Short communication

On the aliasing error upper bound for homogeneous random fields*

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Abstract. The Brown aliasing error upper bound in the sampling cardinal series expansion (SCSE) of weakly stationary non-band-limited (NBL) stochastic processes is extended to NBL homogeneous random fields (HRF). The magnitude of the derived bound is ordered under some smoothness condition upon the field.

1. Introduction

Brown has shown in his introductory paper [1] that a weakly stationary NBL stochastic process \( \{X(t) | t \in \mathbb{R}\} \) approximated by its SCSE
\[ X_w(t) = \sum_{n \in \mathbb{Z}} X(n\pi/w) \text{sinc}(wt-n\pi), \]
for a given bandwidth \( 0 < w < +\infty \), possesses the mean-square aliasing error
\[ \mathbb{E}_w(t) = \mathbb{E}[|X(t) - X_w(t)|^2] \]
bounded above in terms of the spectral density \( f(\lambda) \) of \( X(t) \):
\[ \mathbb{E}_w(t) \leq 4 \int_{|\lambda| > w} f(\lambda) \, d\lambda. \quad (1.1) \]

Here \( \text{sinc}(x) = x^{-1} \sin(x), x \neq 0, \text{sinc}(0) = 1 \). Brown also proved that the constant 4 is sharp even (1.1) holds uniformly in \( t \).

It should be pointed out that (1.1) also holds whenever the spectral distribution \( F(\lambda) \) of \( X(t) \) is continuous at all points \( (2k+1)\pi, k \in \mathbb{Z} \), i.e. \( \mathbb{E}_w(t) \leq 4 \int_{|\lambda| > w} dF(\lambda) \) [5].

Due to the Paley–Wiener theorem, there do not exist signal functions which are simultaneously band-limited and duration-limited, and this fact causes complications in engineering applications, since both limitations are natural for real physical signals [7]. The way between the theory and practice will be bridged by considering signals which are approximately band-limited, on the basis of a given level of accuracy (some applications are given in [2–8]). The point will be to choose the...
bandwidth \( w \) large enough, such that the mean-square aliasing error \( \mathcal{E}_x^2(t) \) becomes sufficiently small. For example, Turbovich was proposed in this purpose the so-called iterative sampling \([9]\).

The main goal of this paper is to extend (1.1) to the class of NBL HRFs and to order the magnitude of the bound with respect to the bandwidth \( W = (w_1, \ldots, w_r) > 0 \) under some smoothness conditions upon the considered field.

2. Preliminaries

Let \( \{\xi(x) | x \in \mathbb{R}^r\} \) be a zero-mean, \( r \)-dimensional NBL HRF defined on the probability space \((\Omega, \mathcal{F}, P)\). If \( K(\tau) = E\xi(x + \tau)\xi^*(x) \) is the correlation function of \( \xi(x) \), we have the spectral representations

\[
\xi(x) = \int_{\mathbb{R}^r} e^{i<\lambda, x>} dZ(\lambda),
\]

\[
K(\tau) = \int_{\mathbb{R}^r} e^{i<\lambda, \tau>} d\Phi(\lambda),
\]

where \( <a, b> \) denotes the inner product \( \sum_{j=1}^r a_jb_j \), \( Z(\lambda) \) is the spectral field and \( \Phi(\lambda) \) is the spectral distribution function of \( \xi(x) \), \( d\Phi(\lambda) = E|dZ(\lambda)|^2 \).

Let us introduce the multiple SCSE \( \xi_w(x) \) of \( \xi(x) \) to the given choice of bandwidth \( W = (w_1, \ldots, w_r) \),

\[
\xi_w(x) = \sum_{j=1}^r \sum_{n=-\infty}^\infty \xi(x^n) \prod_{k=1}^r \text{sinc}(w_kx_k - n_k\pi),
\]

say, where \( x^n \) runs over the set

\[
\{(n_1\pi/w_1, \ldots, n_r\pi/w_r) | n_j \in \mathbb{Z}\}.
\]

Denote \( |s| = \sum_{j=1}^r s_j \geq 0 \), \( s_j \) nonnegative integers, the mixed-exponent, i.e., for some \( \alpha = (\alpha_1, \ldots, \alpha_r) \) it is \( \alpha^{(s)} \triangleq \prod_{k=1}^r \alpha_k^{s_k}, \gamma \in \mathbb{R}. \) Finally let us take

\[
\mathcal{H} \triangleq \times_{j=1}^r (-w_j, w_j).
\]

For each real \( x_1 \) define \( (e^{ix_1\lambda_k})_{2w_k} \) as the \( 2w_k \)-periodic extension of \( \exp(ix_1\lambda_k) \) from \(( -w_k, w_k) \) to \( \mathbb{R}, k=1, \ldots, r. \)

3. Derivation of the bound

The main tool in the derivation is the following spectral representation of the multiple SCSE of the considered field \( \xi(x) \) proved in detail in \([6]\).

**PROPOSITION 1.** \( \Phi(\lambda) \) is continuous on the lattice \( \text{Lat}(W) \triangleq \{((2n_1+1)w_1, \ldots, (2n_r+1)w_r) | n_j \in \mathbb{Z}\} \) if and only if

\[
\xi_w(x) = \int_{\mathbb{R}^r} \prod_{k=1}^r (e^{i\lambda_kx_k})_{2w_k} dZ(\lambda).
\]

Now, we state and prove our main results.

**THEOREM.** Let \( \Phi(\lambda) \) be continuous on \( \text{Lat}(W) \). Then

(i) \( \mathcal{E}_w^2(x) \triangleq E|\xi(x) - \xi_w(x)|^2 \leq 4 \int_{|\lambda| > W} d\Phi(\lambda), \)

(ii) if \( \xi(x) \) is \(|s|\)-fold differentiable, \( |s| > 0 \), then it follows that

\[
\mathcal{E}_w^2(x) = O(W^{-2|s|}) = O\left(\prod_{k=1}^r w_k^{-2s_k}\right).
\]

**PROOF.** (i) On account the isometry between the Hilbert-spaces \( H(\xi) \triangleq \langle \xi(x) | x \in \mathbb{R}^r \rangle \) and \( L_2(\mathbb{R}^r; d\Phi) \triangleq \{\varphi | \mathbb{R}^r; \varphi^2d\Phi\} \) we get by \( (3.1) \)

\[
\mathcal{E}_w^2(x) = \int_{\mathbb{R}^r \setminus \mathcal{H}} \left| \sum_{k=1}^r (e^{i\lambda_kx_k})_{2w_k} \right|^2 d\Phi(\lambda)
\]

\[
\leq 4 \int_{\mathbb{R}^r \setminus \mathcal{H}} d\Phi(\lambda).
\]

The proof, that the constant 4 cannot be improved, would be identical to the same kind of proof in \([1]\) for the two-dimensional case.

(ii) The existence of the \(|s|\)-th mixed-derivative of \( \xi(x) \) is equivalent to

\[
\frac{\partial^{2|s|}}{\partial x^{2|s|}} K(0) = \frac{\partial^{2|s|} K(0)}{\partial x_1^{2s_1} \cdots \partial x_r^{2s_r}} \leq \infty.
\]

Therefore we get
Finally, we are interested in the mean-square convergence of \( \xi_w(x) \) to \( \xi(x) \) as \( \tilde{w} \to \infty \). Let us denote \( \tilde{w} = \min_{1 \leq j \leq r}(w_j) \).

**PROPOSITION 2.** \( \lim_{\tilde{w} \to \infty} \xi_w(x) = \xi(x) \).

**PROOF.** If \( |s| > 0 \), the statement directly follows from (3.3). If \( |s| = 0 \), it is not hard to see that \( \int_{\mathbb{R}\setminus B} d\Phi(\lambda) \) monotonically vanishes as \( \tilde{w} \to \infty \). \( \square \)

**REFERENCES**


