Output-Feedback Adaptive Neural Control for Stochastic Nonlinear Time-Varying Delay Systems With Unknown Control Directions

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Abstract—This paper presents an adaptive output-feedback neural network (NN) control scheme for a class of stochastic nonlinear time-varying delay systems with unknown control directions. To make the controller design feasible, the unknown control coefficients are grouped together and the original system is transformed into a new system using a linear state transformation. Then, the Nussbaum function technique is incorporated into the backstepping recursive design technique to solve the problem of unknown control directions. Furthermore, under the assumption that the time-varying delays exist in the system output, only one NN is employed to compensate for all unknown nonlinear terms depending on the delayed output. Moreover, by estimating the maximum of NN parameters instead of the parameters themselves, the NN parameters to be estimated are greatly decreased and the online learning time is also dramatically decreased. It is shown that all the signals of the closed-loop system are bounded in probability. The effectiveness of the proposed scheme is demonstrated by the simulation results.

Index Terms—Adaptive output feedback control, neural network (NN), stochastic nonlinear systems, time-varying delays, unknown control directions.

I. INTRODUCTION

RECENTLY, stochastic nonlinear control has received more and more attention due to the existence of stochastic disturbances in many practical systems. With the proposition of many important concepts of stochastic stability theory [1], quite a number of research results on deterministic nonlinear systems have been extended to stochastic nonlinear systems, for example, Sontag's stabilization formula, nonlinear optimality, and backstepping technique [2], [3], where the backstepping approach was developed to the point of a step-by-step design procedure in [4]. The main technical obstacle in the Lyapunov design for stochastic systems is that the Itô stochastic differentiation involves not only the gradient but also the higher order Hessian term. Pan and Basar [5] were the first to solve the stochastic stabilization problem for a class of strict-feedback systems based on a risk-sensitive cost criterion. By employing the quadratic Lyapunov functions, Deng et al. [2], [3] extended the backstepping design method to output-feedback stabilization and state-feedback stabilization of stochastic nonlinear systems, respectively. However, the combined problem of the control of stochastic nonlinear systems with nonlinear uncertainties simultaneously is still a cumbersome issue.

As well known, both neural network (NN) and fuzzy logic system (FLS) have been proved to be a useful tool for solving the control problem of uncertain systems with unknown nonlinear functions in practical applications [6]. By combining NNs or FLSs with the backstepping design technique, several adaptive NN or adaptive fuzzy backstepping control schemes have been developed for several classes of stochastic nonlinear systems with mismatched conditions [7]–[11]. For example, a novel adaptive fuzzy backstepping control scheme was proposed based on the observer design in [8] for a class of stochastic nonlinear strict-feedback systems. In [9] and [10], two constructive output-feedback control approaches were developed based on adaptive NNs for two classes of stochastic nonlinear systems, respectively. In addition, by combining backstepping technique with stochastic small-gain approach, a novel robust adaptive fuzzy output feedback controller was presented for a class of stochastic nonlinear systems in [11]. Recently, [12] and [13] extended the above results to a class of stochastic nonlinear large-scale systems, respectively. More recently, a novel adaptive control scheme was proposed for a class of uncertain nonlinear stochastic systems based on fuzzy NN approximation in [14].

However, when approximating the unknown smooth functions using either NNs or FLSs, the number of parameters to be tuned online, i.e., the neural weights of hidden units in a NN, or the fuzzy weights in a FLS, will grow rapidly with the dimension of the argument vector of the function to be approximated [15], which causes the explosion of learning parameters. Consequently, it makes the complex NNs or FLSs online at http://ieeexplore.ieee.org.

Manuscript received July 2, 2013; revised June 13, 2014; accepted June 23, 2014. This work was supported in part by the National Natural Science Foundation of China under Grant 61373137, in part by the Program for Liaoning Excellent Talents in University Foundation of China under Grant 51179019, Grant 61374114, and Grant 2014. This work was supported in part by the National Natural Science Foundation of China under Grant 2011CB302801, in part by the Fundamental Research Funds for the Central Universities under Grant 315–2014–321, in part by the Macau Science and Technology Development Foundation, Macau, China, under Grant 008/2010/A1, and in part by the Multiyear Research Grants. (Corresponding author: Zifu Li.)

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Digital Object Identifier 10.1109/TNNLS.2014.2343638
unsuitable for the real-time systems that are sensitive to time delays, and the time-consuming process is unavoidable during the implementation of these control algorithms. This problem was pointed out by Fischle and Schroder [16] and first solved by Yang and Ren [17] in their pioneering work, where the so-called minimal learning parameter (MLP) algorithm containing much less online adaptive parameters were constructed by fusion of traditional backstepping technique and radial-basis-function (RBF) NNs. Recently, by combining dynamic surface control and MLP techniques, Li et al. [18] first proposed an algorithm that can simultaneously solve both problems of the explosion of learning parameters and the explosion of computation complexity. Then, a novel adaptive neural output feedback controller based on reduced-order observer was explored for a class of uncertain nonlinear Single Input Single Output systems in [19]. More recently, by estimating the maximum of NN parameters, Yu et al. [20] designed an adaptive neural controller containing only one adaptive parameter. Nevertheless, little work has been done to investigate the adaptive NN or fuzzy output-feedback control problem for stochastic nonlinear systems with unknown control directions.

It is well known that the control directions, defined as signs of the control gains, are normally required to be known a priori in adaptive control literature. However, the unknown control directions often exist in many practical nonlinear systems. It is also a major source of resulting in instability of the control systems. When the signs of control gains are unknown, the adaptive control problem becomes much more difficult, since we cannot decide the direction along which the control operates. This problem has been remained open till the Nussbaum-type gain was first introduced in [21] for adaptive control of a class of first-order linear systems. Later on, the Nussbaum gain was extended to adaptive control of nonlinear systems with unknown control directions incorporated with different techniques. For example, Liu [22] investigated the output-feedback adaptive stabilization for a class of nonlinear systems with unknown control directions using the linear state transformation technique. Wen and Ren [23] proposed a state observer-based adaptive neural control scheme for the systems with unknown control directions and unmeasurable states. To handle the unknown control directions in stochastic nonlinear systems, Wang et al. [24] and Yu et al. [25] proposed adaptive fuzzy or neural backstepping control methods for stochastic nonlinear systems using the Nussbaum function technique under state feedback framework, respectively. Recently, Wang et al. [26] developed an adaptive fuzzy output-feedback control scheme for a class of stochastic nonlinear systems with unknown control direction, in which only one control coefficient was assumed unknown. However, little work was dedicated to the unknown control direction problem for stochastic nonlinear systems along with time delays.

In many practical control systems, such as biological systems, microwave oscillators, nuclear reactors, network systems and so on, there often exists time delay, which is frequently a source of instability, and the control performance of these systems is often degraded. Over the past years, the problem of adaptive control design along with the stability analysis for the time-delay systems has been a popular topic and significant progress has been achieved [27]–[40]. For example, Ge et al. [33] and Hong et al. [34] successfully constructed state feedback controllers for nonlinear time-delay systems, respectively. Hua et al. [35] and Tong et al. [36] investigated the output-feedback control problems for nonlinear time-delay systems, respectively. Then, Chen et al. [38] developed an adaptive consensus control scheme for a class of nonlinear multiagent time-delay systems with the help of NNs approximation. Recently, Chen and Jiao [39] addressed the problem of adaptive NN output-feedback control for a class of uncertain stochastic nonlinear strict-feedback systems with time-varying delays, where the problem of nonlinear observer design was solved by introducing the circle criterion. More recently, without controllable linearization and by first employing the adding-a-power-integer technique to solve the stabilization problem of stochastic time-delay systems, Chen et al. [41] investigated the state-feedback stabilization problem for a class of lower-triangular stochastic time-delay nonlinear systems. However, to the best of the authors knowledge, there is no result reported on the output-feedback control problem of stochastic nonlinear systems with both time-varying delays and unknown control directions.

Motivated by the above observations, incorporating the linear state transformation with MLP techniques, an adaptive backstepping output-feedback neural control scheme is proposed for a class of stochastic nonlinear systems with both time-varying delays and unknown control directions. The unknown control coefficients are grouped together and the original system is transformed into a new system by employing the linear state transformation technique. Then, the Nussbaum-type gain function is used to deal with the unknown parameters caused by the unknown control directions in the new system. In addition, an RBF NN is used to approximate all unknown nonlinear terms depending on the delayed output. Still, the MLP technique is used to alleviate the computational burden by estimating the maximum of NN parameters. Compared with the existing results, the main contributions in this paper lie in the following: 1) for the first time, the linear state transformation technique is introduced to the stochastic nonlinear time-varying delay systems with unknown control directions. Moreover, unlike [23] and [39], the assumption on the value of the unknown control coefficients is not needed, both control singularity and unknown control direction can be tackled in this paper; 2) all the unknown output-dependent functions are grouped into a suitable unknown function that is compensated only by one NN. This simplifies the design procedure and reduces the computation loads dramatically; and 3) by estimating the maximum of NN parameters instead of the parameters themselves, the parameters to be estimated are greatly decreased. Hence, the exploding of learning parameters is solved efficiently.

The rest of this paper is organized as follows. Section II provides some notations and preliminary knowledge. The problem formulation is presented in Section III. In Section IV, the adaptive output-feedback control design and stability analysis are presented for the stochastic nonlinear time-varying delay systems with unknown directions. Section V gives
some simulation examples to illustrate the effectiveness of the method proposed in this paper, followed by the conclusion presented in Section VI.

II. NOTATIONS AND PRELIMINARY KNOWLEDGE

The following notations will be used throughout this paper. $R^n$ denotes the set of all nonnegative real numbers. $R^m$ denotes the real $n$-dimensional space. $R^{m \times n}$ denotes the real $n \times r$ matrix space. For a given vector or matrix $X$, $X^T$ denotes its transpose. $T_r(X)$ denotes its trace when $X$ is square. $|X|$ denotes the Euclidean norm of a vector $X$, and the corresponding induced norm for matrix $X$ is denoted by $\|X\|$. $\|X\|_F$ denotes the Frobenius norm of matrix $X$ defined by $\|X\|_F = \sqrt{Tr(X^T X)}$ with properties $\|X\| \leq \|X\|_F$ and $\|XY\|_F \leq \|X\|_F \|Y\|_F$. $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the minimal eigenvalue and maximal eigenvalue of symmetric real matrix $X$, respectively. $C^1$ denotes the set of all functions with continuous $1 \text{st}$ partial derivatives, $C^{2,1}(R^n \times [-d, \infty); R^n)$ denotes the family of all nonnegative functions $V(x,t)$ on $R^n \times [-d, \infty)$, and $C^2$ in $x$ and $C^1$ in $t$. $C^{2,1}$ denotes the family of all functions with $C^2$ in the first argument and $C^1$ in $t$.

Consider an $n$-dimensional stochastic time-delay system

$$dx = f(t,x(t), x(t-d(t)))dt + g(t,x(t), x(t-d(t)))dw \quad (\forall t \geq 0) \quad (1)$$

with initial condition $\{x(\sigma) : -d \leq \sigma \leq 0\} = \xi \in C_0^b \times \{(-d, 0); R^n\}$, where $d(t) : R_+ \rightarrow [0, d]$ is a Borel measurable function. $f : R^r \times R^n \times R^n \rightarrow R^n$ and $g : R^r \times R^n \times R^n \rightarrow R^{m \times n}$ are locally Lipschitz, and $w$ is $r$-dimensional standard Brownian motion defined on the complete probability space $(\Omega, F, \{F_t\}_{t \geq 0}, P)$ with $\Omega$ being a sample space, with $F$ being a $\sigma$ field, $\{F_t\}_{t \geq 0}$ being a filtration, and $P$ being the probability measure.

Define a differential operator $L$ known as infinitesimal generator for twice continuously differentiable function $V(x,t) \in C^{2,1}$ as follows:

$$LV = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x(t), x(t-d(t)), t) + \frac{1}{2} Tr \left( g^T \frac{\partial^2 V}{\partial x^2} g \right). \quad (2)$$

**Definition 2.1:** $N(\cdot)$ is an even smooth Nussbaum-type function, if it satisfies

$$\lim_{t \rightarrow -\infty} \sup \frac{(1)}{t} \int_0^t N(u)du$$

$$\lim_{t \rightarrow \infty} \inf \frac{(1)}{t} \int_0^t N(u)du. \quad (3)$$

From the definition, we know that Nussbaum functions should have infinite gains and switching frequencies. There are many continuous function satisfying these conditions, such as $e^{\beta \cdot \cos(u)}$, $\ln(u + 1) \cos \ln(u + 1)$, and $u^2 \cos(u)$. In this paper, the even Nussbaum function $\beta^2 \cos(u)$ is used.

**Lemma 2.1 [24]:** Consider the stochastic nonlinear system (1), and assume that there exists a smooth function $\tilde{z} : R^+ \rightarrow R$, a function $V(x,t) \in C^{2,1}(R^n \times [-d, \infty); R^n)$, and a Nussbaum type even function $N(\cdot)$. Let $\chi$ be a nonnegative random variable, $M(t)$ be a real valued continuous local martingale with $M(0) = 0$ such that

$$V(x,t) \leq \chi + e^{-\chi t} \int_0^t (\delta N(\tilde{z}) \tilde{z} + \tilde{z}) e^{\chi t} dt + M(t). \quad (4)$$

Then, the functions $V(x,t)$, $\tilde{z}(t)$ and $\int_0^t (\delta N(\tilde{z}) \tilde{z} + \tilde{z}) e^{\chi t} dt$ must be bounded in probability.

In this paper, the RBF NN will be used to approximate any unknown continuous function $F(Z)$, namely $F_{nn}(Z) = W^T S(Z)$, where $Z \in \Omega_Z \subset R^q$ is the input vector with $q$ being the input dimension of NNS, $W = [w_1, w_2, \ldots, w_l]^T \in R^l$ is the weight vector with $l > 1$ being the node number of a NN, and $S(Z)$ means the basis function vector with $s_i(Z)$ being chosen as a Gaussian function $s_i(Z) = \exp(-|Z - \mu_i|^2 (Z - \mu_i)/\gamma^2)$, $i = 1, 2, \ldots, l$, where $\mu_i = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{iq}]^T$ is the center of the receptive field, and $\gamma > 0$ is the width of the basis function $s_i(Z)$. It has been proven that an RBF NN can approximate any continuous function over a compact set $\Omega_Z \subset R^q$ to an arbitrary accuracy as

$$F(Z) = W^T S(Z) + \delta(Z). \quad (5)$$

The ideal weight vector $W^*$ is an artificial quantity required for analytical purposes. It is defined as

$$W^* := \arg \min_{W \in R^l} \left\{ \sup_{Z \in \Omega_Z} \left| h_{nn}(Z) - \hat{W}^T S(Z) \right| \right\}.$$

**Assumption 2.1 [25]:** For $\forall Z \in \Omega_Z$, there exists an ideal constant weight vector $W^*$ such that $\|W^*\|_\infty \leq w_{\max}$ and $|\delta(Z)| \leq \delta_{\max}$ with bounds $w_{\max}$ and $\delta_{\max} > 0$. It is obvious that

$$W^T S(Z) + |\delta(Z)| \leq \|W^T S(Z)\| + |\delta(Z)| \leq \sum_{i=1}^l |s_i(Z)| w_{\max} + \delta_{\max} \leq \theta \beta(Z) \quad (6)$$

where

$$\beta(Z) = \sqrt{\left(1 + \frac{1}{\theta} \left[ \sum_{i=1}^l s_i^2(Z) \right] + 1 \right)},$$

$$\theta = \max(\delta_{\max}, w_{\max}).$$

**Assumption 2.2:** For $1 \leq i \leq n$, according to the mean value theorem, the following equalities hold:

$$f_i(y, y(t-d(t))) = \left. \frac{\partial f_i}{\partial y} \right|_{s=y(t-d(t))} \int_{\bar{y}_{hi}}^{\bar{y}_{hi}} f_i(y, y(t-d(t))) \partial s \quad (7)$$

$$h_i(y, y(t-d(t))) = \left. \frac{\partial h_i}{\partial y} \right|_{s=y(t-d(t))} \int_{\bar{y}_{hi}}^{\bar{y}_{hi}} h_i(y, y(t-d(t))) \partial s \quad (8)$$

where $f_i(y, 0) = 0$, $h_i(0) = 0$, for $0 < \bar{y}_{hi}$, $\bar{y}_{hi} < 1$, the unknown functions satisfy

$$\tilde{f}_i(y, y(t-d(t))) = \left. \frac{\partial \tilde{f}_i}{\partial y} \right|_{s=y(t-d(t))} \int_{\bar{y}_{hi}}^{\bar{y}_{hi}} \tilde{f}_i(y, y(t-d(t))) \partial s \quad (9)$$

$$\tilde{h}_i(y, y(t-d(t))) = \left. \frac{\partial \tilde{h}_i}{\partial y} \right|_{s=y(t-d(t))} \int_{\bar{y}_{hi}}^{\bar{y}_{hi}} \tilde{h}_i(y, y(t-d(t))) \partial s \quad (10).$$

**Remark 2.1:** These unknown nonlinear functions depending on the delayed output will be grouped into a suitable unknown function that will be compensated by only one NN.
Consider the following stochastic nonlinear time-varying delay system described by:

\[
\begin{align*}
\dot{x}_i &= \left[ g_i x_{i+1} + \phi_i(y) + f_i(y, y(t - d(t))) \right] dt + h_i(y, y(t - d(t))) d\omega \\
\vdots \\
\dot{x}_n &= \left[ g_n u + \phi_n(y) + f_n(y, y(t - d(t))) \right] dt + h_n(y, y(t - d(t))) d\omega \\
y &= x_1
\end{align*}
\]

(9)

where \( x_i \in R(i = 1, \ldots, n) \), \( u \in R \), and \( y \in R \) are the unmeasured system states, the control input, and the system output, respectively. \( g_i \neq 0 \), \( i = 1, 2, \ldots, n \) called control coefficients, are unknown constants with unknown directions. \( \phi_i(y) \) are known smooth nonlinear functions with \( \phi_i(0) = 0 \), \( f_i(\cdot) : R^2 \rightarrow R \) and \( h_i(\cdot) : R^2 \rightarrow R' \) are unknown locally Lipschitz smooth functions with \( f_i(y, 0) = 0 \) and \( h_i(y, 0) = 0 \). The uncertain time-varying delay \( d(t) : R^+ \rightarrow [0, d] \) satisfies \( d(t) \leq \varsigma < 1 \), with \( \varsigma \) being an unknown constant. The initial condition \( \{ x(s) : -d \leq s \leq 0 \} = \zeta \in C_{F_0}^b \times \left[ (-d, 0); R^n \right] \) is unknown. \( w \) has been defined in (1). Only the system output \( y \) can be available for measurement.

Let \( \zeta_t = x_t / \prod_{j=i}^n g_j \), for \( i = 1, \ldots, n \), then, the new state variables \( \zeta = [\zeta_1, \ldots, \zeta_n] \) are given by

\[
\begin{align*}
\dot{\zeta}_t &= \left[ \zeta_{t+1} + \frac{1}{\prod_{j=i}^n g_j} (\phi_i(y) + f_i(y, y(t - d(t)))) \right] dt \\
&\quad + \frac{1}{\prod_{j=i}^n g_j} h_i(y, y(t - d(t))) d\omega \\
\vdots \\
\dot{\zeta}_n &= \left[ u + \frac{1}{g_n} \phi_n(y) + f_n(y, y(t - d(t))) \right] dt \\
&\quad + \frac{1}{g_n} h_n(y, y(t - d(t))) d\omega.
\end{align*}
\]

(10)

It is obvious that all the control coefficients are known in (10). If all the states of (10) are available, it is easy to design a controller. But, because of the existence of unknown control coefficients in the linear state transformation, all the states of (10) are unavailable. Therefore, a full-order state observer must be established first to estimate the unmeasured states. Then, a novel NN adaptive output-feedback control scheme will be explored based on the designed observers.

Remark 3.4: After the linear state transformation, (9) becomes a strict-feedback uncertain nonlinear system with both unknown parameters \( 1 / \prod_{j=i}^n g_j \) and unknown time-varying delays. Therefore, it is much feasible to design a better controller.

Consider the observers for (10) as follows:

\[
\begin{align*}
\dot{\hat{\zeta}}_t &= \hat{\zeta}_{t+1} - k_i \hat{\zeta}_1 \\
\vdots \\
\dot{\hat{\zeta}}_n &= u - k_n \hat{\zeta}_1
\end{align*}
\]

(11)

where \( k_1, \ldots, k_n \) are optional positive constants such that the following matrix:

\[
A = \begin{bmatrix}
-k_1 & I_{n-1} \\
\vdots & \vdots \\
-k_n & 0 & \cdots & 0
\end{bmatrix}
\]

is asymptotically stable. Denote \( \hat{\zeta} = [\hat{\zeta}_1, \ldots, \hat{\zeta}_n] \) and let \( \tilde{\zeta} = \zeta - \hat{\zeta} \) be the observer error. For convenience, define \( \zeta_{n+1} = u \) and \( \zeta_{n+1} = 0 \). By combining (10) and (11), the observer error dynamics of \( \tilde{\zeta} \) can be obtained as follows:

\[
\begin{align*}
\dot{\tilde{\zeta}} &= \left[ A \tilde{\zeta} + \Phi(y) + F(y, y(t - d(t))) + K \zeta_1 \right] dt \\
&\quad + H(y, y(t - d(t))) d\omega
\end{align*}
\]

(12)

where

\[
\Phi() = \left[ \prod_{j=1}^n g_j \phi_1, \ldots, \phi_n \right]^T \\
K = [k_1, \ldots, k_n]^T \\
F() = \left[ \prod_{j=1}^n g_j f_1, \ldots, f_n \right]^T \\
H() = \left[ \prod_{j=1}^n g_j h_1, \ldots, h_n \right]^T
\]

Hence, there exists a positive definite matrix \( P = P^T \) such that \( A^T P + PA = -I \).
Then, the complete system can be expressed as
\[
\begin{aligned}
    d\hat{z}_n &= \left[ A_1^\varepsilon + \Phi(y) + F(y, y(t - d(t))) + K_1 \right] dt \\
    dy &= \left[ \hat{g}\bar{z}_2 + \phi_1(y) + f_1(y, y(t - d(t))) \right] dt \\
    &\quad + h_1(y, y(t - d(t))) dw \\
    d\hat{z}_i &= \left[ \hat{z}_{i+1} - k_i \bar{z}_i \right] dt \\
    \vdots
    d\hat{z}_{i+n} &= \left[ u - k_n \bar{z}_{i+n} \right] dt
\end{aligned}
\]  
where \( \hat{g} = \prod_{j=1}^n g_j \).

IV. ADAPTIVE OUTPUT-FEEDBACK CONTROL DESIGN AND STABILITY ANALYSIS

In this section, the adaptive backstepping technique will be used to design an adaptive NN output-feedback controller. To simplify the design procedure, some derivations are omitted, and only the main design procedures are given.

Step 1: Following the adaptive backstepping design idea, we define the error variables as follows:
\[
z_1 = y, \quad z_2 = \hat{z}_2 - \alpha_i(y, \hat{z}_1, \hat{z}_2, \hat{\theta}, \hat{l})
\]
where \( \alpha_i(\cdot) \) is a smooth function to be determined later, and \( \hat{z}_1 \) is the estimate of \( \theta \) and the error \( \hat{\theta} = \theta - \hat{\theta} \). \( \hat{l} \) denotes the estimate of \( l \) and the error \( \hat{l} = l - \hat{l} \). \( \hat{l} \) is an unknown constant defined as
\[
l = \max \left\{ \frac{3\hat{g}^4}{\delta_1}, \frac{3\hat{g}^4}{\delta_2}, \frac{3\hat{g}^4}{\delta_3} \right\}
\]
where \( i = 1, \ldots, n, \ j = 2, \ldots, n, \delta_1, \delta_2, \delta_3 \) are known parameters to be designed later. For simplicity, here and hereafter, \( f_1 = f_i(y, y(t - d(t))) \), \( h_1 = h_i(y, y(t - d(t))) \), \( F = F(y, y(t - d(t))) \), and \( H = H(y, y(t - d(t))) \).

Then, the differential of \( z_1 \) is
\[
d\hat{z}_1 = \left[ \hat{g}(\hat{z}_2 + \hat{z}_2) + \phi_1(y) + f_1 \right] dt + h_1 dw. \tag{14}
\]

Remark 4.1: By defining the unknown constant \( l \) rather than limiting the value of unknown parameters \( \hat{g}_i \) in (10), the difficulty caused by the unknown control directions can be overcome using integrator backstepping approach together with tuning function technique and a Nussbaum-type function.

Consider the following Lyapunov function candidate:
\[
V_1 = \frac{b}{2} \left( \hat{z}_1^T \hat{z}_1 \right)^2 + \frac{1}{4} \hat{z}_1^T + \frac{1}{2} H^{-1} l^2. \tag{15}
\]

It follows from (2), (13), and (14) that:
\[
\begin{aligned}
    LV_1 &= -b\hat{z}_1^T P \hat{z}_1 \hat{z}_1^T + 2b\hat{z}_1^T P \hat{z}_1 (\Phi(y) P \hat{z}_1) \\
    &\quad + bTr(2P \hat{z}_1^T P \hat{z}_1 (\Phi(y) P \hat{z}_1)) + \frac{1}{2} \hat{z}_1^T h_1 H^T h_1 \\
    &\quad + 2b\hat{z}_1^T P \hat{z}_1 \left( (K_1, \hat{z}_2) \right) + 2b\hat{z}_1^T P \hat{z}_1 (F(y) P \hat{z}_1) \\
    &\quad + \frac{1}{2} \left[ \hat{g}(\hat{z}_2 + \hat{z}_2) + \phi_1(y) + f_1 \right] - \frac{1}{2} H^{-1} l^2. \tag{16}
\end{aligned}
\]

Then, using (16) and (A.1)–(A.8) in Appendix A, one has
\[
\begin{aligned}
    LV_1 &\leq \hat{z}_1^T \left[ \hat{g} \hat{a}_1 + \phi_1(y) + z_1 \left( 2\hat{l} + \frac{3}{4\lambda_{11}} + \frac{3}{4\lambda_{12}} \right) \right] \\
    &\quad + \frac{1}{4} \hat{z}_1^T \hat{z}_1^T - b\hat{z}_1^T P \hat{z}_1 \hat{z}_1^T + \Lambda_1 - H^{-1} l^2. \tag{17}
\end{aligned}
\]

where \( \tau_1 = 2\Gamma \hat{z}_1^T \). According to the well-known mean value theorem, we can get \( \phi_1(y) = \frac{y}{\hat{y}} \).

Step 2: Define \( z_3 = \hat{z}_3 - a_2(y, z_1, \hat{z}_1, \hat{z}_2, \hat{\theta}, \hat{l}) \) with \( a_2(\cdot) \) being a smooth function to be determined later. Then from (13), the differential of \( z_2 \) is
\[
\begin{aligned}
    dz_2 &= \left[ \hat{z}_3 - a_2 \hat{z}_1 - \frac{\delta a_1}{\delta y} (\hat{g}(\hat{z}_2 + \hat{z}_2) + \phi_1(y) + f_1) \\
    &\quad - \frac{\delta a_1}{\delta \hat{\theta}} (\hat{g}(\hat{z}_2 + \hat{z}_2) + \phi_1(y) + f_1) \right] dt \\
    &\quad - \frac{1}{2} \frac{\delta^2 a_1}{\delta y^2} h_1 h_1^T - \frac{\delta a_1}{\delta \hat{z}_1} \hat{z}_1 dt - \frac{\delta a_1}{\delta \hat{\theta}} \hat{\theta} dt \tag{18}.\end{aligned}
\]

Consider the following Lyapunov function candidate:
\[
V_2 = V_{i-1} + \frac{1}{4} \hat{z}_2^T. \tag{19}
\]

The derivative of \( V_2 \) satisfies
\[
\begin{aligned}
    LV_2 \leq &\ 
    \hat{z}_3^T \left[ \hat{g} \hat{a}_1 + \phi_1(y) + z_1 \left( 2\hat{l} + \frac{3}{4\lambda_{11}} + \frac{3}{4\lambda_{12}} \right) \right] \\
    &\quad - b\hat{z}_1^T P \hat{z}_1 \hat{z}_1^T + \Lambda_2 + \frac{1}{4} \delta_{\lambda_2} (a_2 - \Delta_2) \\
    &\quad + \frac{1}{4} \lambda_{24} \hat{z}_2^T - \frac{H^{-1}}{l_i} l_i. \tag{21}
\end{aligned}
\]

where \( \tau_2 = \tau_1 + \Gamma \hat{z}_2^T (\delta a_1/\delta y)^4 (2 + \epsilon_i^4) \).

Step 1: (3 \leq i \leq n - 1) Define the error variable \( \hat{z}_i = \hat{z}_i - a_{i-1}, \hat{z}_{i+1} = \hat{z}_{i+1} - a_i \), where \( a_i(\cdot) \) is a smooth function to be determined later. Then from (13), the differential of \( \hat{z}_i \) is
\[
\begin{aligned}
    d\hat{z}_i &= \left[ \hat{z}_{i+1} - k_i \hat{z}_i - \frac{\delta a_{i-1}}{\delta y} (\hat{g}(\hat{z}_2 + \hat{z}_2) + \phi_1(y) + f_1) \right] dt \\
    &\quad - \frac{\delta a_{i-1}}{\delta \hat{\theta}} (\hat{g}(\hat{z}_2 + \hat{z}_2) + \phi_1(y) + f_1) \\
    &\quad - \frac{1}{2} \frac{\delta^2 a_{i-1}}{\delta y^2} h_1 h_1^T - \frac{\delta a_{i-1}}{\delta \hat{z}_1} \hat{z}_1 dt \tag{22}.\end{aligned}
\]

Consider the following Lyapunov function candidate:
\[
V_i = V_{i-1} + \frac{1}{4} \hat{z}_i^T. \tag{23}
\]
Similar to the Step 2, and noting (A.9)–(A.14), (A.16), and (A.18) in Appendix A, the differential of $V_i$ satisfies

$$LV_i \leq z_1^3 \left[ \tilde{g}a_1 + \phi_1(y) + z_1 \left( 2\tilde{t} + \frac{3}{4\lambda_{11}} + \frac{3}{4\lambda_{12}} \right) \right]$$

$$- b\zeta^T P_{\tilde{z}} \tilde{z}^2 + z_2^2 [a_2 - \Delta_2] + \sum_{j=3}^{n} z_j^2 [a_j - \Delta_j]$$

$$+ \lambda_i + \frac{1}{4} \sum_{j=3}^{n} \lambda_j \zeta^2 - \tilde{l}^T \Gamma^{-1} \tilde{l} - \tau_i$$

(24)

where $\tau_i = \tau_{i-1} + \Gamma \zeta^4 (\zeta_1 - \tilde{\zeta}_1) / 4$. (2 + $\tilde{\zeta}^4 / 4$).

Step n: Define the error variable $z_n = \zeta_n - \alpha_{n-1}$. Then, from (13), the differential of $z_n$ is

$$dz_n = \left[ u - k_n \hat{\zeta} - \frac{\partial \alpha_{n-1}}{\partial y} (\tilde{\zeta}_2 + \hat{\zeta}_2) + \phi_1(y) + f_1 \right]$$

$$- \sum_{i=1}^{n-1} \frac{\partial \zeta_{i-1}}{\partial y} (\zeta_i - k_i \hat{\zeta}) - \frac{\partial \alpha_{n-1}}{\partial \theta} - \frac{\partial \alpha_{n-1}}{\partial \tilde{l}}$$

$$- \frac{1}{2} \frac{\partial^2 \alpha_{n-1}}{\partial y^2} h_1 h_1^T d + \frac{\partial \alpha_{n-1}}{\partial y} h_1 d w.$$  

(25)

Consider the following Lyapunov function candidate:

$$V_n = V_{n-1} + \frac{1}{4} e^2.$$  

(26)

Repeating the similar operation in the former steps, and with the help of Itô formula and (A.17) in Appendix A, the differential of $V_n$ satisfies

$$LV_n \leq z_1^3 \left[ \tilde{g}a_1 + \phi_1(y) + z_1 \left( 2\tilde{t} + \frac{3}{4\lambda_{11}} + \frac{3}{4\lambda_{12}} \right) \right]$$

$$- b\zeta^T P_{\tilde{z}} \tilde{z}^2 + \left[ \frac{1}{4} \sum_{j=1}^{n} \delta_j + \frac{3}{2} b \|P\|_2 e^{\frac{3}{4}} \right]$$

$$+ \delta_4 \left( 2 \|b\|_2 + b \|P\|_2 e^{\frac{3}{2}} \right) + \frac{3}{2} b \|P\|^2 e^4$$

$$+ \sum_{i=1}^{n} z_i^3 [a_i - \Delta_i] + \left[ u - \alpha_n \right] + y^4 \lambda \beta(Z)$$

$$+ y^4 (t - d(t))e^{-r \tilde{z}} \psi(y, y(t - d(t)))$$

$$- \frac{1}{1 - \zeta} \psi(y, y(t)) - \tilde{l}^T \Gamma^{-1} \tilde{l} - \tau_n$$

(27)

where $r$ is a known positive scalar, $\tilde{d} = \max(d(t))$, $\psi(y, y(t - d(t))) = \psi_1(y, y(t - d(t))) + \psi_2(y, y(t - d(t)))$, which satisfies

$$\psi_1(y, y(t - d(t))) = \frac{1}{2} e^{\frac{3}{4} \tilde{d}^2} b \|P\|^2 \sum_{j=1}^{n} \left( \frac{1}{\prod_{j=i}^{n} g_j} \tilde{g}_j \right)^4$$

$$+ \frac{1}{2} e^{\frac{3}{4} \tilde{d}^2} (2 \|b\|_2 + b \|P\|_2 e^{\frac{3}{2}} \lambda_{\max}(p))$$

$$\times \sum_{i=1}^{n} \left( \frac{1}{\prod_{j=i}^{n} g_j} \tilde{h}_i \right)^4$$

(28)

$$\psi_2(y, y(t - d(t))) = \frac{1}{4} e^{\frac{3}{4} \tilde{d}^2} \sum_{i=1}^{n} \left( \frac{1}{\prod_{j=i}^{n} g_j} \tilde{g}_j \right)^4$$

$$+ \frac{1}{4} e^{\frac{3}{4} \tilde{d}^2} \sum_{i=1}^{n} \lambda_j \tilde{h}_i^4.$$  

(29)

Now, define a new smooth function as follows:

$$f = e - \frac{1}{4} \sum_{i=2}^{n} \delta_{i-1}^3 + \frac{1}{2} e^{\frac{3}{4}} b \|P\|^2 \sum_{i=1}^{n} \left( \frac{1}{\prod_{j=i}^{n} g_j} \tilde{g}_j \phi_i(y) \right)^4$$

$$+ \frac{1}{2} e^{\frac{3}{4}} b \|P\|^2 \sum_{i=1}^{n} \left( \lambda_j \tilde{h}_i \phi_i(y) \right)^4 + \frac{1}{1 - \zeta} \psi(y, y(t)).$$

Since $\gamma_i$ are unknown control coefficients, $f$ cannot be directly implemented to construct the virtual controller $\alpha_1$. Thus, the RBF NN approximation property will be employed here so that $f$ can be approximated by an RBF NN in the form of (6) on the compact set $\Omega_\zeta$ as follows:

$$f = \theta \beta(Z).$$  

(30)

Now, consider the following positive definite function for the whole closed-loop system:

$$V = V_n + \frac{1}{2} \lambda^{-1} \tilde{d}^2 + \frac{1}{1 - \zeta} \int_{t-d(t)}^{t} e^v \psi(y, y(v)) dv$$  

(31)

where $\lambda$ is a design parameter and $\psi(\cdot)$ is a positive continuous function to be determined. Notice that $d(t) \leq \zeta < 1$, the differential of $V$ satisfies

$$LV \leq z_1^3 \left[ \tilde{g}a_1 + \phi_1(y) + z_1 \left( 2\tilde{t} + \frac{3}{4\lambda_{11}} + \frac{3}{4\lambda_{12}} \right) \right]$$

$$+ \hat{\theta} \left( \beta(Z) \tan \left( \frac{z_1^4 \beta(Z)}{\sigma} \right) \right)$$

$$- \left[ b\zeta_{\min}(p) - \frac{3}{2} b \|P\|_2 e^{\frac{3}{4}} \right] - \frac{1}{4} \sum_{i=1}^{n} \delta_j - \frac{3}{2} b \|P\|^2 e^{\frac{3}{2}}$$

$$- \frac{3}{2} b \|P\|^2 e^4 - \epsilon_4 \left( 2 \|b\|_2 + b \|P\|_{\lambda_{\max}(p)} \right) \right] \tilde{z}^4$$

$$+ z_2^3 [a_2 - \Delta_2] + \sum_{i=3}^{n} z_3^3 [a_i - \Delta_i] + \left[ u - \Delta_u \right]$$

$$- \lambda^{-1} \hat{\theta} \left( \beta(Z) \tan \left( \frac{z_1^4 \beta(Z)}{\sigma} \right) \right)$$

$$- r \left( \frac{1}{1 - \zeta} \int_{t-d(t)}^{t} e^v \psi(y, y(v)) dv \right) - \frac{1}{2} \omega \tilde{t}^2$$

$$- \frac{1}{2} \tilde{t} \hat{\theta}^2 + D - \tilde{l}^T \Gamma^{-1} \left[ \tilde{l} - (-\alpha \Gamma \tilde{l} + \tau_n) \right]$$

(32)

where $D = 1 / 2 \omega \tilde{t}^2 + 1 / 2 \tilde{t} \hat{\theta}^2 + 0.2785 \omega \theta$. From (32), we can design the virtual control law, the actual control law and the adaptive laws as follows:

$$\alpha_1 = N(\zeta) \eta$$  

(33)
\[a_2 = -c_2\ddot{z}_2 + k_2\ddot{\xi}_1 + \frac{\partial a_1}{\partial \dot{y}}(\ddot{y}_1) + \frac{\partial a_1}{\partial \xi_1}(\dot{\xi}_2 - k_1\dot{\xi}_1) + \frac{\partial a_1}{\partial \dot{\theta}} \dot{\theta} + \frac{\partial a_1}{\partial \dot{l}} \dot{l} + \frac{\partial a_1}{\partial \dot{z}_2} - z_2\]
\[
\times \left[ \frac{1}{4}\delta_1 + \frac{1}{4}\delta_2 + \frac{3}{4}\delta_{22} + \frac{3}{4}\delta_{21} \right] \left[ \frac{\partial a_1}{\partial y} \frac{4}{7} \right] \left[ \frac{\partial^2 a_1}{\partial y^2} \frac{2}{7} \right] + \frac{1}{4}\lambda_{22} \left[ \frac{\partial^2 a_1}{\partial y^2} \frac{2}{7} \right] + \frac{3}{4}\lambda_{21} \left[ \frac{\partial a_1}{\partial y} \frac{4}{7} \right] + i\left[ \frac{\partial a_1}{\partial y} \frac{4}{7} \right] (2 + \dot{a}_1) \right]
\]
\[a_i = -c_i\dot{z}_i + k_i\dot{\xi}_i + \frac{\partial a_{i-1}}{\partial \dot{y}}(\ddot{y}_i) + \frac{\partial a_{i-1}}{\partial \xi_j}(\dot{\xi}_j - k_j\dot{\xi}_j) + \frac{\partial a_{i-1}}{\partial \dot{\theta}} \dot{\theta} + \frac{\partial a_{i-1}}{\partial \dot{l}} \dot{l} + \frac{\partial a_{i-1}}{\partial \dot{z}_2} - z_i \left[ \frac{1}{4}\delta_{1i-1,4} \right]
\]
\[
\times \frac{3}{4}\lambda_{i4} + \frac{1}{4}\delta_2 + \frac{3}{4}\lambda_{i1} \left[ \frac{\partial a_{i-1}}{\partial y} \frac{4}{7} \right] + \frac{1}{4}\lambda_{i3} \left[ \frac{\partial^2 a_{i-1}}{\partial y^2} \frac{2}{7} \right] + \frac{3}{4}\lambda_{i2} \left[ \frac{\partial a_{i-1}}{\partial y} \frac{4}{7} \right] + i\left[ \frac{\partial a_{i-1}}{\partial y} \frac{4}{7} \right] (2 + \dot{a}_i) \right]
\]
\[u = -c_n\dot{z}_n + k_n\dot{\xi}_n + \frac{\partial a_{n-1}}{\partial \dot{y}}(\ddot{y}_i) + \frac{\partial a_{n-1}}{\partial \dot{\theta}} \dot{\theta} + \frac{\partial a_{n-1}}{\partial \dot{l}} \dot{l} + \frac{\partial a_{n-1}}{\partial \dot{z}_2} - z_n \left[ \frac{1}{4}\delta_{1n-1,4} \right]
\]
\[
\times \frac{3}{4}\lambda_{n4} + \frac{1}{4}\delta_2 + \frac{3}{4}\lambda_{n1} \left[ \frac{\partial a_{n-1}}{\partial y} \frac{4}{7} \right] + \frac{1}{4}\lambda_{n3} \left[ \frac{\partial^2 a_{n-1}}{\partial y^2} \frac{2}{7} \right] + \frac{3}{4}\lambda_{n2} \left[ \frac{\partial a_{n-1}}{\partial y} \frac{4}{7} \right] + i\left[ \frac{\partial a_{n-1}}{\partial y} \frac{4}{7} \right] (2 + \dot{a}_n) \right]
\]
where \(c = \min\{2c_0, 4c_i, z_i^\prime \dot{\lambda}, \omega \Gamma\}, i = 1, 2, \ldots, n, and\)
\[
c_0 = b\lambda_{\text{min}}(P) - \frac{1}{2} \sum_{i=1}^{n} \delta_j - \frac{3}{2} b\|P\|^4 - \frac{3}{2} b\|P\|^4
\]
(40)

**Theorem 4.1:** Consider the closed-loop system consisting of (9), the observer (11), the control law (36), and the adaptive laws (37) and (38). Under the Assumptions 2.1 and 2.2, the following properties can be obtained:
1) all the involved signals are bounded in probability 2) the error signal converges to the compact set defined as \( \Omega_\varepsilon := \{z \in \mathbb{R} | E(\sum_{i=1}^{n} z_i^4) \leq 4\varepsilon\}. \)

**Proof:** See Appendix B.

### V. Simulation Example

In this section, two simulation examples will be given to illustrate the effectiveness of the proposed control method.

**Example 5.1:** Consider the following second-order stochastic nonlinear system:
\[
\begin{align*}
\frac{dx_1(t)}{dt} &= [g_1 x_2(t) + y^2(t) + y(t)y(t - d(t))]dt + \sin(y(t - d(t)))dw \\
\frac{dx_2(t)}{dt} &= [g_2 u + y(t) + \sin(y(t)y(t - d(t))]dt + y(t - d(t))e^{-y^2(t - d(t))}dw \\
y(t) &= x_1(t)
\end{align*}
\]
where \(g_1\) and \(g_2\) represent the unknown control direction coefficients, \(d(t) = 2 + 0.5 \cos(t)\).

Manipulating the linear state transformations with \(\zeta_1 = x_1/(g_1g_2)\) and \(\zeta_2 = x_2/g_2\), the following system can be obtained:
\[
\begin{align*}
d\zeta_1(t) &= \left[ \zeta_2(t) + \frac{1}{g_1 g_2} y^2(t) + \frac{1}{g_1 g_2} y(t)(y(t - d(t)) \right]dt + \frac{1}{g_1 g_2} \sin(y(t - d(t)))dw \\
d\zeta_2(t) &= \left[ u + \frac{1}{g_2} y(t) + \frac{1}{g_2} \sin(y(t)y(t - d(t)) \right]dt + \frac{1}{g_2} y(t - d(t))e^{-y^2(t - d(t))}dw \\
dy(t) &= \left[ g_1 g_2 \zeta_2(t) + y^2(t) + y(t)y(t - d(t)) \right]dt + \sin(y(t - d(t)))dw.
\end{align*}
\]

Design the observer for (42) as follows:
\[
\begin{align*}
\dot{\zeta}_1 &= \dot{\zeta}_2 - \dot{\zeta}_1 \\
\dot{\zeta}_2 &= u - \dot{\zeta}_1.
\end{align*}
\]

Define the error variables \(z_1 = y\) and \(z_2 = \dot{\zeta}_2 - a_1(y, \dot{\zeta}, \dot{\theta}, \dot{\bar{l}})\). By virtue of the design procedures and results given in Section IV, the signal \(\zeta_i\), the adaptive output feedback virtual
control law \( \alpha_1 \), the actual control law \( u \), and the adaptive laws are given by

\[
\dot{\eta} = z_1 \left[ c_1 + 2\dot{l} + \frac{3}{4}\dot{\lambda}_{11} + \frac{3}{4}\dot{\lambda}_{12} + \dot{\theta} \beta(Z) \tanh\left( \frac{z_1^4 \beta(Z)}{\sigma} \right) \right] + y^2 (t) \\
u = -c_2 z_2 + \dot{\zeta}_1 + \frac{\partial \alpha_1}{\partial y}(y) + \frac{\partial \alpha_1}{\partial \zeta_1}(\dot{\zeta}_2 - \dot{\zeta}_1) + \frac{\partial \alpha_1}{\partial \theta} \dot{\theta} \\
+ \frac{\partial \alpha_1}{\partial l} \dot{l} + \frac{\partial \alpha_1}{\partial \zeta} \dot{\zeta}
\]

\[
- z_2 \left[ \frac{1}{4} \dot{\theta}_{22} + \frac{3}{4} \dot{\lambda}_{24} + \frac{3}{4} \dot{\lambda}_{21} \frac{\partial \alpha_1}{\partial y} \frac{4}{2} + \frac{1}{4} \dot{\lambda}_{23} \frac{\partial^2 \alpha_1}{\partial y^2} \right] z_2^2 \\
+ \frac{3}{4} \dot{\lambda}_{22} \frac{\partial \alpha_1}{\partial y} \frac{4}{2} + 2 (2 + \dot{\alpha}_1^4) \\
\dot{l} = -\omega \Gamma \dot{l} + 2 \Gamma z_1^4 + \Gamma z_2^2 \frac{\partial \alpha_1}{\partial y} \frac{4}{2} (2 + \dot{\alpha}_1^4) \\
\dot{\theta} = -\lambda' \dot{\theta} + \lambda z_1^4 \beta(Z) \tanh\left( \frac{z_1^4 \beta(Z)}{\sigma} \right).
\]

In this simulation, one RBF NN is used to approximate the unknown function, which contains 100 nodes and the width of the basis function is chosen as five. The Nussbaum function is chosen as \( N(\zeta) = \zeta^2 \cos(\zeta) \). The following initial conditions and suitable parameters can be chosen as \( x_1(0) = 0.5, \quad \dot{z}_2(0) = 0.5, \quad \dot{\zeta}_1(0) = 0.7, \quad \dot{\zeta}_2(0) = 1, \quad \zeta(0) = 0.5, \quad \dot{\theta}(0) = 0.8, \quad \dot{l}(0) = 0.5, \quad c_1 = c_2 = 1, \quad \lambda_{11} = \lambda_{12} = 5, \quad \sigma = 0.05, \quad \delta_2 = 0.05, \quad \lambda_{21} = \lambda_{22} = \lambda_{23} = 200, \quad \lambda_{24} = 1, \quad \omega = 400, \quad \Gamma = 0.01, \quad \lambda = 0.02, \quad \lambda' = 200. \) If the unknown control coefficients are chosen as \( g_1 = -0.8 \) and \( g_2 = -0.5 \), the simulation results are shown in Figs. 1–6. Then, changing the unknown control coefficients as \( g_1 = +0.8 \) and \( g_2 = +0.5 \), the simulation results, with the same initial conditions, the parameters, the controller and the adaptive laws, are shown in Figs. 7–12. From the simulation results, it can be seen that the proposed controller in this paper is effective.

Example 5.2: To further illustrate the effectiveness of our results, we consider the following practical example of a pendulum system with stochastic disturbances [42]:

\[
ml\ddot{q} = -mg \sin q - klq + \frac{1}{l} u
\]  

(44)

where \( u \) is the torque applied to the pendulum, \( q \) is the antclockwise angle between the vertical axis through the pivot point and the rod, \( g \) is the gravity acceleration, and the constants \( k, l, \) and \( m \) denote a coefficient of friction,
The length of the rod, and the mass of the bob, respectively. It is assumed that the constant \( l \) is unknown. Since there are not stochastic disturbances and time delays in the pendulum systems, we introduce the stochastic disturbances and the time-delay coefficient \( s \), which satisfy \( s \in [0, 1] \). Let \( x_1 = q/k, x_2 = ml\dot{q} + klq \). The nonlinear stochastic system with time delays can be expressed as follows:

\[
\begin{align*}
    dx_1 &= (g_1 x_2 - s \frac{k}{m} x_1 - (1 - s)x_1 (t - d(t))) dt \\
    &\quad + 0.5(x_1(t - d(t)))^2 d\omega \\
    dx_2 &= (g_2 u - smg \sin kx_1 + (1 - s) \sin(x_1(t - d(t)))) dt \\
    &\quad + 0.5(x_1(t - d(t)))^2 d\omega \\
    y &= x_1
\end{align*}
\]  

where the time-delay coefficient is chosen as \( s = 0.5 \), \( g_1 = 1/mkl \) and \( g_2 = 1/l \) represent the unknown control direction coefficients. \( d(t) = 0.5(1 + \sin t(t)) \) is the time delay. After the same linear state transformation as the Example 5.1, (45) can be rewritten in the form of (10).
Then, the following system can be obtained:

\[
\begin{align*}
    d\zeta_1(t) &= \left[ \frac{1}{2} \frac{k}{m} x_1 - \frac{1}{2} (1-s)x_1(t-d(t)) \right] dt \\
    d\zeta_2(t) &= \left[ u - \frac{1}{2} \frac{k}{m} x_1 + \frac{1}{2} (1-s) \sin(x_1(t-d(t))) \right] dt \\
    dy(t) &= \left[ \frac{1}{2} \frac{k}{m} x_1 - \frac{1}{2} (1-s)x_1(t-d(t)) \right] dt \\
        &+ \frac{1}{2} \sin(x_1(t-d(t))) d\omega.
\end{align*}
\]

(46)

Based on the proposed control scheme, the foregoing control problem is easily solved. First, the observer is designed as follows:

\[
\begin{align*}
    \dot{\hat{\xi}} &= \hat{\xi} - \hat{\zeta} \\
    \dot{\zeta} &= u - \hat{\zeta}.
\end{align*}
\]

(47)

Then, define the error variables \( z_1 = y, z_2 = \hat{\zeta} - a_1 \), and follow the same manipulations in Section IV, the signal \( \hat{\xi} \), the virtual control law \( a_1 \), the actual control law \( u \), and the adaptive laws can be given as follows \( \hat{\xi} = z_1^3, a_1 = N(\hat{\xi}) \eta, \bar{a}_1 = a_1/y \):

\[
\begin{align*}
    \eta &= z_1 \left[ c_1 + \frac{2}{4} \frac{k}{m} x_1 + \frac{3}{4} \frac{k}{m} x_1 + \frac{1}{2} \beta(Z) \tanh \left( \frac{z_1^4 \beta(Z)}{\sigma} \right) \right] \\
    u &= -c_2 z_2 + \dot{\hat{\xi}}_1 + \frac{\partial a_1}{\partial y} \phi_1(y) + \frac{\partial a_1}{\partial \xi} (\hat{\zeta} - \hat{\zeta}) + \frac{\partial a_1}{\partial \theta} \hat{\theta} \\
    &+ \frac{\partial a_1}{\partial \varphi} \hat{\varphi} + \frac{\partial a_1}{\partial \xi} \hat{\xi} \\
    &- z_2 \left[ \frac{1}{4} \frac{\partial a_1}{\partial y} + \frac{1}{4} \frac{\partial a_1}{\partial \varphi} + \frac{3}{4} \frac{\partial a_1}{\partial \xi} \right] ^{1/4} \\
    &+ \frac{1}{4} \frac{\partial a_1}{\partial y} \left( \frac{\partial a_1}{\partial y} \right)^{1/2} z_2^2 + \frac{3}{4} \frac{\partial a_1}{\partial \varphi} \left( \frac{\partial a_1}{\partial \varphi} \right)^{1/2} \\
    &+ \left( \frac{\partial a_1}{\partial \xi} \right)^{1/4} \left( 2 + \hat{a}_1 \right)^{1/4} \\
    \dot{\theta} &= -\lambda \dot{\varphi} + \lambda z_1^4 \beta(Z) \tanh \left( \frac{z_1^4 \beta(Z)}{\sigma} \right).
\end{align*}
\]

Similar to the Example 5.1, in this simulation, one RBF NN is taken for the unknown function, which contains 100 nodes and the width of the basis function is chosen as five. The Nussbaum function is chosen as \( N(\hat{\xi}) = \hat{\xi}^2 \cos(\hat{\xi}) \). The following initial conditions and suitable parameters can be chosen as \( m = 2, k = 0.98, l = 1, g = 9.8, x_1(0) = 1, \hat{\xi}(0) = 0.5, a_1 = 0.01, a_2 = 0.01, \bar{a}_1 = 0.01, \bar{a}_2 = 0.01, \lambda = 0.1, \lambda' = 10 \). The simulation results are shown in Figs. 13–15, from which we can see that the control performance is still very well. To demonstrate the superiority of the algorithm proposed in this paper, a comparative simulation on (45) will be also given using the algorithm proposed in [42]. The initial conditions and suitable design parameters are chosen as \( x_1(0) = 0.1, \hat{\xi}(0) = 0.1, x_2(0) = \hat{\xi}_2(0) = 0, \bar{a}_1 = 0.01, \bar{a}_2 = 0.01, \lambda = 0.1, \lambda' = 10 \). All the meaning of the design parameters can be founding [42], which will not be explained in this paper. The simulation results are shown in Figs. 16–18. Comparing the results in Figs. 16–18 with the results in Figs. 13–15, it is obvious that it takes more time to keep (45) stable using the algorithm in [42] than the algorithm proposed in this paper. This indicates that the algorithm proposed in this paper has a better transient performance than the algorithm proposed in [42].
where $\varepsilon > 0$, the constants $p > 1$ and $q > 1$ satisfy $(p - 1)(q - 1) = 1$, and $(x, y) \in \mathbb{R}^2$. Applying these inequalities leads to

$$2b_{\infty}^T P_{\infty} (\Phi^T(y) P_{\infty}) \leq \frac{1}{2\varepsilon_0^2} b \|P\|^2 \sum_{i=1}^{n} \left( \frac{1}{\prod_{j=i}^{n} g_j} \phi_i(y) \right)^4$$

(A.1)

$$2b_{\infty}^T P_{\infty} (F^T P_{\infty}) \leq \frac{3}{2b} \|P\|^2 \sum_{i=1}^{n} \left( \frac{1}{\prod_{j=i}^{n} g_j} f_i \right)^4$$

(A.2)

$$2b_{\infty}^T P_{\infty} ((K\zeta)^T P_{\infty}) \leq \frac{3}{2b} \|P\|^2 \sum_{i=1}^{n} (k_i\zeta_i)^4$$

(A.3)

$$bTr\{ (2P_{\infty}^T P + \zeta_{\infty}^T P_{\infty} P) H H^T \}$$

$$\leq \frac{1}{\varepsilon_4} \left[ 2b \|P\| + b \|P\| \lambda_{\max}(P) \right] \sum_{i=1}^{n} \left( \frac{1}{\prod_{j=i}^{n} g_j} h_i \right)^4$$

$$+ \varepsilon_4 [2b \|P\| + b \|P\| \lambda_{\max}(P)] \zeta_{\infty}^4$$

(A.4)

$$\dot{\zeta}_{\infty}^2 \leq \frac{1}{4} \delta_{11} \zeta_{\infty}^4 + \frac{3}{4\bar{\delta}_{11}} \bar{g}^2 \bar{z}_{\infty}^4$$

(A.5)

$$\dot{\zeta}_{\infty}^2 \leq \frac{1}{4} \delta_{12} \zeta_{\infty}^4 + \frac{3}{4\bar{\delta}_{12}} \bar{g}^2 \bar{z}_{\infty}^4$$

(A.6)

$$\dot{z}_i^4 f_i \leq \frac{3}{4\lambda_{11}} \zeta_{\infty}^4 + \frac{1}{4}\lambda_{11} y^4(t - d(t))$$

(A.7)

$$\frac{3}{2} \zeta_{\infty}^2 h_i \bar{h}_i^T \leq \frac{3}{4\lambda_{12}} y^4(t - d(t)) \bar{h}_i^T (y, y(t - d(t))) + \frac{3}{4\lambda_{12}} \zeta_{\infty}^4$$

(A.8)

$$-\frac{\partial a_{i-1}}{\partial y} \bar{g}^2 \bar{z}_{\infty}^2 \leq \frac{1}{4} \delta_{11} \zeta_{\infty}^4 + \frac{3}{4\bar{\delta}_{11}} \bar{g}^2 \left( \frac{\partial a_{i-1}}{\partial y} \right) \frac{1}{4} \zeta_{\infty}^4$$

(A.9)

$$-\frac{\partial a_{i-1}}{\partial y} \bar{g}^2 \bar{z}_{\infty}^2 \leq \frac{1}{4} \delta_{12} \zeta_{\infty}^4 + \frac{3}{4\bar{\delta}_{12}} \bar{g}^2 \left( \frac{\partial a_{i-1}}{\partial y} \right) \frac{1}{4} \zeta_{\infty}^4$$

(A.10)

$$-\frac{\partial a_{i-1}}{\partial y} \bar{g}^2 \bar{z}_{\infty}^2 \leq \frac{1}{4} \delta_{12} \zeta_{\infty}^4 + \frac{3}{4\bar{\delta}_{12}} \bar{g}^2 \left( \frac{\partial a_{i-1}}{\partial y} \right) \frac{1}{4} \zeta_{\infty}^4$$

(A.11)

VI. CONCLUSION

In this paper, the output-feedback adaptive neural control has been investigated for a class of stochastic nonlinear systems with time-varying delays and unknown control directions. Through a linear state transformation, the unknown control coefficients are grouped together and the original system is transformed into a new system that makes the control design feasible. The proposed adaptive neural controller contains only one adaptive parameter that is relevant to the given NNs such that the online learning time is dramatically decreased. Notably, only one NN is employed to compensate for all the unknown functions. The stability analysis guarantees that all the signals in the closed-loop system are bounded in probability. The simulation examples demonstrate the performance of the proposed approach. Future works will focus on the virtual control gain function and the function $\phi(x_i)$ rather than $\phi(y)$, which will make the design more challenging.

APPENDIX A

In this appendix, we use Young’s inequality [43]

$$xy \leq \frac{p}{p} |x|^p + \frac{1}{q} |y|^q$$

Fig. 16. Response of $x_1$ and $\hat{x}_1$ in [42].

Fig. 17. Response of $x_2$ and $\hat{x}_2$ in [42].

Fig. 18. Control input $u$ in [42].
\[
\frac{1}{2} \frac{\partial^2 a_{i-1}}{\partial y^2} z_i^2 h_{i-1}^T \leq \frac{3}{2} \frac{\partial a_{i-1}}{\partial y} \frac{\partial z_i}{\partial y} \left( \frac{3}{4} \lambda_{13} z_i^2 \right) + \frac{1}{4} \lambda_{13} y^4 (t - d(t)) h_{i-1}^T
\]
(13)

\[
\frac{3}{2} z_i^2 \left( \frac{\partial a_{i-1}}{\partial y} \right)^2 h_{i-1}^T \leq \frac{3}{4} \frac{\partial a_{i-1}}{\partial y} \left( \frac{\partial z_i}{\partial y} \right)^4 + \frac{3}{4} \frac{\partial a_{i-1}}{\partial y} \left( \frac{\partial z_i}{\partial y} \right)^2
\]
(A.14)

Then, we can define

\[
\Delta_2 = k_2 \hat{c} + \frac{\partial a_1}{\partial y} \phi_1(y) + \frac{\partial a_1}{\partial \theta} \hat{c} + \frac{\partial a_1}{\partial \theta} \hat{c}
\]

\[
+ \frac{1}{4} \delta_1 + \frac{1}{4} \delta_2 + \frac{3}{4} \lambda_{23} \left( \frac{\partial a_1}{\partial y} \right)^4
\]

\[
+ \frac{3}{4} \lambda_{24} + i \left( \frac{\partial a_1}{\partial y} \right)^4 \left( 2 + \frac{3}{2} \right)
\]
(15)

\[
\Delta_i = k_i \hat{c} + \frac{\partial a_{i-1}}{\partial y} \phi_1(y) + \sum_{j=1}^{i-1} \frac{\partial a_{j-1}}{\partial \theta} \left( \hat{c}_{j+1} - k_j \hat{c}_j \right)
\]

\[
+ \frac{\partial a_{i-1}}{\partial \theta} \hat{c} + \frac{\partial a_{i-1}}{\partial \theta} \hat{c}
\]

\[
- \left[ \frac{1}{4} \lambda_{14} + \frac{3}{4} \lambda_{13} \right] + \frac{1}{4} \delta_1 + \frac{3}{4} \lambda_{14} \left( \frac{\partial a_{i-1}}{\partial y} \right)^4
\]

\[
+ \frac{1}{4} \lambda_{13} \left( \frac{\partial a_{i-1}}{\partial y} \right)^2 \left( 2 + \frac{3}{2} \right)
\]

\[
+ \left( \frac{\partial a_{i-1}}{\partial y} \right)^4 \left( 2 + \frac{3}{2} \right)
\]
(A.16)

\[
\Delta_n = k_n \hat{c} + \frac{\partial a_{n-1}}{\partial y} \phi_1(y) + \sum_{j=1}^{n-1} \frac{\partial a_{j-1}}{\partial \theta} \left( \hat{c}_{j+1} - k_j \hat{c}_j \right)
\]

\[
+ \frac{\partial a_{n-1}}{\partial \theta} \hat{c} + \frac{\partial a_{n-1}}{\partial \theta} \hat{c}
\]

\[
- \left[ \frac{1}{4} \lambda_{n1} + \frac{3}{4} \lambda_{n2} \right] + \frac{1}{4} \delta_n + \frac{3}{4} \lambda_{n1} \left( \frac{\partial a_{n-1}}{\partial y} \right)^4
\]

\[
+ \frac{1}{4} \lambda_{n2} \left( \frac{\partial a_{n-1}}{\partial y} \right)^2 z_n^2
\]

\[
+ \left( \frac{\partial a_{n-1}}{\partial y} \right)^4 \left( 2 + \frac{3}{2} \right)
\]
(A.17)

\[
\Lambda_i = y^4 b \| P \|^2 \left[ \frac{1}{2} \sum_{m=1}^{n} \left( \frac{1}{\Pi_{j=m}^n g_j} \phi_m(y) \right)^4 + \frac{1}{2} \sum_{m=1}^{n} \left( \frac{1}{\Pi_{j=m}^n g_j} \right)^4 \right]
\]

\[
+ \left[ \frac{1}{4} \sum_{i=1}^{n} \delta_j + \frac{3}{2} b \| P \|^2 \epsilon_1 + \epsilon_4 (2 b \| P \|)
\]

\[
+ b \| P \| \lambda_{\max} (p) \left( \frac{3}{2} b \| P \|^2 \epsilon_2 \right)
\]

\[
+ \frac{3}{2} b \| P \|^2 \epsilon_3 \left( 2 \epsilon_3 \right) + y^4 (t - d(t))
\]

\[
\times \left[ \frac{1}{2} b \| P \|^2 \sum_{m=1}^{n} \left( \frac{1}{\Pi_{j=m}^n g_j} \right)^4 + \frac{1}{4} \sum_{j=1}^{i} \lambda_j \bar{f}_1^4
\]

\[
+ \frac{3}{4} \lambda_{j2} \bar{f}_1^4 + \frac{1}{4} \sum_{j=2}^{i} \lambda_j \bar{f}_1^4 \right].
\]
(A.18)

**APPENDIX B**

**PROOF OF THEOREM 4.1**

1) Define \( W(t, x) = V(t, x) e^{ct} \). Applying [Itô’s formula to the function \( W(t, x) \) yields

\[
d(V(t, x) e^{ct}) = e^{ct} (V(t, x) + LV) dt
\]

\[
+ e^{ct} \left( \frac{\partial V}{\partial \xi^1} h_1 - \sum_{i=2}^{n} \frac{\partial V}{\partial \xi_i} h_i + \frac{\partial V}{\partial H} \right) dw
\]

\[
\leq e^{ct} ((\tilde{g} N(\xi) + 1) \hat{\xi} + D) dt
\]

\[
+ e^{ct} \left( \frac{\partial V}{\partial \xi^1} h_1 - \sum_{i=2}^{n} \frac{\partial V}{\partial \xi_i} h_i + \frac{\partial V}{\partial H} \right) dw.
\]
(B.1)

Integrating (B.1) over \((0, t)\), we obtain

\[
V(t, x) \leq e^{-ct} V(0, x(0)) + \frac{D}{c}
\]

\[
+ \int_0^t e^{-c(t-s)} ((\tilde{g} N(\xi) + 1) \hat{\xi}) ds
\]

\[
+ \int_0^t e^{-c(t-s)} \left( \frac{\partial V}{\partial \xi^1} h_1 - \sum_{i=2}^{n} \frac{\partial V}{\partial \xi_i} h_i + \frac{\partial V}{\partial H} \right) dw(s).
\]
(B.2)

The stochastic integral (second integral) in (B.2) is local martingale. Then, according to Lemma 1, the conclusion can be obtained that the functions \( V(t, x), \hat{\xi}, (t, \hat{\xi}) \), and \( \int_0^t e^{-c(t-s)} ((\tilde{g} N(\xi) + 1) \hat{\xi}) ds \) must be bounded in probability.

2) Taking expectations on (B.2) and applying the Founsd lemma, we have

\[
EV(t, x) \leq EV(0, x(0)) + \frac{D}{c} + \sigma
\]
(B.3)
where \( \tilde{\sigma} = \sup \int_0^t \left( \| \tilde{g}(N_c(z)) \| + 1 \right) \tilde{c} ds. \) Then
\[
\frac{1}{4} E \sum_{i=1}^{4} z_i^4 \leq EV(t, x) + EV(0, x(0)) + \frac{D}{c} + \tilde{\sigma}.
\]

Denote \( \Xi = EV(0, x(0)) + \frac{D}{c} + \tilde{\sigma}, \) then \( E \sum_{i=1}^{4} z_i^4 \leq 4 \Xi. \) Therefore, there exists a compact set \( \Omega_c, \) such that \( z_i \in \Omega_c. \)

Thus, the proof is completed.

REFERENCES


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