Further results on generalized Kapteyn-type expansions

Tibor Pogány

University of Rijeka, Faculty of Maritime Studies, Studentska 2, HR-51000 Rijeka, Croatia

A R T I C L E   I N F O

Article history:
Received 8 March 2007
Received in revised form 10 March 2008
Accepted 24 March 2008

In memoriam Csörgő Sándor, dear friend and great scholar

Keywords:
Bessel function of first kind
Bounds for \( J_\nu(z) \)
Positive zeros \( j_\nu, s \) of \( J_\nu(z) \)
Generalized Kapteyn expansion
Kapteyn series

A B S T R A C T

A new inequality is derived for the Bessel function of the first kind \( J_\nu(z) \), \( \nu \in \mathbb{R} \), and a discussion of the consequent convergence is given for the generalized Kapteyn-type expansion, improving certain results obtained recently by the author.

© 2008 Elsevier Ltd. All rights reserved.

1. Introduction with a short review of bounds for \( J_\nu(z) \)

The series representation of the Bessel function of the first kind \( J_\nu(z) \) of order \( \nu \) reads

\[
J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\nu + 2n}}{\Gamma(\nu + n + 1) n!} 
\]

(1)

In various fields of mathematics, mathematical physics and related applications the problem of bounding Bessel functions arises. Hansen [1, p. 31] gave the bound

\[
|J_0(z)| \leq 1, \quad |J_\nu(z)| \leq 1/\sqrt{2} \quad (r \in \mathbb{N}, z \in \mathbb{R}) .
\]

(2)

Exton [2] used the bound \( J_{N+2k}((N + 2k)z) \leq 1 \), \( N + 2k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N} \). Love [3, Eq. (11)] reported

\[
|J_\nu(x)| < \left( \frac{x}{2} \right)^{\nu} \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu + 1/2)} \quad (x > 0, \Im(\nu) > -1/2).
\]

Landau [4] gave in a sense best possible bounds for \( J_\nu(x) \) with respect to \( \nu \geq 0 \) and \( x \in \mathbb{R} \); these are

\[
|J_\nu(x)| \leq b_\nu x^{-1/3}, \quad b_\nu = \sqrt{2} \sup_{x \in \mathbb{R}^+} \text{Ai}(x),
\]

(3)

\[
|J_\nu(x)| \leq c_\nu |x|^{-1/3}, \quad c_\nu = \sup_{x \in \mathbb{R}^+} x^{1/3} J_0(x),
\]

(4)

where Ai(·) stands for the familiar Airy function

\[
\text{Ai}(-x) := \frac{\sqrt{x}}{3} \left[ J_{-1/3} \left( 2x^{3/2}/3 \right) + J_{1/3} \left( 2x^{3/2}/3 \right) \right].
\]
By the assumptions of the lemma we have
\[ \sup_{x \in [0, \pi]} \sqrt{x} J_{\nu}(x) \leq b_1 \sqrt{\nu^{1/3} + \frac{\alpha_1}{\nu^{1/3}} + \frac{3\alpha_1^2}{10\nu}} := d_\nu \ (\nu > 0), \] (5)
where \( \alpha_1 = 2^{-1/3} a_1 \), and \( a_1 \) is the smallest positive zero of the Airy function \( \text{Ai}(x) \), while \( b_1 \) is Landau's constant, described in (3).

A uniform bound was obtained by Krasikov [6] as well:
\[ J_{\nu}(x) \leq \frac{4(4x^2 - (2\nu + 1)(2\nu + 5))}{\pi (4x^2 - \mu)^{3/2} - \mu} =: \eta_\nu(x) \quad \left( x > \sqrt{\mu + \mu^{2/3}/2, \nu > -1/2} \right), \] (6)
where \( \mu = (2\nu + 1)(2\nu + 3) \). This estimate is sharp in the sense that
\[ J_{\nu}(x) \geq \frac{4(4x^2 - (2\nu + 1)(2\nu + 5))}{\pi (4x^2 - \mu)^{3/2} + \mu}, \]
at all points between every two consecutive zeros of \( J_{\nu}(x) \) [6, Theorem 2]. Krasikov remarked that the inequalities (3) and (4) are sharp only in the transition region, i.e. for \( x \) around the least positive zero \( j_{\nu, 1} \) of \( J_{\nu}(x) \). He pointed out that (6) provides a sharp bound in the whole oscillatory region; however, in the transition region this estimate becomes very poor and should be replaced with another estimate [6]. Because \( \eta_\nu(x) \) is positive and monotone decreasing on \( ((\mu + \mu^{2/3})/2, \infty) \), according to Krasikov's bound (6), in [7] the bound
\[ |J_{\nu}(x)| \leq \eta_\nu(x) := \frac{\sqrt{\pi}}{\sqrt{x}} \chi_{(0, \infty)}(x) = \sqrt{\frac{x}{\pi}} \left( 1 - \chi_{(0, \infty)}(x) \right) \left( A_\mu > \frac{1}{2} \right) \left( \mu + \mu^{2/3} \right), \] (7)
has been used, where \( \chi_{(0, \infty)}(x) \) stands for the characteristic function (or indicator) of the set \( S \). Finally, further inequalities can be found on the website [8]; unfortunately, the two most general of them, the fourth and fifth, are erroneous.

In this note we derive a bound for \( |J_{\nu}(z)| \) when \( z \) belongs to the closed Cassinian oval \( C_{\nu, \lambda} \); see (12). With the help of this estimate we study the convergence rate of a generalized Kapteyn-type expansion whose terms contain Bessel functions of the first kind, relaxing the earlier results, and simultaneously extending the convergence domain. Such series were considered earlier by Exton [2], and recently the author [9].

2. The inequality

Consider the well-known product representation formula [1, p. 550]
\[ J_{\nu}(z) = \frac{(z/2)^{\nu}}{\Gamma(\nu + 1)} \prod_{s=1}^{\infty} \left( 1 - \frac{z^2}{J_{\nu,s}^2} \right), \quad (z \in \mathbb{C}, \nu \notin \mathbb{N}), \]
where \( j_{\nu,s} \) denotes the \( s \)-th zero of \( J_{\nu}(z) \), the zeros being numbered in increasing order of their real parts. By the celebrated Lommel theorem \( "J_{\nu}(x) has an infinity of real zeros, for any given real value of \( \nu " \) [1, p. 478]. Therefore, here and in what follows, we will assume that \( \nu \) is real, so \( 0 < j_{\nu, 1} < j_{\nu, 2} < \cdots \). Finally, let us write
\[ \mathbb{D}_\eta = \{ z; |z| < \eta \} \]
for the closed centered disc having diameter \( 2\eta \), while the open unit disc is \( \mathbb{D} = \{ z; |z| < 1 \} \).

**Lemma 1.** Let \( 0 \leq \lambda \leq 1 \). Then for all \( \zeta - 1 \in \mathbb{D}_{(1-\lambda)/(1+\lambda)} \) there holds the inequality
\[ |1 - \zeta| \leq e^{-\lambda |\zeta|}. \] (8)

**Proof.** By the assumptions of the lemma we have
\[ |1 - \zeta| \leq \frac{1 - \lambda}{1 + \lambda}, \]
and at the same time \( \zeta \in \mathbb{D}_{2/(1+\lambda)} \). Indeed, by direct calculation we get
\[ |\zeta| \leq |1 - \zeta| + 1 \leq \frac{2}{1 + \lambda}. \]
Hence
\[ |1 - \zeta| + \lambda |\zeta| \leq 1, \] (9)
that is
\[ |1 - \zeta| \leq 1 - \lambda |\zeta| \leq e^{-\lambda |\zeta|}. \] (10)
Here we point out that (9) is not redundant (the intermediate expression in (10) is positive), since \( \mathbb{D}_{2/(1+\lambda)} \subset \mathbb{D}_{1/\lambda} \), because \( \lambda^{-1} > 2(1 + \lambda)^{-1} \). \( \square \)
In the next step we take \( \zeta = \left( \frac{z}{j_{\nu,1}} \right)^2 \) and for some fixed \( \lambda \in (0, 1) \) we conclude that

\[
|j_\nu(z)| \leq \frac{(|z|/2)^\nu}{|\Gamma(\nu + 1)|} \prod_{s=1}^\infty \left| 1 - \frac{z^2}{j_{\nu,s}^2} \right| \\
\leq \frac{(|z|/2)^\nu}{|\Gamma(\nu + 1)|} \prod_{s=1}^\infty \left( 1 - \frac{\lambda |z|^2}{j_{\nu,s}^2} \right) \\
\leq \frac{(|z|/2)^\nu}{|\Gamma(\nu + 1)|} \exp \left( -\lambda |z|^2 \sum_{s=1}^\infty \frac{1}{j_{\nu,s}^2} \right),
\]

(11)

which holds for all \( z = x + iy \) coming from the closed Cassinian oval

\[
\mathcal{C}_{\nu,\lambda} := \left\{ z : |x^2 - j_{\nu,1}^2| \leq j_{\nu,1}^2, \frac{1 - \lambda}{1 + \lambda} \right\}.
\]

(12)

Now, by the estimate (11) we deduce

\[
|j_\nu(z)| \leq \frac{|z|^\nu e^{-\lambda \sigma^{(2)}_\nu} |z|^2}{2^\nu |\Gamma(\nu + 1)|},
\]

(13)

where

\[
\sigma^{(2)}_\nu = \sum_{s=1}^\infty j_{\nu,s}^{-2}\nu
\]

denotes the so-called Rayleigh function; bearing in mind that \( \sigma^{(2)}_\nu = (4(\nu + 1))^{-1} \), [1, p. 550] we prove:

**Theorem 1.** Let \( 0 < \lambda < 1 \), \( \nu \in \mathbb{R} \). Then the following bilateral bounding inequality holds:

\[
|j_\nu(z)| \leq \frac{(|z|/2)^\nu}{|\Gamma(\nu + 1)|} \exp \left( -\frac{\lambda |z|^2}{4(\nu + 1)} \right) \quad (z \in \mathcal{C}_{\nu,\lambda}).
\]

(14)

For negative integer \( \nu \) the right-hand bound in (14) terminates.

**Remark 1.** It is worth mentioning here that recently András and Baricz [10] proved the analogue of (14) for when \( z \) belongs to \( \mathbb{D} \) (see the article [11] too). Namely, they proved [10, Corollary 1] that for all \( \nu > -1 \) the radius of the smallest disc which contains the image region \( \mathcal{J}_\nu(\mathbb{D}) \) is \( I_\nu(1) \), where

\[
\mathcal{J}_\nu(z) = (2/z)^\nu \Gamma(\nu + 1) j_\nu(z) \quad \text{and} \quad I_\nu(z) = (2/z)^\nu \Gamma(\nu + 1) I_\nu(z),
\]

and where \( I_\nu \) stands for the modified Bessel function. Moreover, in [10, Corollary 1] the authors have been proved that if \( \nu_1, \nu_2 \in \mathbb{C} \) such that \( \Re(\nu_1) > \Re(\nu_2) \geq -1/4 + 3\zeta/6 \), then \( \mathcal{J}_{\nu_1}(\mathbb{D}) \subset \mathcal{J}_{\nu_2}(\mathbb{D}) \). Thus, it would be of interest to see what happens with the monotonicity of the image region \( \mathcal{J}_\nu(\mathcal{C}_{\nu,\lambda}) \).

**Corollary 1.** Let \( 0 < \lambda < 1 \), \( \nu \in \mathbb{R} \). Then we have

\[
|\cos z| \leq \exp \left( -\frac{\lambda |z|^2}{2} \right) \quad \left( z \in \mathcal{C}_{-1/2,\lambda} = \left\{ z : |z - \pi/2|^2 \leq \frac{\pi^2}{4} \frac{1 - \lambda}{1 + \lambda} \right\} \right)
\]

(15)

\[
\left| \frac{\sin z}{z} \right| \leq \exp \left( -\frac{\lambda |z|^2}{6} \right) \quad \left( z \in \mathcal{C}_{1/2,\lambda} = \left\{ z : |z - \pi|^2 \leq \frac{\pi^2}{4} \frac{1 - \lambda}{1 + \lambda} \right\} \right).
\]

(16)

**Proof.** The inequality (14) can be written as

\[
|\mathcal{J}_\nu(z)| \leq \exp \left( -\frac{\lambda |z|^2}{4(\nu + 1)} \right).
\]

Recalling that \( \mathcal{J}_{-1/2}(z) = \cos z \) and \( \mathcal{J}_{1/2}(z) = (\sin z)/z \) we easily deduce (15) and (16), where \( \lambda \in (0, 1) \) and \( j_{-1/2,1} = \pi/2 \) and \( j_{1/2,1} = \pi \). \( \square \)
3. Convergence discussion

Any series of the type

$$
\sum_{n=0}^{\infty} \beta_n J_{\nu+n} ((v+n)z) \quad (z \in \mathbb{C})
$$

(17)

in which \(v, \beta_n\) are constants and \(J_{\mu}(z)\) stands for the first-kind Bessel function of order \(\mu\) is called a Kapteyn series [1, 17.1]. By certain formal manipulations Exton [2, Eqs.(1.1), (4.1)] derives the following Kapteyn-type series:

$$
\frac{\Gamma'(a_1 + v/2) \cdots \Gamma'(a_n + v/2)}{\Gamma'(b_1 + v/2) \cdots \Gamma'(b_k + v/2)} \left(\frac{z}{2}\right)^v = \nu^2 \sum_{k=0}^{\infty} \frac{\Gamma(v+k)}{(v+2k)^{v+1}} k! \mathcal{X}_{\nu+2k} \left[\frac{(a)}{(b)}\right] (v+2k)z
$$

(18)

where

$$
\mathcal{X}_{\nu} \left[\frac{(a)}{(b)}\right] := \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n \frac{\nu \Gamma(v+n+k)}{(v+1+k)n!} \mathcal{X}_{\nu+2k} \left[\frac{(a)}{(b)}\right] (v+2k)z
$$

(19)

Here the case \(n = 0\) corresponds to \(\mathcal{X}_{\nu} \equiv J_{\nu}\). Exton examined the convergence of the series on the right of (18), concluding that the series converges absolutely and uniformly for all real \(z\) for \(\Re(d) < 1\), where

$$
d := \sum_{j=1}^{n} (a_j - b_j).
$$

Very recently the author has improved Exton’s result to \(\Re(d) < 4/3\) using the bounds (3)–(5); see [9, Theorem 1]. Now, we shall relax this result in a certain complex domain, making use of the newly derived bound (14).

**Theorem 2.** Denote by \(0 < \rho_1 < \rho_2\) the real solutions of the equation

$$
\rho e^{-\lambda \rho^2 /4} = 2e^{-1} \quad (0 < \lambda < 1).
$$

(20)

Assume that

$$
z \in \bigcup_{\nu \in \mathbb{C}, k} D_k \cap \mathcal{E}_{\nu,k} \quad (A := [0, \rho_1) \cup (\rho_2, \infty))
$$

and let \(\nu, \Re(a_j) + v/2 \notin \mathbb{N}^- = \{0, -1, -2, \ldots\}. Then, the series

$$
\mathcal{X}_{\nu} \left[\frac{(a)}{(b)}\right] := \sum_{n=0}^{\infty} \frac{\Gamma(v+n+k)}{(v+1+k)n!} \mathcal{X}_{\nu+2k} \left[\frac{(a)}{(b)}\right] (v+2k)z
$$

(21)

converges absolutely and uniformly for all \(d = \sum_{j=1}^{n} (a_j - b_j) \in \mathbb{C}\).

Moreover, for \(|z| \in (\rho_1, \rho_2)\) the series (21) converges absolutely and uniformly when

$$
\Re(d) < \frac{3}{2}.
$$

(22)

**Proof.** By the well-known asymptotic expansion result

$$
\Gamma(z) = \sqrt{2\pi z^{1/2} e^{-z} \left(1 + \mathcal{O}(z^{-1})\right)} \quad (|\arg z| < \pi, \ |z| \to \infty)
$$

(23)

for \(k\) large enough, we have

$$
\mathcal{X}_{\nu+2k} \left[\frac{(a)}{(b)}\right] (v+2k)z \sim k^d J_{\nu+2k} ((v+2k)z).
$$

Therefore, it suffices to examine the convergence of the auxiliary series

$$
\mathcal{X}_{\nu} \left[\frac{(a)}{(b)}\right] := \sum_{k=0}^{\infty} \frac{\Gamma(v+k) k^d}{(v+2k)^{v+1} k!} J_{\nu+2k} ((v+2k)z) \quad \left(\Re(d) := \sum_{k=0}^{\infty} \Re(A_k)\right)
$$

(24)

such that it is equiconvergent with \(\mathcal{X}_{\nu} \left[\frac{(a)}{(b)}\right] \). Applying (14) to \(J_{\nu+2k} ((v+2k)z)\), we conclude that

$$
|J_{\nu+2k} ((v+2k)z)| \leq \frac{(v+2k)^{v+2k} |z|^{v+2k}}{2^{v+2k} |\Gamma(v+2k+1)|} \exp \left\{ \frac{\lambda(v+2k)^2 |z|^2}{4(v+2k+1)} \right\},
$$

(25)
By the Stirling formula and by (23) we find that

\[
\left| a_n \right| \leq \frac{(v + k)^{v+k-1/2}(v + 2k)^{2k-1}|z|^{v+2k}}{e^{4k} k^{2-\gamma(d)+1/2} \Gamma(v + 2k + 1)} \exp \left\{ -\lambda (v + 2k)^2 |z|^2 \right\} \frac{4(v + 2k + 1)}{4(v + 2k + 1)}
\]

\[
= \mathcal{O} \left[ \left( v + k \right)^{v+k-1/2}(v + 2k)^{2k-1} \exp \left\{ -\left( 2k + \frac{1}{v + 2k + 1} \right) \lambda |z|^2 \right\} \right]
\]

\[
= \mathcal{O} \left[ \left( 1 + v/k \right)^{v+k-1/2} \left( 1 + (v+1)/(2k) \right)^{2k-1} \exp \left\{ -\lambda |z|^2 k / 2 \right\} \right]
\]

\[
= \mathcal{O} \left( |z|^{2k} k^{\gamma(d)-5/2} e^{-\lambda |z|^2 k / 2} \right),
\]

that is, \( a_n |z| \) converges absolutely and a fortiori uniformly when the power series

\[
\sum_{k=1}^{\infty} \frac{|z|^{2k}}{k^{\gamma(d)-5/2}} e^{-\lambda |z|^2 k / 2}
\]

converges. This is the case (using e.g. the Cauchy test) when

\[
\lim_{k \to \infty} \left[ \frac{|z|^{2k}}{k^{\gamma(d)-5/2}} \right]^{1/k} = \left( \frac{|z|^{2k}}{k^{\gamma(d)-5/2}} \right)^{2/5} < 1.
\]

Intersecting the domain \( \xi_{\lambda, z} \) with the solution region of (26), we deduce the first assertion of the theorem.

Besides, solving (20) we locate \( 0 < \rho_1 < 1, \rho_2 > 2 \). Setting \( |z| \in (\rho_1, \rho_2) \) the convergence question of (21) reduces to the convergence of

\[
\sum_{k=1}^{\infty} \frac{|z|^{2k}}{k^{\gamma(d)-5/2}} = \xi \left( 5/2 - \gamma(d) \right),
\]

where \( \xi (\cdot) \) is the Riemann Zeta. So, \( \gamma(d) < 3/2 \) secures the convergence of \( a_n \xi_{\lambda, z} \).

Acknowledgements

I would like to express my gratitude to the unknown reviewer for drawing my attention to the Ref [10] and suggesting to me the corollary. The present investigation was supported by the Ministry of Sciences, Education and Sports of Croatia under Research Project Number 112-2352818-2814.

References