The Poisson sum formulae associated with the fractional Fourier transform

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A B S T R A C T

The theorem of sampling formulae has been deduced for band-limited or time-limited signals in the fractional Fourier domain by different authors. Even though the properties and applications of these formulae have been studied extensively in the literature, none of the research papers throw light on the Poisson sum formula and non-band-limited signals associated with the fractional Fourier transform (FrFT). This paper investigates the generalized pattern of Poisson sum formula from the FrFT point of view and derived several novel sum formulae associated with the FrFT. Firstly, the generalized Poisson sum formula is obtained based on the relationship of the FrFT and the Fourier transform; then some new results associated with this novel sum formula have been derived; the potential applications of these new results in estimating the bandwidth and the fractional spectrum shape of a signal in the fractional Fourier domain are also proposed. In addition, the results can be seen as the generalization of the classical results in the Fourier domain.

1. Introduction

As a generalization of the classical Fourier transform, the fractional Fourier transform (FrFT) has received much attention in recent years. It has been applied in several areas, including optics, quantum physics and in the signal processing community [1–4]; and its relationship with the Fourier transform can be found in [5–8]. The definition of the discrete FrFT and the fast computational methods of FrFT have been proposed by different researchers from different perspectives [9–11]. The well known operations and relations [such as Hilbert transform; convolution and product operations; uncertainty principle] in traditional Fourier domain have also been extended to the fractional Fourier domain by different authors [12–17]. The research achievements of the FrFT in the signal processing community has been proposed [18,19]. For further properties and applications of FrFT in optics and signal processing community, one can refer to [3,4].

The properties and applications of the sampling formulae in the traditional Fourier domain have simultaneously been studied extensively in the literature, and the extension of the sampling signal analysis for band-limited or time-limited signals in the fractional Fourier domain have been deduced recently [7,8,20,21]. But, so far none of the research papers throw light on the extension of the traditional Poisson sum formula [22–24], and also no results regarding the non-band-limited signals associated with the fractional Fourier domain have been reported as yet. It is therefore, worthwhile as well as interesting to investigate the extension of the Poisson sum formula and non-band-limited signals analysis associated with the FrFT.

The objective of this paper is to generalize the traditional Poisson sum formula in the Fourier domain to the fractional Fourier domain. In order to obtain the desired results for a signal \( x(t) \), we apply one of the
relationships between the FrFT and the Fourier transform to achieve the innovative result. The paper is organized as follows, the preliminaries are presented in Section 2, the main results of the paper in Section 3, the potential applications of the results in Section 4 and finally the conclusion and future working directions are given in Section 5.

2. Preliminaries

2.1. The FrFT

The FrFT of a signal \( x(t) \) with angle \( \alpha \) is defined as [3]

\[
X_\alpha(u) = \int_{-\infty}^{\infty} K_\alpha(u, t)x(t)\,dt
\]

(1)

where

\[
K_\alpha(u, t) = \begin{cases} 
A_\alpha \exp \left( \frac{i^2}{2} \cot \alpha - jut \csc \alpha \right), & \alpha \neq k\pi \\
+\frac{i^2}{2} \cot \alpha, & \alpha = k\pi \\
\delta(t-u), & \alpha = 2k\pi \\
\delta(t+u), & \alpha + \pi = 2k\pi
\end{cases}
\]

\[
A_\alpha = \sqrt{\frac{1-j \cot \alpha}{2\pi}}
\]

In order to obtain new results, this paper deals with the case when \( \alpha \neq k\pi \), and the following definitions have been introduced.

A signal \( x(t) \) is said to be band-limited with respect to \( \Omega_x \) in FrFT domain, when

\[
X_\alpha(u) = 0 \quad \text{for} \quad |u| > \Omega_x
\]

(2)

where \( \Omega_x \) is called the bandwidth of signal \( x(t) \) in the fractional Fourier domain. At the same time, a signal \( x(t) \) is called chirp-periodic with period \( T \) and order \( \alpha \) if it satisfies the following equation [8].

\[
e^{j(\pi/2)\cot \alpha} x(t) = e^{j(\pi-\alpha^2/2)\cot \alpha} x(t+T)
\]

(3)

The FrFT can be seen as the generalization of the traditional Fourier transform; when \( \alpha = \pi/2 \), it reduces to the Fourier transform. Further, there exists the following important relationships between the FrFT and the Fourier transform for a signal \( x(t) \). Without loss of generality, we assume \( \sin \alpha > 0 \) in the proceeding sections.

**Lemma 1.** Suppose the FrFT of a signal \( x(t) \) for order \( \alpha \) is \( X_\alpha(u) \), the Fourier transform operator is represented as \( \mathfrak{F} \). If we let \( g(t) = x(t)e^{j\pi/24\alpha^2} \cot \alpha \), then the following relations hold:

\[
X_\alpha(u)e^{-j(\pi/2)\cot \alpha} x(t) = \mathfrak{F}(g(t))(u \csc \alpha) \]

(4)

\[
X_\alpha(v \sin \alpha)e^{-j(v \sin \alpha^2/2)\cot \alpha} x(t) = \mathfrak{F}(g(t))(v) = G(v)
\]

(5)

**Proof.** It is easy to verify by the definition of FrFT and FT. \( \square \)

2.2. The Poisson sum formulae

The Poisson sum formula demonstrates that the sum of infinite samples in time domain of a signal \( x(t) \) is equivalent to the sum of infinite samples of \( X(u) \) in the Fourier domain. Mathematically, the Poisson sum formula can be represented as follows:

\[
\sum_{k=-\infty}^{\infty} x(t + kT) = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} X \left( \frac{n}{\tau} \right) e^{jn/\tau} \]

(6)

or

\[
\sum_{k=-\infty}^{\infty} x(kT) = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} X \left( \frac{n}{\tau} \right)
\]

(7)

where \( X(u) \) is the traditional Fourier transform of signal \( x(t) \).

The aim of this paper is to extend the above mentioned Poisson sum formula to the fractional Fourier domain, in other words, we derive the relationship of infinite time samples and the infinite fractional domain samples based on the facts reflected by Lemma 1 in the following sections.

3. The main results

3.1. The generalized Poisson sum formulae

Suppose the FrFT of a signal \( x(t) \) for order \( \alpha \) is \( X_\alpha(u) \), the Poisson sum formula of signal \( x(t) \) associated with the FrFT of order \( \alpha \) can be derived as follows:

**Theorem 1.** The Poisson sum formula of a signal \( x(t) \) in the fractional Fourier domain of order \( \alpha \) can be derived as in Eqs. (8) and (9).

\[
\sum_{k=-\infty}^{\infty} x(t + kT)e^{j(\pi/2)\cot \alpha} x(2k\pi+\alpha^2/2) = \frac{1}{\tau} e^{j(\pi/2)\cot \alpha} x
\]

\[
\times \left\{ \sum_{n=-\infty}^{\infty} e^{-j\pi/2\cot \alpha} x(n \sin \alpha \tau/\tau) \mathfrak{F}(x(t))(\alpha \tau/\tau) \right\}
\]

(8)

and

\[
\sum_{k=-\infty}^{\infty} x(kT)e^{j(\pi/2)\cot \alpha} (n\tau)^\alpha
\]

\[
= 1/\tau \sum_{n=-\infty}^{\infty} X_\alpha(k \sin \alpha \tau/\tau)e^{-j(\pi/2)\cot \alpha} x(n \sin \alpha \tau/\tau)
\]

(9)

**Proof.** If we let \( g(t) = x(t)e^{j\pi/24\alpha^2} \cot \alpha \), then from Lemma 1 we obtain

\[
X_\alpha(v \sin \alpha)e^{-j(v \sin \alpha^2/2)\cot \alpha} x(t) = \mathfrak{F}(g(t))(v) = G(v)
\]

(10)

From the traditional Poisson sum formula for a signal \( g(t) \) in the Fourier domain, we obtain

\[
\sum_{k=-\infty}^{\infty} g(t + kT) = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} G \left( \frac{n}{\tau} \right) e^{jn/\tau} \]

(11)

Eq. (8) can be derived via Eqs. (10) and (11). Eq. (9) can be deduced by letting \( \tau = 0 \) in Eq. (8). \( \square \)
Eqs. (8) and (9) can be seen as the generalization of the traditional Poisson sum formula associated with the FrFT of order α. It is clear from the above equation that the infinite sum of the periodic phase-shift replica of a signal x(t) is equal to the infinite sum of the periodic phase-shift replica of Xα(u).

Further, the band-limited signal is one of the most important types of signals that have received much attention in the literature. Assuming a signal x(t) is band-limited to Ωα in the fractional Fourier domain for angle α (α ≠ kπ + π/2, k ∈ ℤ), then from the renowned results of Xia [7] x(t) is not band-limited in the traditional Fourier domain. The conventional band-limited signal processing methods of traditional Fourier domain would therefore not be suitable for the processing of the signal x(t). So it is worthwhile to investigate the Poisson sum formulae for the band-limited signal in the FrFT domain. Theorem 2 as follows answers this problem.

**Theorem 2.** Suppose a signal x(t) is band-limited to Ωα in the fractional Fourier domain of order α, then the Poisson sum formulae derived in Theorem 1 can be rewritten as following according to the replica period τ.

(a) When sin α/τ > Ωα, the Poisson sum reduces to the following:

\[ \sum_{k=-\infty}^{\infty} x(t + k\tau)e^{j(1/2)\cot(2\pi t^2 \tau)} = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} e^{-j(1/2)cot(2\pi n^2 \tau)} \{ X_\alpha(0) + \sum_{k=1}^{\infty} e^{-j(1/2)cot(2\pi n^2 \tau)} \} \]

(b) When Ωα/2 < sin α/τ < Ωα, the Poisson sum formula reduces to:

\[ \sum_{k=-\infty}^{\infty} x(t + k\tau)e^{j(1/2)\cot(2\pi t^2 \tau)} = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} e^{-j(1/2)cot(2\pi n^2 \tau)} \{ X_\alpha(0) + \sum_{k=1}^{\infty} e^{-j(1/2)cot(2\pi n^2 \tau)} \}
\]

(c) When Ωα/(n + 1) < sin α/τ < Ωα/n, the Poisson sum can be reduced to:

\[ \sum_{k=-\infty}^{\infty} x(t + k\tau)e^{j(1/2)\cot(2\pi t^2 \tau)} = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} e^{-j(1/2)cot(2\pi n^2 \tau)} \{ X_\alpha(0) + \sum_{k=1}^{\infty} e^{-j(1/2)cot(2\pi n^2 \tau)} \}
\]

Theorem 2 derives the Poisson sum formulae associated with the FrFT for the band-limited signal x(t) with respect to the Poisson sum formula (Eq. (8)). The following Theorem gives the sum formula with respect to Eq. (9).

**Proof.** (a) Since x(t) is a Ωα band-limited signal in the fractional Fourier domain and sin α/τ > Ωα, the Xα(n sin α/τ) in the right hand side of Eq. (8) equal to zero when n ≠ 0. Thus, it is easy to derive the right hand side of Eq. (8) as

\[ \frac{1}{\tau} \sum_{n=-\infty}^{\infty} e^{-j(1/2)cot(2\pi n^2 \tau)} \{ X_\alpha(0) + \sum_{k=1}^{\infty} e^{-j(1/2)cot(2\pi n^2 \tau)} \}
\]

Substituting Eq. (15) into Eq. (8) yields the final result.

(b) Similar with the proof of (a), when Ωα/2 < sin α/τ < Ωα, the sum part in the right hand side of Eq. (8) now becomes

\[ \sum_{n=-\infty}^{\infty} e^{-j(1/2)cot(2\pi n^2 \tau)} \{ X_\alpha(0) + \sum_{k=1}^{\infty} e^{-j(1/2)cot(2\pi n^2 \tau)} \}
\]

(c) It is easy to proof just by noting that there are only 2n+1 nonzero value of Xα(u), and the summation part in right hand side of Eq. (8) can be rewritten as

\[ \sum_{n=-\infty}^{\infty} e^{-j(1/2)cot(2\pi n^2 \tau)} \{ X_\alpha(0) + \sum_{k=1}^{\infty} e^{-j(1/2)cot(2\pi n^2 \tau)} \}
\]

The final results will be derived by substituting Eq. (17) into Eq. (8). □

Theorem 3. Assume a band-limited signal x(t) in the fractional Fourier domain is sampled at period τ and multiplied with a chirp discretely as x(kt)e^{j(1/2)cot(2\pi k^2 \tau)}, then the sum of this formula is found to be finite, i.e.,

\[ \sum_{k=-\infty}^{\infty} x(k\tau)e^{j(1/2)cot(2\pi k^2 \tau)} = \begin{cases} \frac{1}{\tau} X_\alpha(0), & \text{if } \sin \alpha/\tau > \Omega_\alpha \\ \frac{1}{\tau} \{ X_\alpha(0) + \sum_{k=1}^{\infty} e^{-j(1/2)cot(2\pi k^2 \tau)} \}, & \text{if } \Omega_\alpha/2 < \sin \alpha/\tau < \Omega_\alpha \\ \frac{1}{\tau} \{ X_\alpha(0) + \sum_{k=1}^{\infty} e^{-j(1/2)cot(2\pi k^2 \tau)} \}, & \text{if } \Omega_\alpha/(n+1) < \sin \alpha/\tau < \Omega_\alpha/n \\ \frac{1}{\tau} X_\alpha(0), & \text{if } \Omega_\alpha/n < \sin \alpha/\tau < \Omega_\alpha/(n-1) \end{cases} \]
**Proof.** It is easy to obtain these results just by letting \( t = 0 \) in Theorem 2. \( \square \)

The above theorems relate the infinite sum of samples in the time domain to the finite sum of samples in the fractional Fourier domain. When the samples are highly correlated, in other words, if the replica period is small enough to satisfy \( \tau < \sin \omega_{fs} \), then the summation of Eq. (18) converges to a constant proportional equivalent to \( \frac{1}{\tau} \). The significance of this point is reflected in the following Corollary 1.

**Corollary 1.** A band-limited signal \( x(t) \) in the fractional Fourier domain is sampled at period \( \tau \) and multiplied with a chirp discretely as \( x(t) e^{j(1/2)\tau t} \), if \( \tau < \sin \omega_{fs} \), then the discrete sum of \( x(t) e^{j(1/2)\tau t} \) becomes identical to the integral of \( x(t) e^{j(1/2)\tau t} \), i.e.,

\[
\sum_{k=-\infty}^{\infty} x(t) e^{j(1/2)\tau t} = \frac{1}{\tau} \int_{-\infty}^{\infty} x(t) e^{j(1/2)\tau t} dt
\]  

**Proof.** It is easy to verify this via the Theorem 3 and the definition of FrFT. \( \square \)

**Corollary 2.** When \( \omega = \pi/2 \), the results derived in Theorems 1 and 2 reduces to the well known results in the Fourier domain.

**Proof.** It is easy to derive this via the relationship between the FrFT and the Fourier transform. \( \square \)

In this sense, the well known results in the Fourier domain can be seen as a special case of our achieved results, and Theorems 1–3 can be classified as the generalization of the classical Poisson sum formula associated with the fractional Fourier domain. It is clear that the sum of the replica of a signal \( x(t) \) in the time domain required in the proving of the Poisson sum formulae associated with the FrFT can be presented as following:

\[
\sum_{k=-\infty}^{\infty} x(t + k\tau) e^{j(1/2)\tau (t + k\tau)}
\]  

This formula (Eq. (20)) plays an important role in the derivation of the generalized Poisson sum formulae associated with the FrFT. In the next subsection, we will investigate this formula in detail and obtain some interesting properties of this formula.

### 3.2. Properties of Eq. (20)

In order to present our ideas more clearly, we let

\[
y(t) = \sum_{k=-\infty}^{\infty} x(t + k\tau) e^{j(1/2)\tau (t + k\tau)}
\]  

from Eq. (21), \( y(t) \) can be seen as the infinite sum of the periodic phase-shift replica of signal \( x(t) \). The following theorems give the properties of \( y(t) \).

**Theorem 4.** Suppose a signal \( x(t) \) is band-limited to \( \Omega_x \) in the fractional Fourier domain of order \( \alpha \), and \( y(t) = \sum_{k=-\infty}^{\infty} x(t + k\tau) e^{j(1/2)\tau (t + k\tau)} \), then \( y(t) \) is a chirp-periodic signal with period \( \tau \).

**Proof.** By the definition of the chirp-periodicity of Eq. (3), we obtain

\[
y(t + \tau) e^{j(1/2)\tau t} = \sum_{k=-\infty}^{\infty} x(t + \tau + k\tau) e^{j(1/2)\tau t} = e^{j(1/2)\tau t} \sum_{k=-\infty}^{\infty} x(t + k\tau) e^{j(1/2)\tau t} = y(t) e^{j(1/2)\tau t}
\]  

This proves the chirp-periodicity of the signal. \( \square \)

Based on Theorem 4 we know that \( y(t) \) is a chirp-periodic signal in the fractional Fourier domain. So we use the series decomposition definition proposed in [8,25] to derive the series representation of this signal. The following theorem deals with the fractional Fourier domain series [8] and the band-limited signal.

**Theorem 5.** Suppose a signal \( x(t) \) is band-limited to \( \Omega_x \) in the fractional Fourier domain of order \( \alpha \), and \( y(t) = \sum_{k=-\infty}^{\infty} x(t + k\tau) e^{j(1/2)\tau (t + k\tau)} \), then \( x(t) \) is a band-limited signal in fractional Fourier domain with order \( \alpha \), if and only if \( y(t) \) has a finite number of nonzero fractional Fourier series coefficients for any \( \tau \).

**Proof.** To prove the necessary condition, the nth fractional Fourier series coefficient \( c_{n,\alpha} \) of signal \( y(t) \) can be deduced from the fractional Fourier series definition in [8,25] as

\[
c_{n,\alpha} = \int_{0}^{\tau} y(t) e^{j(1/2)\tau t} \cos(jn\alpha(\Omega_x)\tau) dt
\]  

Eq. (23) reduces to

\[
c_{n,\alpha} = \frac{\sin \omega \cos \omega}{\pi} \int_{0}^{\tau} y(t) e^{j(1/2)\tau t} \cos(jn\alpha(\Omega_x)\tau) dt
\]

let \( \lambda = t + k\tau \), then \( c_{n,\alpha} \) reduces to

\[
c_{n,\alpha} = \frac{\sin \omega \cos \omega}{\pi} \int_{0}^{\tau} y(t) e^{j(1/2)\tau t} \cos(jn\alpha(\Omega_x)\tau) dt
\]  

from Eq. (24), \( y(t) \) can be seen as the infinite sum of the periodic phase-shift replica of signal \( x(t) \). The following theorems give the properties of \( y(t) \).
let $F^* = 2\pi \sin \alpha /\tau$, Eq. (24) changes to

$$c_{n,z} = \frac{\sqrt{(\sin z - j \cos z) /\tau}}{K_z} \times \int_{-\infty}^{+\infty} K_z(x(z)) e^{j(1/2)\cot z (a(t + \alpha^2) \cos z - j \cos z \sin z)} \sin z \, dz$$

$$= \frac{\sqrt{(\sin z - j \cos z) /\tau}}{K_z} X_z \left( \frac{2\pi}{\alpha} \sin \alpha \right) \frac{2\pi}{\alpha}$$

Since $x(t)$ is a $\Omega_z$ band-limited signal in the fractional Fourier domain for angle $\alpha$, that is to say

$$X_z(u) = \int_{-\infty}^{+\infty} x(t) K_z e^{j(1/2)\cot z (a(t + \alpha^2) - j \cos z \sin z)} \sin z \, dt = 0, \quad |u| > \Omega_z$$

Comparing Eqs. (25) and (26), we obtain

$$c_{n,z} = 0 \quad \text{when} \quad n > \frac{\pi \Omega_z \csc \alpha}{2\pi}$$

The necessary condition is proved.

To prove the sufficient condition, let us assume that $c_{n,z} = 0$ for $n > N$, where $N$ is any finite integer. From Eq. (27)

$$X_z(u) = \int_{-\infty}^{+\infty} x(t) K_z e^{j(1/2)\cot z (a(t + \alpha^2) - j \cos z \sin z)} \sin z \, dt = 0, \quad |u| > 2\pi N / (\tau \csc \alpha)$$

Hence, $x(t)$ is band-limited having the following bandwidth

$$\Omega_z < 2\pi N / (\tau \csc \alpha)$$

3.3. Physical interpretation

Before going to the next section, let us concentrate on the physical interpretation of the theorems derived. These theorems state that by shifting discretely any signal $x(t)$ to left and right positions in the time domain and multiplying by a chirp discretely, the resultant summation is a chirp-periodic signal $y(t)$. This paper derived some interesting results about the signal $x(t)$ and the chirp-periodic signal $y(t)$. We chose the signal $x(t)$ to be band-limited to $\Omega_z$ as an example to show the physical interpretation of the derived results.

(1) $y(t)$ is a chirp-periodic signal with period $\tau$; further, $y(t)$ has a finite number of nonzero fractional series according to the definition of the fractional Fourier series proposed in [8,25].

(2) The value of $y(t)$ is related to $\tau$ and the bandwidth $\Omega_z$ of the signal $x(t)$. When $\tau < \Omega_z$, the value of $y(t)$ is only related with $X_z(0)$, that is $y(t) = (1/\tau) X_z(0) e^{j(1/2)\cot z t \cos z}$; When $\tau$ becomes larger and satisfies $\Omega_z / 2 < \sin \alpha / \tau < \Omega_z$, the value of $y(t)$ is related with $X_z(t)$ and $X_z(\pm \sin \alpha / \tau)$; when $\tau$ is increased further, the value of $y(t)$ is related to more values of $X_z(u)$. These facts can be used to estimate the band width of a signal in the fractional Fourier domain.

4. Potential applications

In this section, the potential applications of the derived results above in estimating the bandwidth and the fractional shape of a signal are discussed.

4.1. The bandwidth estimation

Band-limited signal processing is one of the most important research topics in signal processing community, and the first step in processing these signals is to decide whether it is band-limited or not. So it is worthwhile and interesting to analyze and estimate the bandwidth in the FrFT domain. Based on the above derived results, the bandwidth of the signal can be estimated as discussed below. Suppose the bandwidth $\Omega_z$ of a signal $x(t)$ associated with the FrFT is unknown, the $\Omega_z$ estimation method can be summarized as following:

(1) Select an arbitrary $\tau$ and deduce $y(t)$ from the signal $x(t)$. If signal $y(t)$ is chirp-periodic and with a finite number (suppose $N$) of fractional Fourier series, then from Theorems 4 and 5 $x(t)$ is a band-limited signal in the fractional Fourier domain of order $\alpha$. From Eq. (28), the bandwidth of the signal satisfy $\Omega_z < 2\pi N \sin \alpha / \tau$.

(2) Therefore the results of Theorem 2 is utilized to estimate the signal bandwidth as following; if $y(t)$ has a property similar to that of part (c) in Theorem 2, then the bandwidth of $\Omega_z$ of signal $x(t)$ is in the range of $n \sin \alpha / \tau < \Omega_z < (n + 1) \sin \alpha / \tau$;

(3) If $\tau$ is kept decreasing as well as evaluating $y(t)$, at the junction point of having property of part (b) and part (a), then the bandwidth of signal $x(t)$ is found out to be $\Omega = \sin \alpha / \tau$.

4.2. The fractional spectrum estimation

Besides the bandwidth estimation, the shape of $X_z(u)$ can also be determined by the facts reflected in Theorem 2. Based on this theorem, more samples of $X_z(u)$ in the fractional Fourier domain appears only by increasing $\tau$, that is we can obtain more samples of $X_z(\pm k \sin \alpha / \tau), \quad k = 1, 2, \ldots, n$. Thus a better estimation of $X_z(u)$ can be found.

5. Conclusions

Based on the FrFT and the well known Poisson sum formula in the Fourier domain, this paper investigates the generalized pattern of Poisson sum formula from the FrFT point of view and derives several novel sum formulae associated with the FrFT. Firstly, the generalized Poisson sum formula is obtained based on the relationship of the FrFT and the Fourier transform; then some new results associated with this novel sum formula have been derived; the potential applications of these new results in estimating the bandwidth and the fractional spectrum shape of a signal in the fractional Fourier domain are also
proposed. In addition, the results can be seen as the generalization of the classical results in the fractional Fourier domain. In future the new results can be applied in real practical applications as well as the new results associated with this formula in other transform domain.

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