Application of Signal Space Diversity Over Multiplicative Fading Channels

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Abstract—This letter generalizes the application of signal space diversity (SSD) over multiplicative fading channels, where fading is represented by the product of \( K \) statistically independent Nakagami-\( m \) random variables. The pairwise error probability (PEP) for the systems is first obtained in a closed-form using generalized hypergeometric functions. Based on the obtained PEP expression, it is shown that the error performance over multiplicative fading channels can be significantly improved by using high diversity constellations. Furthermore, by employing SSD with a sufficiently large dimension, it is observed that the adverse effects of multiplicative fading can be practically eliminated without any power nor bandwidth expansion. Simulation results for both uncoded and coded systems are provided to show the agreement with the analysis.

Index Terms—Hypergeometric functions, multiplicative fading, pairwise error probability, signal space diversity.

I. INTRODUCTION

SIGNAL space diversity (SSD) [1] is an effective modulation technique to increase the diversity order and, therefore, the reliability of the wireless transmissions over fading channels without any power nor bandwidth expansion. The application of SSD has been considered extensively for both uncoded and coded systems over a conventional Rayleigh fading channel. In particular, it was shown in [1] that the error performance of rotated lattice constellations over a Rayleigh fading channel becomes insensitive to fading. The result was also extended in [2] to a bandwidth-efficient coded system, called bit-interleaved coded modulation with iterative decoding (BICM-ID).

Recently, attention has been paid to the case of multiplicative fading models, where the overall fading process is represented by \( K \) statistically independent fading processes connected via a product of Nakagami-\( m \) fading processes [3], [4]. This fading model has been shown to match very well with the measurements made in a forest environment [3], in a keyhole propagation with \( K = 2 \) keyholes [5], or in a propagation via diffracting street corners. For example, with \( K = 2 \), one has the cascaded Rayleigh fading channel, which has been considered as a common model for keyhole fading [3], [5]. Apparently, the higher the order of the multiplicative fading model is, the higher the probability that any of the cascaded fading channels is in deep fade becomes. Consequently, the error performance over a multiplicative fading channel is severely degraded.

This letter generalizes the application of SSD over multiplicative fading channels, where fading is represented by the product of \( K \) statistically independent (but not identically distributed) Nakagami-\( m \) random variables. We first obtain the pairwise error probability in a closed-form expression using generalized hypergeometric functions. It is then shown that the condition of high diversity constellation is the most crucial in order to improve the error performance. Performance improvement achieved by using high diversity constellations is thoroughly demonstrated. It is observed that the adverse effect of multiplicative fading can be practically eliminated by employing SSD with a sufficiently large dimension. Various analytical and simulation results for both uncoded and coded system (in the form of BICM-ID) show that the error performance of the systems under consideration can also closely approach that over an AWGN channel.

II. SYSTEM MODEL

Consider a group of \( Nb \) information bits. These bits are mapped to one real or complex \( N \)-dimensional (\( N \)-dim) constellation symbol by some mapping rule \( \xi \) to produce the symbol sequence \( s = \{ s_1, s_2, \ldots, s_N \} \), where \( s_i \in \Omega, \forall i \), where \( \Omega \) is a PAM or QAM constellation of size \( 2^b \). For simplicity, assume that the mapping rule \( \xi \) is implemented independently and identically for each component \( s_i \), i.e., the mapping rule \( \xi \) is employed over the constellation \( \Omega \). The “super” symbol \( s \) in the \( N \)-dim constellation \( \Psi \) is then rotated by an \( N \times N \) rotation matrix \( \mathbf{G} \). The rotated symbol \( \mathbf{x} = [x_1, x_2, \ldots, x_N] \), belonging to a new rotated constellation \( \Psi_r \), is given by \( \mathbf{x} = \mathbf{G}^\dagger \mathbf{s} \).

Over a multiplicative Nakagami-\( m \) fading channel, the \( N \)-dim received signal \( \mathbf{r} \) can then be represented as \( \mathbf{r} = \mathbf{H} \mathbf{s} + \mathbf{w} \). In general, the entries of \( \mathbf{w} = [w_1, \ldots, w_N] \) are i.i.d. circularly symmetric Gaussian random variables with variance \( N_0 \). The matrix \( \mathbf{H} = \text{diag}(h_1, \ldots, h_N) \) contains the fading coefficients in its diagonal, with the assumption that the channel changes independently in each component duration of the \( N \)-dim signal symbol. Furthermore, assume that the receiver has perfect channel state information (CSI).

Each fading coefficient \( h_i \) is represented as \( h_i = \prod_{k=1}^{i-1} \alpha_k \exp(j\phi_k), 1 \leq i \leq N \), where \( \{\phi_k\} \) are i.i.d. over \( (0, 2\pi) \) and \( \{\alpha_k\} \) are i.i.d. Nakagami-\( m \) random variables with fading severity parameters \( \{m_k\} \). The probability density function of \( \alpha_k \) is given as \( f_{\alpha_k}(\alpha) = (2\Gamma(m_k))(m_k/\Gamma_k)^m \alpha^{2m-1} \exp(-m_k\alpha^2/\Gamma_k) \), with \( 0 \leq \alpha \leq m_k \) and \( \Gamma_k = \Gamma(\alpha_k^2) \). Let \( \alpha_i = \|h_i\|^2 \) and assume that the channel gain is normalized such that \( E(\|h_i\|^2) = E(\alpha_i) = \prod_{k=1}^{i} \Gamma_k = 1 \).

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III. PAIRWISE ERROR PROBABILITY ANALYSIS

This section obtains a closed-form expression of the pairwise error probability (PEP) in terms of generalized hypergeometric functions with the technique of moment generating function (MGF). Based on the obtained expression, it is analytically demonstrated that by using a high-order modulation diversity constellation, the error performance over multiplicative Nakagami-$m$ fading channels can approach that over an AWGN channel. The Chernoff bound on the PEP is finally derived to obtain the design criteria of the constellation rotation.

A. Probability Density Function and Moment Generating Function of $a_i$

Using the similar analysis presented in [7] but for the general case of $\{a_k\}$ being Nakagami-$m$ random variables, the probability density function of $a_i$ can be expressed as follows:

$$f_A(a) = \frac{1}{a^{K_0}} e^{\frac{K_0}{a}} \frac{\Gamma(m_k)}{\prod_{k=1}^{K} \Gamma(m_k)} \times \left[ \prod_{k=1}^{K} m_k \Gamma(m_k) \right]^{-1}$$

where $\Gamma(\cdot)$ is the Meijer G-function [8]. The MGF of $a_i$ can then be computed as

$$\Phi_A(s) = \int_0^\infty e^{-sa} f_A(a) da$$

$$= \frac{1}{\prod_{k=1}^{K} \Gamma(m_k)} \frac{\Gamma(1-K_0)}{s} \frac{\prod_{k=1}^{K} m_k \Gamma(m_k)}{\prod_{k=1}^{K} \Gamma(m_k)}$$

where the last equality in (2) comes from [8, 7.813.1] and the fact that $\prod_{k=1}^{K} \Gamma(m_k) = 1$. Alternatively, by using [8, 9.34.8], $\Phi_A(s)$ can be also expressed with a generalized hypergeometric function as follows:

$$\Phi_A(s) = \frac{1}{\prod_{k=1}^{K} \Gamma(m_k)} \frac{\Gamma(1-K_0)}{s} \frac{\prod_{k=1}^{K} m_k \Gamma(m_k)}{\prod_{k=1}^{K} \Gamma(m_k)}$$

It should be pointed out that one can also apply the technique in [4] to first derive the MGF of $a_i$ and then using inverse Laplace transform to obtain the density function of $a_i$.

B. Pairwise Error Probability and Convergence to an AWGN Channel

Consider the two signal points $x$ and $y$ in the rotated constellation $\Psi_e$. Let $e_i = x_i - y_i$ be the difference between the $i$th components of $x$ and $y$. Without loss of generality, assume that the Hamming distance between $x$ and $y$ is $L$ and they differ in the first $L$ components, i.e., $e_i = 0$ for $i > L$. The PEP, $P(x \rightarrow y)$, conditioned on the fading coefficients $h = [h_1, h_2, \ldots, h_N]$ is given as

$$P(x \rightarrow y | h) = \frac{Q\left(\sqrt{2\gamma} \frac{1}{\sqrt{1 - 2\gamma N_0}} \sqrt{\sum_{i=1}^{L} |h_i|} \frac{e_i}{|e_i|} \right)}{1}$$

By using the identity $Q\left(\sqrt{2\gamma} \frac{1}{\sqrt{1 - 2\gamma N_0}} \sqrt{\sum_{i=1}^{L} |h_i|} \frac{e_i}{|e_i|} \right) = 1/\sqrt{1 - 2\gamma N_0} \exp\left(-\gamma \sin^2 \theta \frac{1}{4N_0 \sin^2 \theta} \right) d\theta$, the conditional PEP can be expressed as

$$P(x \rightarrow y | h) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\sum_{i=1}^{L} \frac{a_i |e_i|^2}{4N_0 \sin^2 \theta} \right) d\theta$$

where $a_i = |h_i|^2$. Averaging over the independent random variables $\{a_i\}$ and exchanging the integration and the expectation operations, the unconditional PEP $P(x \rightarrow y)$ can be computed as follows:

$$P(x \rightarrow y) = \frac{1}{\pi} \int_0^{\pi/2} \prod_{i=1}^{L} \Phi_A\left(\frac{|e_i|^2}{4N_0 \sin^2 \theta} \right) d\theta$$

where $\Phi_A(\cdot)$ is the moment generating function given in (3).

Equation (6) shows that the unconditional PEP $P(x \rightarrow y)$ can be effectively computed via a single integral. Over an AWGN channel, by performing similar analysis, the PEP can be computed as [1]

$$P_{\text{AWGN}}(x \rightarrow y) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\sum_{i=1}^{L} \frac{|e_i|^2}{4N_0 \sin^2 \theta} \right) d\theta$$

It was shown in [1] that when $L \rightarrow \infty$, the PEP in (6) for $K = 1$ and $m_1 = 1$, i.e., the PEP over a conventional Rayleigh fading channel, can approach the PEP of the Gaussian channel in (7). In the following, the result is generalized to the case of multiplicative Nakagami-$m$ fading channels.

For simplicity, assume that $|e_i|^2 = 1$ for $1 \leq i \leq L$. Let SNR be the average signal-to-noise-ratio, defined as $\text{SNR} = L/N_0$. Then the PEP in (6) for a multiplicative Nakagami-$m$ fading channel is rewritten as

$$P(x \rightarrow y) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\sum_{i=1}^{L} \frac{|e_i|^2}{4N_0 \sin^2 \theta} \right) d\theta$$

Similarly, the PEP in (7) simplifies to

$$P_{\text{AWGN}}(x \rightarrow y) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{\text{SNR}}{4 \sin^2 \theta} \right) d\theta$$

Define $X = \frac{4 \sin^2 \theta}{\text{SNR}}$. To show that (8) approaches (9) when $L \rightarrow \infty$, one needs the following Lemma regarding the generalized hypergeometric function.

**Lemma:** Let $X$ be a positive number. Then

$$\lim_{L \rightarrow \infty} \left[ \frac{1}{X} \right]^{L} = \exp\left(\frac{-1}{X} \right)$$

Similarly, the PEP in (7) simplifies to

$$P_{\text{AWGN}}(x \rightarrow y) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{LX \prod_{k=1}^{K} m_k}{4N_0 \sin^2 \theta} \right) d\theta$$

Define $X = \frac{4 \sin^2 \theta}{\text{SNR}}$. To show that (8) approaches (9) when $L \rightarrow \infty$, one needs the following Lemma regarding the generalized hypergeometric function.

**Lemma:** Let $X$ be a positive number. Then

$$\lim_{L \rightarrow \infty} \left[ \frac{1}{X} \right]^{L} = \exp\left(\frac{-1}{X} \right)$$

(10)
Proof: By taking the logarithms of both sides of (10) and consider a new variable $Y = L \cdot X$, proving the Lemma is equivalent to showing that $\lim_{Y \to \infty} Y \cdot \log \left[ K_{F_0} \left( m_1, \ldots, m_K; \frac{-1}{Y \prod_{k=1}^{K} m_k} \right) \right] = -1$. Using a generalized hypergeometric series in [8, 9.14.1], one has

$$K_{F_0} \left( m_1, \ldots, m_K; \frac{-1}{Y \prod_{k=1}^{K} m_k} \right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \left( \prod_{k=1}^{K} (m_k)^n \right)}{n! \left( \prod_{k=1}^{K} m_k \right)^n Y^n}$$

(11)

where $(x)_n = \prod_{i=1}^{n-1} (x + i)$ represents the Pochhammer symbol. Using the L'Hopital rule, it then follows that

$$\lim_{Y \to \infty} Y \cdot \log \left[ K_{F_0} \left( m_1, \ldots, m_K; \frac{-1}{Y \prod_{k=1}^{K} m_k} \right) \right] = \lim_{Y \to \infty} Y \cdot \log \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \left( \prod_{k=1}^{K} (m_k)^n \right)}{n! \left( \prod_{k=1}^{K} m_k \right)^n Y^n} \right)$$

$$= \lim_{Y \to \infty} \frac{1}{Y} \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \left( \prod_{k=1}^{K} (m_k)^n \right)}{n! \left( \prod_{k=1}^{K} m_k \right)^n Y^n} \right) \cdot (-1)^{-2}$$

$$= \lim_{Y \to \infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \left( \prod_{k=1}^{K} (m_k)^n \right)}{(n-1)! \left( \prod_{k=1}^{K} m_k \right)^n Y^{n-1}} \right) = -1$$

(12)

which completes the proof.

To illustrate the effect of the order $L$, Fig. 1 presents the PEPs computed with (8) over multiplicative fading with $K = 1, 2, 3$, and $m_k = 1$ for various values of $L$. The PEP over an AWGN channel is also provided for comparison. It can be seen from Fig. 1 that for small values of $L$, there is a significant performance gap between the higher-order and lower-order multiplicative fading models. However, as $L$ becomes sufficiently large, the performance differences almost disappear, and they all closely approach that over an AWGN channel. This implies that the adverse effects of multiplicative fading can be practically eliminated.

The above analysis can also be easily extended to the case of BICM-ID systems over multiplicative fading channels. In particular, similar to the analysis done in [2], it is possible to obtain the closed-form expression of the asymptotic bit-error probability of BICM-ID based on the PEP expression obtained earlier. It is then straightforward to show that by employing SSD with a sufficiently large dimension, the error performance of BICM-ID systems over a multiplicative fading channel can also closely approach that over an AWGN channel. For the brevity of presentation, the analysis of BICM-ID systems is omitted here.

In order to provide an insight on how to obtain a good rotation matrix and as a consequence, a good constellation for this multiplicative fading model, one can use the Chernoff bound $Q \left( \sqrt{2\gamma} \right) < (1/2) \exp (-\gamma)$ in (4) and then taking the average over random variables $\{a_i\}$ as done in the previous subsection. After some manipulations, the following approximation is obtained:

$$P(x \to y) \sim \frac{1}{2} \prod_{i=1}^{L} K_{F_0} \left( m_1, \ldots, m_K; \frac{-\|e_i\|^2}{4N_0 \prod_{k=1}^{K} m_k} \right)$$

(13)

For a fixed average energy of the constellation $\Psi$ and at high enough SNR, $N_0 \to 0$ and $\|e_i\|^2 / 4N_0 \left( \prod_{k=1}^{K} m_k \right) \to \infty$. Following the analysis in [9], it can be seen that $K_{F_0} \left( m_1, \ldots, m_K; \frac{-\|e_i\|^2}{4N_0 \prod_{k=1}^{K} m_k} \right)$ is asymptotically of order $N_0^\alpha$, where $\alpha = \min_i \{m_k\}$, i.e.,

$$K_{F_0} \left( m_1, \ldots, m_K; \frac{-\|e_i\|^2}{4N_0 \prod_{k=1}^{K} m_k} \right) = O(N_0^{-\alpha})$$

As a consequence of the above result, for given $\{m_k\}$, the term $N_0^\alpha$ dominates the PEP in (13) when we fix the average energy of $\Psi$. This means that to achieve a low error probability, the first priority is to maximize the diversity order $L$ of the rotated constellation $\Psi_r$, which is similar to the case of uncoded transmission over a Rayleigh fading channel [1]. Therefore, a good rotation matrix over the conventional Rayleigh fading channel can be applied effectively over a general multiplicative Nakagami-$m$ fading channel.

IV. SIMULATION RESULTS

In this section, simulation results are provided to confirm the convergence to an AWGN channel for both uncoded systems and coded systems. For uncoded systems, we consider the systems employing real cyclotomic rotations constructed in [10] with a throughput of $\eta = 2 \text{bits/symbol}$, which is the same as the throughput provided by conventional QPSK modulation. The maximum likelihood (ML) decoder is implemented for all systems by using a sphere decoder [11]. In the case of coded systems (i.e., BICM-ID systems), QPSK modulation scheme with two-dimensional Gray mapping $\xi$ and a rate-1/2, 4-state convolutional code with generator matrix $g = (5, 7)$ are applied, which results in a throughput of $\eta = 1 \text{bit/symbol}$. The random interleaver has a length 25 600 coded bits.

Simulations results are presented for three cases of multiplicative Nakagami-$m$ fading with $K = 1, 2, 3$, and $m_k = 1, \forall k$. In all figures, the bit-error rate (BER) is plotted versus $E_b/N_0$, where $E_b$ is the energy per information bit.
where $\alpha_1 = \alpha = \exp(j2\pi/2^{t+2})$ and $\alpha_i = \alpha \exp(j2\pi(i-1)/2^{t+2})$ with $N = L = 2^t = 64$. Furthermore, due to its simplicity in implementation but yet effective performance, the suboptimal minimum mean-square error (MMSE) demodulator proposed in [2] is applied.

Fig. 3 presents the error performances after one and ten iterations for BICM-ID systems employing the above $G$ and the MMSE demodulator. The simulation result of error performance over an AWGN channel is also provided to serve as performance benchmarks for the fading channels. Since Gray mapping is used, the error performance over an AWGN channel is obtained with only one iteration. As expected from the analysis, there is only a small gap between the error performances of the three systems over multiplicative fading channels, and they both closely approach the performance of BICM-ID over an AWGN channel. At the practical BER level of $10^{-4}$, the gaps between the error performance over an AWGN channel and those over the multiplicative fading channels with $K = 1$, $K = 2$, and $K = 3$ are only 0.08 dB, 0.18 dB, and 0.43 dB, respectively.

V. CONCLUSIONS

This letter considered the use of signal space diversity over multiplicative Nakagami-$m$ fading channels, where the fading process is represented by the product of $K$ independent but not necessarily identically distributed Nakagami-$m$ random variables. Theoretical analysis and simulation results show that SSD can significantly enhance the error performance of communication systems operating over multiplicative fading channels. By employing SSD with a sufficiently large dimension, it was demonstrated that the BER performances for both uncoded and BICM-ID systems can closely approach that over their counterparts over an AWGN channel.

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