-\varphi_{AB}(T_1)\right)\right]\\&+\theta_{ABC}(T_0,T)\left(1-\overline{\varphi}_B(T_1)+\varphi_{AB}(T_1)-\varphi_B(T_1)\right)\}D(T_0,T)\end{aligned} \tag{A5e}$$

Similarly, we have

$$V(T_1)=E\left\{\left[O(T_1,T_2)\left(X_2+V(T_2)\right)+\Omega(T_1,T_2)R(T_1,T_2)\right]\middle|\mathcal{F}_{T_1}\right\} \tag{A6}$$

Note that $O(T_0,T_1)$ is $\mathcal{F}_{T_1}$-measurable. According to *taking out what is known* and *towe*r properties of conditional expectation, we have

$$\begin{aligned}V(t)&=E\left\{\left[O(T_0,T_1)\left(X_1+V(T_1)\right)+\Omega(T_0,T_1)R(T_0,T_1)\right]\middle|\mathcal{F}_t\right\}=E\left[O(T_0,T_1)X_1\middle|\mathcal{F}_t\right]\\&\quad+E\left[\Omega(T_0,T_1)R(T_0,T_1)\middle|\mathcal{F}_t\right]+E\left\{O(T_0,T_1)E\left\langle\left[O(T_1,T_2)\left(X_2+V(T_2)\right)+\Omega(T_1,T_2)R(T_1,T_2)\right]\middle|\mathcal{F}_{T_1}\right\rangle\middle|\mathcal{F}_t\right\}\\&=\sum_{i=1}^{2}E\left[\left(\prod_{j=0}^{i-1}O(T_j,T_{j+1})\right)X_i\right]+\sum_{i=1}^{2}E\left[\left(\prod_{j=0}^{i-2}O(T_j,T_{j+1})\right)\Omega(T_{i-1},T_i)R(T_{i-1},T_i)\right]\\&\quad+E\left[\left(\prod_{j=0}^{1}O(T_j,T_{j+1})\right)V(T_2)\middle|\mathcal{F}_t\right]\end{aligned} \tag{A7}$$

By recursively deriving from T_2 forward over T_m, where $V(T_m)=X_m$, we have

$$V(t)=\sum_{i=1}^{m}E\left[\left(\prod_{j=0}^{i-1}O(T_j,T_{j+1})\right)X_i\middle|\mathcal{F}_t\right]+\sum_{i=1}^{m}E\left[\left(\prod_{j=0}^{i-2}O(T_j,T_{j+1})\right)\Omega(T_{i-1},T_i)R(T_{i-1},T_i)\middle|\mathcal{F}_t\right] \tag{A8}$$

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