LINEAR ESTIMATION OF STANDARD DEVIATION OF LOGISTIC DISTRIBUTION: THEORY AND ALGORITHM

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Abstract: *The paper presents a theoretical method based on order statistics and a FORTRAN program for computing the variance and relative efficiencies of the standard deviation of the logistic population with respect to the Cramer-Rao lower variance bound and the best linear unbiased estimators (BLUE's) when the mean is unknown. A method based on a pair of single spacing and the 'zero-one' weights rather than the optimum weights are used. A comparison of an estimator based on four order statistics with the traditional estimators is considered.*

Key words: *Order Statistics, Logistic Distribution, Parameter Estimation, Fortran, Relative Efficiencies.*
2000 Math. Subject Classification: *62 G30, 62 F10, 65 C60*

INTRODUCTION

Let $X_{1:n} \leq \ldots \leq X_{n:n}$ denote the order statistics from a random sample of size n from the logistic distribution whose cumulative distribution function is

$$F(x;\mu,\sigma) = \left[1+\exp\left\{-(x-\mu)/\tau\sigma\right\}\right]^{-1},$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$ and $\tau = \sqrt{3}/\pi$. The distribution is absolutely continuous, symmetric about the location parameter, the mean μ, and with scale parameter σ. Dixon [4] introduced the notion of using 'zero-one' weights rather than the optimum weights and noted that high efficiencies are achievable. This work has since then been extended by many authors, notably, Raghunandanan and Srinivasan [11] for the logistic distribution and Wang Cheng-guan [12] for the normal distribution. Raghunandanan and Srinivasan [11] constructed simplified estimators of the location and scale parameters for complete and symmetrically censored samples for sample sizes $4 \leq n \leq 20$.

The principal object of this paper is to develop, based on the work of Wang Cheng-guan [12], a simplified linear estimator of the scale parameter of the population when its location parameter is unknown into a FORTRAN computer program. The desired procedure should be applicable to both censored and uncensored samples and follows the work of Keats *et al.*[8]. This estimator will be shown to have bias less than 0.5% for $n \geq 5$ and this bias reduces rapidly to zero as n increases. The asymptotic variance and asymptotic efficiency are also given. Furthermore, it is shown that the estimator has high relative efficiency with respect to the 'zero-one' linear estimator of the scale parameter (eff_{0-1}) for n between 5 and 20 and we note that this procedure is developed as a compromise between lack of efficiency and quickness and ease of computation.

ESTIMATOR OF STANDARD DEVIATION

The method discussed here is based on a pair of single spacing and the expectation of the sum of consecutive order statistics in a sample of size n. Let a spacing c be defined in relation to the rank i of order statistics $X_{i:n}$ and sample size n as

$$i = [nc + 0.5] \qquad (1)$$

where $[x]$ denotes the greatest integer $\leq x$ and let

$$G(u) = F^{-1}(u) = \tau \log(u/(1-u)), \qquad (2)$$
$$0 < u < 1$$

be the inverse function, then this expectation is given by

$$E(X_{i:n} + X_{i+1:n}) = 2\left\{G(c^*) + G'(c^*)\left(\frac{i}{n} - c^*\right) + \frac{G''(c^*)}{12n^2}\right\} + O(n^{-3}), \quad 1 \leq i \leq n-1. \qquad (3)$$

In this paper, we have used $c^* = i/n$ in all the calculations. Let R_i denote the i-th quasi-range

$$R_i = X_{n-i:n} - X_{i+1:n}. \qquad (4)$$

Table 1: Ranks, Bias, Variances and Efficiencies for various n=5(1)20(5)50, ∞

n	i	i+1	n-i	n-i+1	BIAS	$V(\hat{\sigma}_1)/\sigma^2$	$V(\hat{\hat{\sigma}})/\sigma^2$	eff_1	eff_2	eff_{0-1}
5	1	2	4	5	0.0002	0.1720	0.1704	0.813	0.991	1.000
6	1	2	5	6	0.0001	0.1377	0.1429	0.847	0.995	0.999
7	1	2	6	7	0.0000	0.1160	0.1232	0.861	0.987	0.999
8	1	2	7	8	0.0000	0.1011	0.1052	0.865	0.973	0.999
9	1	2	8	9	0.0000	0.0903	0.0918	0.861	0.956	0.978
10	1	2	9	10	0.0000	0.0820	0.0850	0.853	0.937	0.958
11	1	2	10	11	0.0000	0.0754	0.0788	0.843	0.917	0.943
12	1	2	11	12	0.0000	0.0701	0.0711	0.831	0.898	0.920
13	1	2	12	13	0.0000	0.0658	0.0648	0.818	0.878	0.900
14	1	2	13	14	0.0000	0.0621	0.0634	0.805	0.860	0.881
15	2	3	13	14	0.0000	0.0600	0.0584	0.777	0.826	0.862
16	2	3	14	15	0.0000	0.0559	0.0541	0.782	0.828	0.843
17	2	3	15	16	0.0000	0.0525	0.0527	0.784	0.827	0.830
18	2	3	16	17	0.0000	0.0495	0.0491	0.784	0.825	0.827
19	2	3	17	18	0.0000	0.0470	0.0460	0.783	0.822	0.826
20	2	3	18	19	0.0000	0.0448	0.0439	0.781	0.818	0.818
25	3	4	22	23	0.0000	0.0372	0.0352	0.751	0.779	
30	3	4	27	28	0.0000	0.0310		0.752	0.775	
35	4	5	31	32	0.0000	0.0271		0.736	0.755	
40	4	5	36	37	0.0000	0.0238		0.736	0.752	
45	5	6	40	41	0.0000	0.0214		0.726	0.741	
50	5	6	45	46	0.0000	0.0193		0.726	0.737	
∞					0.0000	1.0227/n		0.684		

Then based on Equations (3) and (4), we propose to use

$$\hat{\sigma}_1 = (R_i + R_{i-1}) \big/ 2E_{i,i+1:n}, \qquad (5)$$

where $E_{i,i+1:n} = -E_{n-i,n-i+1:n}$ due to symmetry,

$$E_{i,i+1:n} \approx 2\left\{G(1-c^*) - \frac{G''(c^*)}{12n^2}\right\}, \quad (6)$$

$$= 2\tau\left\{\log\left(\frac{n-i}{i}\right) + \frac{n(n-2i)}{12i^2(n-i)^2}\right\} \qquad (7)$$

as the estimator of the scale parameter σ. The methods for estimating the location parameter, μ, of logistic distribution is not covered in this paper and readers are referred to Weke *et al.* [13].

EXAMPLE

A single spacing value of 0.10293 has been established from Gupta and Gnanadesikan [5] and is used here in all computations. Let $V(\hat{\sigma}_1)$ (given in the program as VAR) and $V(\hat{\hat{\sigma}})$ be the variances of $\hat{\sigma}_1$ and that of the estimator based on four optimum order statistics, respectively, then we have

$$eff_1 = 9 \Big/ n(3+\pi^2)V(\hat{\sigma}_1)$$

and (8)

$$eff_2 = \frac{Var[BLUE(\sigma)]}{V(\hat{\sigma}_1)} = \frac{V_{opt}}{VAR}$$

as the relative efficiencies of $\hat{\sigma}_1$ with respect to the Cramer-Rao lower bound for the variance of an unbiased estimator of σ and the variance of BLUE of σ, respectively. The efficiencies eff_1 and eff_2 are given in the program as AEF and R.E., respectively. Table 1 gives the ranks of order statistics, the bias and variance of the estimator $\hat{\sigma}_1$ and the efficiencies relative to Cramer-Rao lower variance bound, the BLUE and the 'zero-one' linear estimators of scale parameter for various values of n. For illustration, we consider the case when n = 20 and by using Equations (1) and (5) we have the numerator as 2(1.37563+1.06933) = 4.88992 and denominator as 4.89096 when i = 2. This leads to a bias of 0.00002. This method uses four order statistics instead of the usual two order statistics. It should be mentioned that bias equal to zero does not imply that the estimate in Equation (7) coincides with the expected value of the two consecutive order statistics; it simply means that the error is less than 5×10^{-5}.

In conclusion, we notice that $eff_1 \le eff_2 \le eff_{0-1}$. A comparison with an estimator, abbreviated by $\hat{\hat{\sigma}}$, based on four optimum order statistics given by Chan *et al.* [3] reveals that the variances of the estimator in Equation (5) is smaller than $V(\hat{\hat{\sigma}})$ for most values of n within the range $5 \le n \le 25$. The variances required to calculate eff_{0-1} values are available in Balakrishnan and Cohen [1] page 255 for $2 \le n \le 20$. All computations were performed in double precision by FORTRAN 77 language programs to produce results for values up to n = 100 but to save space we have not included all these cases in our table. The computer programs for the calculation of the means, product moments, variances and covariances of the logistic order statistics and their respective data files necessary for the construction of Table 1 are too huge to be reproduced here. However, their values are found to coincide with those values in Birnbaum and Dudman [2], Gupta *et al*. [6], and Harter and Balakrishnan [7]. The procedure discussed in this paper produces reasonably good results, quick and easy computations. Finally, it is noted that no reference is made to pre-existing tables for the computation of the estimator and its variance and efficiencies.

PROGRAM DESCRIPTION

The FORTRAN program computes the variance of the estimator given in Equation (5), its efficiencies relative to the Cramer-Rao lower variance bound and the BLUE's and gives the variance-covariance matrix of the logistic order statistics as an output. All the features are clearly illustrated in the program, notably,

INPUT FILES: LOGIS1.EXP and LOGIS1.COV are DATA FILES for the expected and covariance values for order statistics. These have been calculated by using a different program. However, similar values can be found in Chan *et al.* [3]. Input $KK2 = 2$; $N0, N6 = 5, n$ and $IJ(I) = i$, $CF1(I) = i + 1$ according to Table 1.

DISPLAY: The lower-triangular matrix of covariance. We have used the Cholesky's inverse matrix and Lloyd's optimal methods (Lloyd [9]) which are similar to Ogawa's method (Ogawa [10]).

EFFICIENCIES: $AEF = eff_1$ and $R.E. = V_{opt} / VAR = eff_2$.

SUBROUTINES: MINV - Employs Cholesky's method to compute inverse of the matrix.

VARML - Output a lower triangular matrix row by row. The lower triangle is stored in a 1-dimensional array.

VLTV - Output a vector with a title.

OUTPUT: LGEST1.VAR

```
N= 20   EXP=  4.88992   NUM=  1.07004   VAR= .04475   AEF= .78136
IJ(K):        1        2        3        4        5        6        7        8        9       10
R1(K):   .00000  .20450  .20450  .00000  .00000  .00000  .00000  .00000  .00000  .00000
  Vopt=   .0366       R.E.=Vopt/VAR= .81795
 COP(K):  -.0729  -.0784  -.0774  -.0726  -.0651  -.0556  -.0447  -.0327  -.0199  -.0063
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PROGRAM LISTING

```
C  VARIL.FOR      by  PATRICK WEKE  at
    HIT/UoN
C  COMPUTE THE VARIANCES of The
    ESTIMATES for THE SCALE
    PARAMETER OF
C  LOGISTIC POPULATION.
C            I=[c*(N+1-2*a)+a]  , a=0.5
   INTEGER N,N0,N6,M,I,J ,I2,J2,L,K, M1, M2
    ,K1,K2,IW,KK,KK2
   INTEGER N1(100),N2(100), N3(100),
    N4(100), N5(100),IJ(100),S(100)
   REAL RN,BB1,BB2,RN1(100),RN2(100) ,
    RN3(100), RN4(100),RN5(100)
  REAL a,c,NUM,DE,SC(10),CF(100),
    CF1(100), W(100),RX(820),R(100)
  REAL VAR,AEF,RY(820),COP(100),R1(100)
   CHARACTER*8 FNAME
C   M2:     NUMBER of ORDER STATISTICS
USED, M1=(M2+1)/2
C    CF(K):  COEFFICIENTS of PAIR of
CONSECUTIVE ORDER STATISTICS
C    CF1(k): COEFFICIENTS of EVERY
ORDER STATISTICS
C    W(K):   WEIGHT of EVERY ORDER
STATISTICS
C    N:      SIZE, STARTING:N0, FINISH:N6
C    R(L) :  THE SUM of EXPECTATION of
ORDER STATISTICS
C    R1(K):  COEFFICIENTS of ESTIMATION
C    RX(K):  THE VARIANCES of ORDER
STATISTICS USED
C    NUM:    NUMERATOR,  EXP:  DE:
DENOMINATOR
C    IJ(K);  RANKS of ORDER STATISTICS
USED
C    SC(K),  c: SPACING, S(K): RANK
C    A:      BLOM'S CONSTANT
C     K1,K2:  THE STARTING, FINISH and
NUMBER of SPACING for N
C     IW:     SWITCH
C     VAR:    VARIANCE of ESTIMATION
C     AEF:    ASYMPTOTIC EFFICIENCY
C     KK:     SWITCH of INTERACTIVE
INPUT
C     KK2:    SWITCH of SPACING and INPUT
for ALL SIZE from N0 to N6
C     COP:    COEFFICIENTS of BLUE
C     Vopt:   VARIANCE of BLUE
C     R.E.:   EFFICIENCY  Vopt/VAR
C     WRITE   FOUR DISK FILES
C     FUNCTION: <1> DISPLAY and SAVE
THE VECTOR of EXPECTATION and THE
MATRIX of
C     COVARIANCES of ORDER STATISTICS
USED
C      <2> COMPUTE COEFFICIENTS of
       ESTIMATION ACCORDING TO THE
       PROPORTION
C     of ITS COEFFICIENTS
C     <3> COMPUTE THE VARIANCE and
EFFICIENCY of ESTIMATION
C     <4> RANK of ORDER STATISTICS MAY
BE OBTAINED by THE SPACING and
C     THE INTERACTIVE INPUT
CCCC   (1): SPACING:
      SC(1)=.10293
      SC(2)=.25
      SC(3)=.125
      SC(4)=0.0625
C     m=5-2=3, K1=2, M1=4, K2=5
      SC(5)=0.03125
      SC(6)=1./FLOAT(2**6)
      SC(7)=1./FLOAT(2**7)
      SC(8)=1./FLOAT(2**8)
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```
      SC(9)=1./FLOAT(2**9)
      SC(10)=1./FLOAT(2**10)
C     WRITE(*,*)' SPACING: ',SC
CCCC   (2): COMPUT THE N1,N2,N3,N4,N5
and RN1,RN2,RN3,RN4,RN5
      a=.5
      I=1
      DO 10 L=2,10
      c=SC(L)
      DO 20 N=2,1024
      IF((c*(FLOAT(N)+1.-2.*a)+a) .GE.
FLOAT(I)) GO TO 25
 20   CONTINUE
 25   RN1(L)=N
      N1(L)=N
      DO 30 N=2,1024
      IF((c*(FLOAT(N)+1.-2.*a)+a).GE.
FLOAT(I+1)) GO TO 35
 30   CONTINUE
C     PAUSE'WARNING !'
 35   RN2(L)=N-1
      N2(L)=N-1
      N3(L)=(N1(L)+N2(L)+1)/2
      RN3(L)=N3(L)
      N4(L)=N3(L)*2
      RN4(L)=N4(L)
      N5(L)=(N1(L)+N3(L))/2
      RN5(L)=N5(L)
 10   CONTINUE
CCCC   (3): THE START, FINISH and
NUMBER of SPACING: K1,K2 for N

OPEN(1,FILE='LOGIS1.EXP',STATUS='OLD')

OPEN(2,FILE='LOGIS1.COV',STATUS='OLD'
)
OPEN(4,FILE='LGEST1.VAR',STATUS='NEW
')
      KK2=0
      WRITE(*,*)' KK2=1    : FINDING RANK
by SPACING'
      WRITE(*,*)' KK2=2    : FINDING RANK
by INTERACTIVE INPUT'
      WRITE(*,*)' KK2=OTHER: FINDING
RANK by VALUE KK, ONE by ONE'
      WRITE(*,*)' THERE ARE THE BLUES
WHEN AND ONLY WHEN KK2=2 OR KK=2'
      WRITE(*,*)' INPUT KK2=?'
      READ(*,*)KK2
      WRITE(*,*)'  INPUT THE STARTING and
FINISH N0,N6 =?'
      READ(*,*)N0,N6
 48   FORMAT(2X,' THE STARTING and
FINISH of SIZE: N0,N6=',2I4)
      WRITE(*,48)N0,N6
      DO 170 N=N0,N6
      RN=FLOAT(N)
      K1=2
      DO 50 L=2,10
C     THE START, FINISH and NUMBER of
SPACING: K1,K2 for N
      IF(N.GE.N5(L)) K2=L
 50   CONTINUE
 60   FORMAT(' N,N5(K2),K1,K2=',5I4)
      WRITE(*,60)N,N5(K2),K1,K2
      DO 65 L=K1,K2
 70   FORMAT('L,N1,N5,N3,N2=',2I5,'   ',4I5)
WRITE(*,70)L,N1(L),N5(L),N3(L),N2(L),
N4(L)
 65   CONTINUE
CC  PDF         f(x)=exp(-x)/[(1 +exp(-x))**2]
C     CDF       F(x)=1/(1 +exp(-x))
C  with ¦Ó=TAO=0.55132890  G(u)  =X=-ln[(1-
u)/u]=ln[u/(1-u)]
C                     G'(u) =1/[u(1-u)]
C                G"(u) =-(1-2u)/[((u(1-u))**2]
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C           1/2**L                          C
C      N1  N5   N3    N2  N4                C
C      RN1 RN5  RN3   RN2 RN4               C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
CCCC (4):
      KK=1
      IF(KK2.EQ.1) GO TO 90
      IF(KK2.EQ.2) GO TO 77
      WRITE(*,*)'INPUT CF1(K)? NO/YES 1/2'
      READ(*,*)IW
      IF(IW.EQ.1) GO TO 90
 77   KK=2
      WRITE(*,*)'INPUT NUMBER of ORDER
STATISTICS USED, M2?'
      READ(*,*)M2
      M1=(M2+1)/2
      DO 75 K=1,M1
      W(K)=1.
 78   FORMAT('IJ(',I2,')=?',' CF1(',I2,')=?')
      WRITE(*,78)K,K
      READ(*,*)IJ(K),CF1(K)
 75   CONTINUE
      GO TO 120
 90   DO 80 L=1,10
      CF(L)=.0
 80   CONTINUE
      DO 82 L=K1,K2
      WRITE(*,*)'INPUT THE COEFFICIENTS
CF(L) =?'
 84   FORMAT('CF(',I2,')=?')
      WRITE(*,84)L
      READ(*,*)CF(L)
 82   CONTINUE
      WRITE(*,86)CF
 86   FORMAT(2X,' CF:',5F9.4/,'      ',5F9.4)
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```
CCCC   (5): THE RANK S(L),CF1(K),W(K),
IJ(K)
      RN=N
      DO 100 L=K2,K1,-1
      S(L)=INT(RN*SC(L)+0.5)
 98   FORMAT(2X,'S(',I3,')=',I3)
      WRITE(*,98)L,S(L)
 100  CONTINUE
      CF1(1)=CF(K2)
      CF1(2)=CF(K2)
      W(1)=1.
      W(2)=1.
      IJ(1)=S(K2)
      IJ(2)=S(K2)+1
      K=2
      DO 110 L=K2-1,K1,-1
      IF(S(L+1).EQ.S(L)) THEN
      CF1(K-1)=CF1(K-1)+CF(L)
      CF1(K)=CF1(K)+CF(L)
      W(K-1)=W(K-1)+1.
      W(K)=W(K)+1.
      GO TO 110
      ELSE
      ENDIF
      IF((S(L+1)+1).EQ.S(L)) THEN
      CF1(K)=CF1(K)+CF(L)
      W(K)=W(K)+1.
      K=K+1
      IJ(K)=S(L)+1
      CF1(K)=CF(L)
      W(K)=1.
      GO TO 110
      ELSE
      ENDIF
      IF((S(L+1)+1).LT.S(L)) THEN
      K=K+1
      IJ(K)=S(L)
      CF1(K)=CF(L)
      W(K)=1.
      K=K+1
      IJ(K)=S(L)+1
      CF1(K)=CF(L)
      W(K)=1.
      ELSE
      ENDIF
 110  CONTINUE
      M1=K
      M2=2*M1
 120  DO 130 J=1,M1
      W(M2+1-J)=W(J)
      IF((M2+1-J).EQ.J) CF1(J)=.0
      CF1(M2+1-J)=-CF1(J)
      IJ(M2+1-J)=N+1-IJ(J)
 130  CONTINUE
 140  FORMAT(10I4)
 150  FORMAT(1X,' N=',I3)
 160  FORMAT(10F9.5)
      WRITE(*,150)N
      WRITE(*,*)' IJ,CF1,W='
      WRITE(*,140)(IJ(K) ,K=1,M2)
      WRITE(*,160)(CF1(K),K=1,M2)
      WRITE(*,160)(W(K)  ,K=1,M2)
CCCC   (6):THE EXPECTATION of ORDER
STATISTICS USED
 180  FORMAT(2I4,F9.5)
      DO 200 I=1,M1
 195  READ(1,180)M,I2,BB1
      IF((M.EQ.N).AND.(I2.EQ.IJ(I)))
      GO TO 210
      GO TO 195
 210  R(I)=BB1
 200  CONTINUE
      DO 212 K=1,M1
      R(M2+1-K)=-R(K)
 212  CONTINUE
      CALL VLTV(R,M1,'EXPECT-L',' 20F9.4 ')
CCCC  (7): FIND OUT THE COVARIANCES
of ORDER STATISTICS USED
 220  FORMAT(3I4,F9.5)
      DO 224 I=1,M1
      DO 228 J=I,M2+1-I
 230  READ(2,220)M,I2,J2,BB2
IF((M.EQ.N).AND.(I2.EQ.IJ(I)).AND.(J2.EQ.IJ
(J))) GO TO 240
      GO TO 230
C     SPACING: from K1=2 to K2
 240  RX((J-1)*J/2+I)=BB2
      RX((M2+1-I-1)*(M2+1-I)/2+M2+1-J)=BB2
 228  CONTINUE
 224  CONTINUE
CCCC  LAST TWO LINES of COV-MATRIX
      DO 234 K=1,M2
      RX(M2*(M2+1)/2 +K)=R(K)
      IF(K.GT.M1) RX(M2*(M2+1)/2 +K)=-
R(M2+1-K)
 234  CONTINUE
      RX((M2+1)*(M2+2)/2)=RN
      DO 238 K=1,M2
      RX((M2+1)*(M2+2)/2 +K)=IJ(K)
 238  CONTINUE
      RX((M2+1)*(M2+2)/2 +M2+1)=K1
      RX((M2+2)*(M2+3)/2)=K2
      CALL VARML(RX,M2+2,'COV---RX','
8F9.5 ',1)
CCCC  (8):  OUTPUT
      WRITE(*,*)'DISPLAY THE LOWER-
TRIANGULAR MATRIX of COV YES/NO 1/2'
      READ(*,*)IW
      IF(IW.NE.1) GO TO 245
      CALL VARML(RX,M2+2,'COV---RX','
8F9.5 ',1)
 245  CONTINUE
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      WRITE(*,*)'WRITE COV FILE? YES/NO 1/2'
      READ(*,*)IW
      IF(IW.NE.1) GO TO 255
      WRITE(*,*)'INPUT THE FILE NAME (LG??.COV) '
      READ(*,260)FNAME
 260  FORMAT(A)
      OPEN(3,FILE=FNAME ,STATUS='NEW')
 270  FORMAT(I4)
      CALL VARML(RX,M2+2,'COV---RX',' 8F9.5 ',2)
      CLOSE(3)
 255  NUM=.0
      DO 280 I=1,M1
      DO 295 K=1,I
      NUM=NUM+W(I)*CF1(K)*RX((I-1)*I/2 +K)*CF1(I)*W(I)
 295  CONTINUE
      DO 300 K=I+1,M2
      NUM=NUM+W(K)*CF1(K)*RX((K-1)*K/2 +I)*CF1(I)*W(I)
 300  CONTINUE
 280  CONTINUE
      EXP=.0
      DO 310 K=1,M1
      EXP=EXP+W(I)*CF1(K)*R(K)
 310  CONTINUE
      EXP=-2.*EXP
      DO 320 K=1,M1
      R1(K)=CF1(K)/EXP
      R1(M2+1-K)=R1(K)
 320  CONTINUE
      CALL VLTV(R1,M2,'COEFIC-L',' 20F9.4 ')
      DE=EXP*EXP
      NUM=2.*NUM
      VAR=NUM/DE
      WRITE(*,*)'By Using The Cramer-Rao Lower Bound, We Obtain:'
      AEF=(9./(3.+3.1415926**2))/(RN*VAR)
 330  FORMAT(2X,'N=',I3,'EXP=',F9.5, 'NUM=', F9.5,'VAR=',F7.5,
     /       '  AEF=',F7.5)
      WRITE(*,330)N,EXP,NUM,VAR,AEF
      WRITE(4,330)N,EXP,NUM,VAR,AEF
      WRITE(*,340)(IJ(K),K=1,M1)
      WRITE(4,340)(IJ(K),K=1,M1)
 340  FORMAT('IJ(K):',10(6X,I3)/,'   ',10(6X,I3))
      WRITE(4,350)(R1(K),K=1,M1)
 350  FORMAT('R1(K): ',10F9.5/,'       ',10F9.5)
      IF(KK2.EQ.2) GO TO 353
      IF((KK.EQ.1).OR.(KK2.EQ.1)) GO TO 375
 353  CALL MINV(RX,RY,R,COP,M2,Vopt)
 360  FORMAT(' Vopt=',F9.4,'   R.E.=Vopt/VAR=',F7.5)
      WRITE(*,360)Vopt,Vopt/VAR
      WRITE(4,360)Vopt,Vopt/VAR
      CALL VLTV(COP,M2,'COP-L   ',' 20F9.4 ')
 370  FORMAT(' COP(K):',10F9.4/,'     ',10F9.4)
      WRITE(4,370)(COP(K),K=1,M2)
      CALL VARML(RY,M2,'INV-MATR',' 8F9.5 ',1)
 375  CONTINUE
CC    TURN TO NEXT SIZE N
 170  CONTINUE
      CLOSE(1)
      CLOSE(2)
      CLOSE(4)
      STOP
      END
C     Cholesky's Inverse Matrix and Lloyd's Optimal-Method (as Ogawa Method)
      SUBROUTINE MINV(RX,RZ,R, COP, M, Vopt)
      DIMENSION RX(*),RY(820),R(M), COP(M), S(100,100)
      REAL  SUM,Vopt,RZ(*)
      INTEGER   M,K,I,J,L
      S(1,1)=SQRT(RX(1))
      DO 10 J=2,M
 10   S(1,J)=RX((J-1)*J/2 +1)/S(1,1)
      DO 20 I=2,M
      L=I-1
      SUM=.0
      DO 30 K=1,L
 30   SUM=SUM+S(K,I)*S(K,I)
      S(I,I)=SQRT(RX((I-1)*I/2 +I)-SUM)
      DO 20 J=1,M
      SUM=.0
      IF(I-J) 25,20,35
 25   DO 40 K=1,L
 40   SUM=SUM+S(K,I)*S(K,J)
      S(I,J)=(RX((J-1)*J/2 +I)-SUM)/S(I,I)
      GO TO 20
 35   S(I,J)=.0
 20   CONTINUE
      WRITE(*,*)'M=',M
      RY(1)=1./S(1,1)
      DO 50 J=2,M
      RY((J-1)*J/2 +J)=1./S(J,J)
      SUM=.0
      L=J-1
      DO 60 I=1,L
 60   SUM=SUM+RY((I-1)*I/2 +1)*S(I,J)
 50   RY((J-1)*J/2 +1)=-SUM/S(J,J)
      DO 70 I=2,M
      DO 70 J=1,M
      IF(I-J) 65,70,75
 65   SUM=.0
      L=J-1
      DO 80 K=I,L
```

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 80   SUM=SUM+RY((K-1)*K/2 +I)*S(K,J)
      RY((J-1)*J/2 +I)=-SUM/S(J,J)
      GO TO 70
 75   CONTINUE
 70   CONTINUE
      CALL VARML(RY,M,'MAT--INV',' 8F9.4
',1)
CC(3) INVERSE MATRIX S of RX
      DO 90 I=1,M
      DO 100 J=1,M
      DO 92 K=1,M
      COP(K)=.0
      IF(K.LT.J) GO TO 92
      COP(K)=RY((K-1)*K/2 +J)
 92   CONTINUE
      SUM=.0
      DO 96 K=1,M
      IF(K.LT.I) GO TO 96
      SUM=SUM +COP(K)*RY((K-1)*K/2 +I)
 96   CONTINUE
      S(I,J)=SUM
      IF(I.GT.J) GO TO 100
      RZ((J-1)*J/2 +I)=S(I,J)
 100  CONTINUE
 90   CONTINUE
CC(4) LOWER TRIANGLE RY of S
      CALL VARML(RZ,M,'MAT--INV',' 8F9.4
',1)
CC(5) CHECK:
      DO 120 I=1,M
      DO 130 J=1,M
      SUM=.0
      DO 140 K=1,J
      SUM=SUM+RX((J-1)*J/2 +K)*S(I,K)
 140  CONTINUE
      DO 145 K=J+1,M
      SUM=SUM+RX((K-1)*K/2 +J)*S(I,K)
 145  CONTINUE
      IF(J.GT.I) GO TO 148
      RY((I-1)*I/2 +J)=SUM
 148  CONTINUE
 130  CONTINUE
 120  CONTINUE
CC(6) Lloyd Method:
      DO 150 I=1,M
      COP(I)=.0
      DO 160 K=1,M
      COP(I)=COP(I) +R(K)*S(I,K)
 160  CONTINUE
 150  CONTINUE
      Vopt=.0
      DO 170 K=1,M
      Vopt=Vopt +R(K)*COP(K)
 170  CONTINUE
      DO 180 K=1,M
      COP(K)=COP(K)/Vopt
 180  CONTINUE
      Vopt=1./Vopt
      CALL VLTV(COP,M,'COEF-OPT',' 6F9.4 ')
      RETURN
      END
C
      SUBROUTINE  VARML(A,M,TITLE, FM,
IW)
      INTEGER     M,IW
      REAL        A(*)
      CHARACTER*8 TITLE,FM
      CHARACTER   F*48,FF(6)*8
      EQUIVALENCE (F,FF(1))
      DATA  F/' (1X,I5,2H) ,   /(8X,          )) '/
      FF(3) = FM
      FF(5) = FM
      IF(IW.EQ.1) WRITE(*,805) TITLE,M,M
      IF(IW.EQ.2) WRITE(3,805) TITLE,M,M
      II = 0
      DO 10 I = 1,M
        I1 = II+1
        II = II+I
      IF(IW.EQ.1) WRITE(*,F) I,(A(IJ),IJ=I1,II)
      IF(IW.EQ.2) WRITE(3,F) I,(A(IJ),IJ=I1,II)
   10 CONTINUE
  800 FORMAT(/1X,2A4,4X,1H(,I5,3H
*,I5,2X,20HSYMMETRICAL MATRIX ))
  805 FORMAT(/1X,A8,4X,1H(,I5,3H
*,I5,2X,20HSYMMETRICAL MATRIX ))
      RETURN
      END
C
      SUBROUTINE  VLTV(V,N,TITLE,FM)
      INTEGER     N
      REAL        V(N)
      CHARACTER*8 TITLE,FM
      CHARACTER   F*48,FF(6)*8
      EQUIVALENCE (F,FF(1))
      DATA  F/'  (/1X,A8,4X,1H(,I5,2H )/(1X,
)) '/
      FF(5) = FM
      WRITE(*,F) TITLE,N,(V(I),I=1,N)
      RETURN
      END
```