An improved robust stabilization method for discrete-time fuzzy systems with time-varying delays

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Received 5 January 2014; received in revised form 25 June 2014; accepted 1 August 2014

Available online 11 August 2014

Abstract

This note focuses on the robust stabilization of discrete-time fuzzy uncertain systems with time-varying delays under a delayed nonparallel distributed compensation scheme. The key idea is twofold: first, the linear matrix inequalities (LMI) proposed here are shown to generalize some previous similar results available in recent literature, and second, the design of control parameters is decoupled from the proposed fuzzy-basis dependent Lyapunov–Krasovskii functional (FBDLKF) by means of Finsler’s lemma. Finally, a numerical example is provided to illustrate the effectiveness of this method.

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1. Introduction

Over the past decades, Takagi–Sugeno (T–S) fuzzy models [1] have been extensively accepted by the control community, since they represent a powerful tool to deal with the robust stability analysis and stabilization of nonlinear systems [2]. In fact, a large number of complex nonlinear systems (i.e., the internal combustion engine system [3] and the quadrotor helicopter [4]) can be represented by a weighted sum of linear subsystems, blended together with some nonlinear scalar functions satisfying the convex sum property. Thus, the advantages of T–S models are twofold: (1) T–S fuzzy models provide a systematic procedure to exactly represent in a compact set nonlinear models (for instance, by means of the sector nonlinearity approach [5]), and (2) the last

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advances in linear system theory can be integrated into the framework of nonlinear control. Concretely, the stabilization of nonlinear discrete-time systems via T–S models has been intensively investigated under the so-called Parallel Distributed Compensation (PDC) technique (see [6] and references therein). More recently, in [7], a nonparallel distributed compensation scheme (denoted as non-PDC) is demonstrated to improve the robust performance with respect to PDC [8,9]. Also, it is well established that the use of fuzzy-basis-dependent Lyapunov functions (FBDLF) generalizes the single quadratic case [10,11].

On the other hand, the time-varying delays are frequently encountered in many practical control systems. Their presence can be related to the own physical nature of the plant (mass or energy transport delays, i.e., in hydraulic or pneumatic systems) or communication delays (i.e., network-based control systems [12]). It is well known that time delays can be a source of poor performance or even instability [13]. Hence, the study of time delayed systems has received much attention and various analysis and synthesis methods have been developed over the last few years (see [14] and references therein).

In the context of discrete-time systems with state delays, the use of Lyapunov–Krasovskii functionals (LKF) has been widely used for stability analysis and stabilization [15,16]. Moreover, LKF has been applied to T–S fuzzy systems with uncertain time-constant delays [17] and unknown time-varying delays with known bounds (see [18] and references therein). Other problems like the robust $H_\infty$ filtering design for T–S fuzzy systems with time-varying delays have also been investigated under LKF (see [19] and references therein). Very recently, the stability analysis and the stabilization of time-varying delayed systems were addressed by using the small-gain theorem and the I/O approach [20]. The latter method transforms the system with time-varying delays into a new one formed by the feedback interconnection of a subsystem with time-constant delays and some norm-bounded feedback subsystem comprising time-varying delay uncertainties. Thus, by virtue of the scaled small gain theorem, only the first subsystem needs to be considered to ensure the stability of the original system. These same ideas were also extended to the context of T–S fuzzy continuous systems [21] and discrete systems [22].

The motivation of this work is to improve the robust performance in the control design of discrete-time fuzzy uncertain systems with time-varying delays. The key idea lies in the combination of the non-PDC control scheme [7] with the incorporation of some delayed terms in the control law. Therefore, a more powerful control design method based on the small-gain theorem and the I/O approach is developed, generalizing some previous results proposed in recent literature [22]. Also, by means of Finsler’s lemma, the control parameters are decoupled from the decision variables linked to the FBDLF and all the control gains can be easily computed by solving a set of LMI constraints.

This paper is organized as follows: the next section provides the notation and useful materials, Section 2 describes the problem statement and previous results given in the literature, Section 3 exposes the main result, Section 4 gives a numerical example and finally, Section 5 gathers some conclusions and perspectives.

**Preliminaries and notations:** The notation $P > 0$ means that $P$ is positive definite, $0_{(m \times n)}$ stands for a matrix with $m$ rows and $n$ columns with all entries equal to zero, and $0_{(n)}$ and $I_{(n)}$ denote, respectively, the zero and identity matrices of dimension $n$. The symbol $\text{diag}(\cdots)$ stands for a block-diagonal matrix and the shortcut $(* \cdot)$ denotes the transpose term (for instance, $X + (* \cdot) = X + X^T$, for any matrix $X$). Also, $\|G\|_\infty$ denotes the $L_2$-induced norm of a transfer function matrix or a general operator.

Define the sample index $k \in \mathbb{N}^+$. The shortcut $X_{z,k}$ is used to denote $X_{z,k} = \sum_{i=1}^{n} \mu_i(z_k)X_i$, and $X_{z,k}^{(p)} = \sum_{i=1}^{n} \mu_i(z_{k+p})X_i$, where $z_k \in \mathbb{R}^q$ is some sample-dependent variable, $X_i$ denotes any
matrices \(X_i, i = 1, \ldots, r\), \(p \in \mathcal{N}\), and \(\mu_k(\cdot)\) denotes a positive function \(\mathcal{R}^q \to \mathcal{R}\), such that \(0 \leq \mu_k(\cdot) \leq 1\) and \(\sum_{i=1}^{r} \mu_i(z) = 1\). Analogously, \(X_{i,j} = \sum_{i=1}^{r} \mu_i(z)_{i,j} X_i X_j\) and the straightforward generalization for multiple sums. On the other hand, we define the set \(\mathbb{I}_r = \{1, 2, \ldots, r\}\). The next lemmas are useful in this paper: Lemma 1 is a sufficient condition to ensure the negativeness (or positiveness) of multiple convex sum and Lemma 2 is known as the elimination lemma:

**Lemma 1** (Tanaka and Wang [5]). The inequality \(\Gamma < 0\) is fulfilled providing the following conditions hold, for \(s, t, l, i, j \in \mathbb{I}_r^2\)

\[
\begin{align*}
\Gamma_{stii} &< 0 \\
\Gamma_{stij} + \Gamma_{slij} &< 0, \quad i < j
\end{align*}
\]  

(1)

**Remark 1.** In order to prove the negativeness of the multiple convex sum \(\Gamma < 0\), the relaxation lemma (Lemma 1) has been chosen for the sake of comparison in the numerical example. However, other more powerful relaxations might be chosen leading to less conservativeness (see [23,24,11]).

**Lemma 2** (Boyd et al. [25]). Given the symmetric matrix \(\Omega\) and the matrices \(U, W, V\) of suitable dimensions, \(\Omega + U W V^T + (*) < 0\) holds if and only if the following LMI constraints hold:

\[
\begin{align*}
N_u \Omega N_u^T &< 0 \\
N_v \Omega N_v^T &< 0
\end{align*}
\]  

(2)

where \(N_u, N_v\) are null basis of \(U, V\), respectively (that is, \(N_u U = 0\) and \(N_v V = 0\)).

2. Problem statement

Let us consider the following discrete-time T–S fuzzy system with unknown time-varying delay \(h_1 \leq d_k \leq h_2\):

\[
\begin{align*}
x_{k+1} &= \tilde{A}_{z,k} x_k + \tilde{A}_{dc,k} x_k - d_k + \tilde{B}_{z,k} u_k \\
x(k) &= \eta_1(x), \quad -h_1 \leq \kappa \leq 0 \\
\tilde{A}_{z,k} &= A_z + \Delta_{A_{z,k}}, \quad \tilde{A}_{dc,k} = A_{dc} + \Delta_{A_{dc,k}}, \quad \tilde{B}_{z,k} = B_z + \Delta_{B_{z,k}}
\end{align*}
\]  

(3)

where, following the notation defined above \((A_z = \sum_{i=1}^{r} \mu_i(z_k) A_i, B_z, \text{ and so on})\), the scalar function \(\mu_i(z_k)\) denotes the normalized membership function of the T–S fuzzy model, evaluated on the premise variables \(z_k\). The value \(r\) is the number of rules of the T–S fuzzy model. On the other hand, the lower and upper bounds of delay \((h_1 \text{ and } h_2, \text{ respectively})\), are assumed to be known. The function \(\eta_1(x)\) represents the initial state condition.

The system uncertainties are defined in the usual norm-bounded form as

\[
(\Delta_{A_{z,k}} \Delta_{A_{dc,k}} \Delta_{B_{z,k}}) = \gamma F_z \Delta_t (E_{A_z} E_{A_{dc}} E_{B_z})
\]  

(4)

for known matrices \(F_z \in \mathcal{R}^{n \times l_z}, E_{A_i} \in \mathcal{R}^{l_z \times n}, E_{A_{dc}} \in \mathcal{R}^{l_z \times n}, E_{B_i} \in \mathcal{R}^{l_z \times m}\), where \(\Delta_t \in \mathcal{R}^{l_z \times l_z}\) is a time-varying unknown matrix that satisfies the \(l_z\)-induced norm bound \(\Delta_t^T \Delta_t \leq I\) and \(\gamma\) is a positive scalar that determines the size of uncertainties, if so wished for later optimization.

Also, let us consider the Parallel Distributed Compensation (PDC) control law:

\[
u_k = K_z x_k
\]  

(5)
The robust stability analysis of system (3) can be guaranteed for a prescribed tolerance to model uncertainties in $\gamma$ by virtue of the following theorem:

**Theorem 1.** System in Eq. (3) with control law (5) is robustly asymptotically stable if there exist matrices $P_i$, $Q_{1i}$, $Q_{2i}$, $Z_1$, $Z_2$, $S > 0$, ($1 \leq i \leq r$) satisfying Eq. (1) with

$$
\Gamma_{stij} = \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} & 0_{(4n\times l1)} & \Sigma^T_{14} \\
(*) & \Sigma_{22} & \gamma \Sigma^T_{23} & 0_{(3n\times l2)} \\
(*) & (*) & -I_{(l1)} & 0_{(l1\times l2)} \\
(*) & (*) & (*) & -I_{(l2)}
\end{pmatrix}
$$

$$
\Sigma_{11} = \begin{pmatrix}
\phi_1 & Z_1 & Z_2 & 0_{(n)} \\
(*) & -Q_{1i} - Z_1 & 0_{(n)} & 0_{(n)} \\
(*) & (*) & -Q_{2i} - Z_2 & 0_{(n)} \\
(*) & (*) & (*) & -S
\end{pmatrix}
$$

$$
\phi_1 = -P_i + Q_{1i} + Q_{2i} - Z_1 - Z_2 \\
\Sigma_{12} = (\Sigma^T_1, \Sigma^T_2, \Sigma^2_2)
$$

$$
\Sigma_1 = \begin{pmatrix}
A_i + B_i K_j & \frac{1}{2} A_{di} & \frac{1}{2} A_{di} & \frac{\tau}{2} A_{di}
\end{pmatrix}
$$

$$
\Sigma_2 = \begin{pmatrix}
A_i + B_i K_j & -I & \frac{1}{2} A_{di} & \frac{1}{2} A_{di} & \frac{\tau}{2} A_{di}
\end{pmatrix}
$$

$$
\Sigma_{14} = \begin{pmatrix}
E_{Ai} + E_{Bi} K_j & \frac{1}{2} E_{Adi} & \frac{1}{2} E_{Adi} & \frac{\tau}{2} E_{Adi}
\end{pmatrix}
$$

$$
\Sigma_{22} = \text{diag}(-P^{-1}_s, -Z^{-1}_1, -S^{-1}) \\
\Sigma_{23} = (F^T_i, F^T_i, F^T_i), \quad Z_d = h_1^2 Z_1 + h_2^2 Z_2, \quad \tau = h_2 - h_1
$$

**Proof.** The proof can be addressed by considering the FBDLKJF (see [22]):

$$
V_k = V_{1k} + V_{2k} + V_{3k}
$$

$$
V_{1k} = x^T_k P_k x_k
$$

$$
V_{2k} = \sum_{i=k-h_1}^{k-1} x^T_i Q_{1i}^{(i)} x_i + \sum_{i=k-h_2}^{k-1} x^T_i Q_{2i}^{(i)} x_i
$$

$$
V_{3k} = \sum_{i=-h_1}^{-1} \sum_{j=k+i}^{k-1} \nu^T_i Z_1 \nu_j + \sum_{i=-h_2}^{-1} \sum_{j=k+i}^{k-1} \nu^T_i Z_2 \nu_j
$$

$$
\nu_k = x_{k+1} - x_k
$$

(7)

On the other hand, a robust stabilizing controller can be found by virtue of the following theorem:

**Theorem 2 (Li et al. [22]).** System in Eq. (3) with control law (5) is robustly asymptotically stable if, for a given scalar $\varepsilon > 0$, there exist matrices $P_i$, $Q_{1i}$, $Q_{2i}$, $Z_1$, $Z_2$, $S > 0$, ($1 \leq i \leq r$), some

...
matrices \( Y_j, M \) satisfying Eq. (1) with
\[
\Gamma_{stlij} = \begin{pmatrix}
\Sigma_{11} & \hat{\Sigma}_{12} & 0_{(4n\times l_1)} & \hat{\Sigma}_{14}^T \\
(* ) & \hat{\Sigma}_{22} & \hat{\Sigma}_{23}^T & 0_{(3n\times l_2)} \\
(* ) & (* ) & -\rho I_{(l_1)} & 0_{(l_1\times l_2)} \\
(* ) & (* ) & (* ) & -I_{(l_2)}
\end{pmatrix}
\]
\[
\hat{\Sigma}_{12} = \begin{pmatrix}
\hat{\Sigma}_{1}^T \\
\frac{1}{2}A_{di}M \\
\frac{1}{2}A_{di}M \\
\frac{1}{2}E_{Adi}M
\end{pmatrix}
\]
\[
\hat{\Sigma}_{1} = \begin{pmatrix}
A_iM + B_iY_j \\
(\gamma - I)M + B_iY_j \\
E_{Adi}M + E_{Bi}Y_j \\
\end{pmatrix}
\]
\[
\hat{\Sigma}_{14} = \begin{pmatrix}
E_{Adi}M + E_{Bi}Y_j \\
(\gamma - I)M + B_iY_j \\
\end{pmatrix}
\]
\[
\hat{\Sigma}_{14} = \begin{pmatrix}
E_{Adi}M + E_{Bi}Y_j \\
(\gamma - I)M + B_iY_j \\
\end{pmatrix}
\]
\[
\hat{\Sigma}_{14} = \begin{pmatrix}
E_{Adi}M + E_{Bi}Y_j \\
(\gamma - I)M + B_iY_j \\
\end{pmatrix}
\]
\[
\hat{\Sigma}_{14} = \begin{pmatrix}
E_{Adi}M + E_{Bi}Y_j \\
(\gamma - I)M + B_iY_j \\
\end{pmatrix}
\]
\[
\hat{\Sigma}_{14} = \begin{pmatrix}
E_{Adi}M + E_{Bi}Y_j \\
(\gamma - I)M + B_iY_j \\
\end{pmatrix}
\]
Moreover, the controller gains (5) can be obtained as \( K_i = Y_iM^{-1} \).

**Proof.** See [22] for details. □

**Corollary 1.** Let us consider \( \rho \) as the decision variable in (8). Then, the following convex optimization problem allows obtaining a controller that optimizes tolerance to model uncertainties for a given delay interval \( \tau = h_2 - h_1 \):
\[
\min \rho \quad \text{s.t.} \ (1).
\]
where \( \Gamma_{stlij} \) is defined in Eq. (8). The achieved tolerance to model uncertainties is \( \gamma = \rho^{-1/2} \).

3. Main results

The first part of this section concerns the robust stability analysis of system (3) under the following delayed non-PDC control law:
\[
u_k = Y_zH_z^{-1}x_k + \frac{1}{2} \sum_{f=1}^2 Y_{dfz}H_z^{-1}x(k - h_f)
\]

**Robust stability analysis:** The following theorem gives a sufficient condition based on LMI to ensure robust stability of system (3), for a given control law in the form (9).

**Theorem 3.** The closed-loop formed by system in Eq. (3) with the control law (9) is robustly asymptotically stable if there exist matrices \( P_i, Q_{1i}, Q_{2i}, Z_1, Z_2, S > 0 \) \((1 \leq i \leq r)\) and some matrix \( W \) satisfying Eq. (1) with
\[
\Gamma_{stlij} = \begin{pmatrix}
\Xi_1 & \gamma WF_i & \Xi_2^T \\
(* ) & -I_{(l_1)} & 0_{(l_1\times l_2)} \\
(* ) & (* ) & -I_{(l_2)}
\end{pmatrix}
\]
\[
\Xi_1 = \Xi_1^T + WX_{ij} + (WX_{ij})^T
\]
and the upper bound delay (\(\Delta\)) Now, let us consider the proof. The proof is addressed by the following input–output (I/O) approach: first, we define the new input \(w_{dk}\) that approximates the time-varying delayed state by the average between the lower and the upper bound delay \((h_1\) and \(h_2\), respectively):

\[
w_{dk} = \frac{2}{\tau} \left\{ x(k-d_k) - \frac{1}{2} x(k-h_1) - \frac{1}{2} x(k-h_2) \right\}
\]

We define \(\Delta_d : \nu \rightarrow w_d\) as the time-varying delay operator, verifying \(\|\Delta_d\|_\infty \leq 1\) [22, Proposition 1], i.e., \(w_{dk} = \Delta_d k \nu_k\), where \(\nu_k = x_{k+1} - x_k\) and \(\|\Delta_d\|_\infty \leq 1\). Therefore, applying the model transformation based on the I/O approach proposed in [22], the T–S fuzzy system in Eq. (3) can be reformulated into the interconnection of the time-constant delayed subsystem \(G_s\) and the feedback subsystem \(\Delta\):

\[
\begin{bmatrix}
  x_{k+1} \\
  \nu_k
\end{bmatrix} = \begin{bmatrix}
  \Gamma^A_{z,k} & \frac{1}{2} \tilde{A}_{d,k} \\
  \Gamma^D_{z,k} & \frac{1}{2} \tilde{A}_{d,k}
\end{bmatrix} \begin{bmatrix}
  \hat{x}_k \\
  w_{dk}
\end{bmatrix}
\]

\[
[w_{dk}] = (\Delta_d)[\nu_k]
\]

\[
\begin{align*}
\Gamma^A_{z,k} &= \left( \tilde{A}_{z,k} + \tilde{B}_{z,k} Y H_z^{-1} \right) \frac{1}{2} \left( \tilde{A}_{d,z,k} + \tilde{B}_{d,z,k} Y_{d,z} H_z^{-1} \right) \\
\Gamma^D_{z,k} &= \left( \tilde{A}_{z,k} - I + \tilde{B}_{z,k} Y H_z^{-1} \right) \frac{1}{2} \left( \tilde{A}_{d,z,k} + \tilde{B}_{d,z,k} Y_{d,z} H_z^{-1} \right) \\
\hat{x}_k &= (x^T_{k} x^T_{k-h_1} x^T_{k-h_2})^T
\end{align*}
\]

Now, let us consider \(V_k\) defined in Eq. (7). The return difference \(\Delta V_k = V_{k+1} - V_k\) can be put as follows:

\[
\begin{align*}
\Delta V_k &= \Delta V_{1k} + \Delta V_{2k} + \Delta V_{3k} \\
\Delta V_{1k} &= x^T_{k+1} P^{(1)} x_{k+1} - x^T_{k} P^T x_k
\end{align*}
\]
\[ \Delta V_{2k} = \sum_{l=1}^{2} \{ x_{k}^{T} Q_{lc} x_{k} \} - x_{k}^{T} Q_{lc} (x_{k-h_l}) \]

\[ \Delta V_{3k} \leq \nu_{k}^{T} Z_{d} \nu_{k} - \sum_{l=1}^{2} \{ x_{k} - x(k-h_l) \}^{T} Z_{l} \{ x_{k} - x(k-h_l) \} \]

(14)

Now, let us define the function \( J(k) \), for some matrix \( S > 0 \), given by

\[ J(k) = \nu_{k}^{T} S \nu_{k} - w_{d_{k}}^{T} S w_{d_{k}} \]

(15)

Then, proving \( \Delta V_{k} + J(k) < 0 \) for all \( k \), decomposing \( S = T^{T} T \), letting \( T_{y} = diag(T, I) \) and \( T_{w} = diag(T, I) \), by virtue of scaled small-gain theorem, \( \| T_{y} \circ G_{s} \circ T_{w}^{-1} \|_{\infty} < 1 \) holds and the closed-loop is stable, following the same baseline as in [20] because the infinity norm of system \( \Delta \) (12) is lower than 1, such as those that arises from the definition of \( \Delta_{d_{l}} \). After applying the suitable algebraic manipulations, we have

\[ \Delta V_{k} + J(k) = \hat{\eta}_{k}^{T} \Omega \hat{\eta}_{k} < 0 \]

\[ \hat{\eta}_{k} = \begin{pmatrix} x_{k+1}^{T} \ x_{k}^{T} \ x_{k-h_1}^{T} \ x_{k-h_2}^{T} \ w_{d_{k}}^{T} \end{pmatrix} \]
\[ \Omega = \begin{pmatrix} \hat{\phi}_{1} & -(Z_{d} + S) & 0 & 0 & 0 \\
(\ast) & \hat{\phi}_{2} & Z_{1} & Z_{2} & 0 \\
(\ast) & (\ast) & \hat{\phi}_{3} & 0 & 0 \\
(\ast) & (\ast) & (\ast) & \hat{\phi}_{4} & 0 \\
(\ast) & (\ast) & (\ast) & (\ast) & -S \end{pmatrix} \]

\[ \hat{\phi}_{1} = P_{z}^{(1)} + Z_{d} + S, \quad \hat{\phi}_{2} = -P_{z} + Z_{d} + S + Q_{1z}^{(-h_1)} + Q_{2z}^{(-h_2)} - Z_{1} - Z_{2} \]
\[ \hat{\phi}_{3} = -Q_{1z}^{(-h_1)} - Z_{1}, \quad \hat{\phi}_{4} = -Q_{2z}^{(-h_2)} - Z_{2} \]

(16)

Define \( \dot{X}_{z,k} = (-I \Gamma_{z,k} (\tau/2) \hat{A}_{d_{z,k}}) \). Taking into account that \( \dot{X}_{z,k} \hat{\eta}_{k} = 0 \), by Finsler’s Lemma [26], the inequality (16) is equivalent to

\[ \Omega + W \dot{X}_{z,k}^{A} + (\ast) < 0 \]

(17)

for some matrix \( W \). Taking into account Eq. (4), the above inequality can be put as

\[ \Omega + W \dot{X}_{z,k}^{A} + (\ast) + \gamma W F_{z} \Delta_{k} \hat{\Gamma}_{z}^{E} + (\ast) < 0 \]

\[ \dot{X}_{z}^{A} = \begin{pmatrix} -I_{A_{z}} + B_{z} Y_{z} H_{z}^{-1} \ 1/2 (A_{dc} + B_{z} Y_{d_{z}} H_{z}^{-1}) \ 1/2 (A_{dc} + B_{z} Y_{d_{z}} H_{z}^{-1}) \ \tau/2 A_{dc} \end{pmatrix} \]
\[ \dot{\Gamma}_{z}^{E} = \begin{pmatrix} 0 \ E_{A_{z}} + E_{B_{z}} Y_{d_{z}} H_{z}^{-1} \ 1/2 (E_{A_{dc}} + E_{B_{z}} Y_{d_{z}} H_{z}^{-1}) \ 1/2 (E_{A_{dc}} + E_{B_{z}} Y_{d_{z}} H_{z}^{-1}) \ \tau/2 E_{A_{dc}} \end{pmatrix} \]

(18)

Now, by virtue of the cross-product lemma, for some scalar \( \mu > 0 \), the following inequality holds:

\[ \gamma W F_{z} \Delta_{k} \hat{\Gamma}_{z}^{E} + (\ast) < \mu (\gamma W F_{z} (\gamma W F_{z})^{T} + \mu^{-1} (\hat{\Gamma}_{z}^{E})^{T} \hat{\Gamma}_{z}^{E}) \]

(19)

\(^{1}\)The dimensions of identity matrices in \( T_{y} \) and \( T_{w} \) may be different if \( \Delta_{k} \) were not square.
Then, applying the Schur complement and the above inequality, Eq. (18) becomes
\[
\begin{pmatrix}
\Omega + W \tilde{X}^A + (\star) + \mu^{-1} (\bar{F}_y^E)^T \bar{F}_y^E & \gamma W F z \\
(\star) & -\mu^{-1} I
\end{pmatrix} < 0
\] (20)

Note that, without loss of generality, the scalar \( \mu \) can be set at \( \mu = 1 \). Then, applying again Schur complement in Eq. (20) we obtain
\[
\Gamma_5 < 0
\]
\[
\Gamma_5 = \begin{pmatrix}
\Omega + W \tilde{X}^A + (\star) & \gamma W F z \\
(\star) & -I
\end{pmatrix}
\] (21)

Finally, the nonnegativity of the multiple convex sum (21) is ensured by virtue of Lemma 1. The proof is completed. \( \Box \)

The next corollary demonstrates that Theorem 3 is a generalization of Theorem 1:

**Corollary 2.** A feasible solution satisfying the matrix inequalities in Theorem 1 can be found if there exists a feasible solution satisfying the matrix inequalities in Theorem 3.

**Proof.** Let us consider the following values in (17): \( W = (W_1^T \ W_2^T \ 0_{(n)} \ 0_{(n)} \ 0_{(n)}), \ H_z = I, \ Y_{z} = K_z, \ Y_{dz} = 0, \ Y_{dz} = 0. \) Thus, Eq. (17) can be rewritten as \( \Omega + U W V^T + (\star) < 0 \), where
\[
U = \begin{pmatrix}
I_{(n)} & 0_{(n)} & 0_{(n)} & 0_{(n)} \\
0_{(n)} & I_{(n)} & 0_{(n)} & 0_{(n)}
\end{pmatrix}^T, \quad W = \begin{pmatrix}
W_1 \\
W_2
\end{pmatrix}
\]
\[
V^T = \begin{pmatrix}
-I & \tilde{A}_{z,k} + \tilde{B}_{z,k} K_z & \frac{1}{2} \tilde{A}_{dz,k} & \frac{1}{2} \tilde{A}_{dz,k}
\end{pmatrix}
\] (22)

By virtue of Lemma 2 and the Schur complement, the proof can be straightforwardly completed, taking the following null basis of \( U \) and \( V \):
\[
N_U^T = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}, \quad N_V^T = \begin{pmatrix}
\tilde{A}_{z,k} + \tilde{B}_{z,k} K & \frac{1}{2} \tilde{A}_{dz,k} & \frac{1}{2} \tilde{A}_{dz,k} & \frac{1}{2} \tilde{A}_{dz,k}
\end{pmatrix}
\] (23)

**Robust stabilization:** Note that Theorem 3 does not allow to design the parameters of the control law (9) by using LMI techniques. Thereafter, this result is adapted for robust stabilization:

**Theorem 4.** System in Eq. (3) with the control law (9) is robustly asymptotically stable if, for some scalar \( \sigma > 0 \), there exist matrices \( \tilde{P}_i, \ \tilde{Q}_{i1}, \ \tilde{Q}_{i2}, \ \tilde{Z}_1, \ \tilde{Z}_2, \ \tilde{S} > 0, \ H_i, \ Y_i, \ Y_{d1i}, \ Y_{d2i} (1 \leq i \leq r) \)
such that Eq. (1) hold with

$$
\Gamma_{stij} = \begin{pmatrix}
\tilde{z}_1 & \tilde{W}_i & \tilde{z}_2^T \\
\star & -\rho I_{(1)}(1) & \star \\
\star & \star & -I_{(2)}(2)
\end{pmatrix}
$$

$$
\tilde{z}_1 = \tilde{z}_1^\prime + \tilde{W}_i \tilde{x}_i + (\tilde{W}_i \tilde{x}_i)^T
$$

$$
\tilde{z}_1^\prime = \begin{pmatrix}
\tilde{\phi}_1 & -\tilde{\phi}_2 & \tilde{X}_i \\
\star & \star & \star \\
\star & \star & \star \\
\star & \star & \star \\
\star & \star & \star
\end{pmatrix}
$$

$$
\tilde{x}_i = \begin{pmatrix}
-H_j & A_h i H_j & B_i Y_j & \frac{1}{2} \left(A_{dhi} H_j + B_i Y_{dji}\right) & \frac{1}{2} \left(A_{dhi} H_j + B_i Y_{dji}\right) & \frac{\tau}{2} A_{dhi} H_j \\
0 & E_{Ah i} H_j & E_{Bi} Y_j & \frac{1}{2} \left(E_{Ah i} H_j + E_{Bi} Y_{dji}\right) & \frac{1}{2} \left(E_{Ah i} H_j + E_{Bi} Y_{dji}\right) & \frac{\tau}{2} E_{Ah i} H_j
\end{pmatrix}
$$

$$
\rho = \gamma^{-2}
$$

**Proof.** The proof can be addressed by choosing $W = (W_i^T W_i^T 0 \_n 0 \_n 0 \_n)^T$ with $W_1 = (H_{d1}^{-1})^T$ and $W_2 = \sigma W_1$. Applying the congruence transformation by pre- and post-multiplying both sides of Eq. (18) by $\text{diag}(H_{d1}, H_{d2}, H_{d3}, H_{d4}, \gamma^{-1} I_{(1)}, I_{(2)})$ and defining $\tilde{P}_z = H_{d1}^T P_z H_{d2}$, $\tilde{Q}_{1z} = H_{d1}^T Q_{1z} H_{d2}$, $\tilde{Q}_{2z} = H_{d1}^T Q_{2z} H_{d2}$, $\tilde{Z}_1 = H_{d1}^T Z_1 H_{d2}$, $\tilde{Z}_2 = H_{d1}^T Z_2 H_{d2}$ and $\tilde{S} = H_{d1}^T S H_{d2}$, the proof is completed. \(\square\)

**Corollary 3.** Let us consider $\rho$ as decision variable in Eq. (24). Then, the following convex optimization problem allows obtaining a controller that optimizes tolerance to model uncertainties for a given delay interval $\tau = h_2 - h_1$:

$$
\min_{\rho} \quad \text{s.t. } (1)
$$

where $\Gamma_{stij}$ is defined in Eq. (24). The achieved tolerance to model uncertainties is $\gamma = \rho^{-1/2}$.

**Remark 2.** Note that, by means of Finsler’s lemma, the decision variables linked to the controller parameters are decoupled from the FBDLKF by the suitable election of the free matrices $W_1$ and $W_2$. However, the scalar $\sigma$ in Theorem 4 has been introduced in order to put the stabilization problem in terms of LMI. By choosing this value properly (i.e., by simple line search), it is possible to achieve better robust performance. For this purpose, some procedures are employed in several works in which the stabilization problem is addressed by using Finsler’s Lemma [26].

**Remark 3.** Note that the number of LMI constraints in Corollary 3 is not increased with respect to Corollary 1. However, the number of decision variables in Corollary 3 (\(1.5(r+1)(n^2+n)+rn^2+3rnn+1\)) is greater than Corollary 1 (\((1.5(r+1)(n^2+n)+n^2+rmn+1)\)) due to the new control parameters defined in Eq. (9), namely $H_{d1i}, Y_{d1i}, Y_{d2i}, i = 1, \ldots, r$. 


Remark 4. The proposed control law (9) includes two delayed terms, corresponding to the lower and the upper bound delay ($h_1$ and $h_2$, respectively). However, it is expected that less conservative results may be obtained combining the proposed non-PDC scheme with some delay-partitioning technique [15,27], leading to an extension of Eq. (9) with multiple delayed components. Of course, the reduction of conservatism is at expenses of some increment of computational complexity in the control design method. The improvement of robust performance keeping a reasonable trade-off with computational cost in the control design is mentioned as a matter of future research.

4. Numerical examples

Let us consider the following Henon system, already studied in [22]. The T–S fuzzy discrete-time model (3) represents approximately the nonlinear system of the proposed example, with two rules ($r=2$), and taking $c=0.8$ and $m=0.6$ (such as in the proposed example), and the following system matrices

$$
A_1 = \begin{pmatrix} cm & 0.3 \\ c & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -cm & 0.3 \\ c & 0 \end{pmatrix},
$$

$$
A_{d1} = \begin{pmatrix} (1-c)m & 0 \\ 1-c & 0 \end{pmatrix}, \quad A_{d2} = \begin{pmatrix} -(1-c)m & 0 \\ 1-c & 0 \end{pmatrix},
$$

$$
B_1 = B_2 = (1 \ 0)
$$

Table 1 shows the achieved improvement of the maximum upper bound delay $h_2$ with respect to [22] taking into account different values for the lower bound delay $h_1$. In the comparison, we have also included a non-delayed version of the control law (9) by setting $Y_{d1z} = 0, Y_{d2z} = 0$ (second row in Table 1). These results illustrate that the reduction of conservatism is achieved by two ways: (1) the use of the non-PDC control scheme (9) combined with the decoupled design of the controller parameters $H_z, Y_z, Y_{1z}, Y_{2z}$ achieved by using Finsler’s lemma, and (2) the introduction of delayed components in the control law (9). Thereafter, we consider some model uncertainties on the form (4) with

$$
F_1 = (0.1 \ 0)^T, \quad F_2 = (0.1 \ 0)^T
$$

$$
E_{A1} = (0.1 \ 0), \quad E_{A2} = (0 \ 0)
$$

$$
E_{Ad1} = (0 \ 0), \quad E_{Ad2} = (0.1 \ 0)
$$

$$
E_{B1} = 0, \quad E_{B2} = 0.1
$$

where $\gamma$ is the robust index performance to be optimized. If we consider $1 \leq d_k \leq 4$, Corollary 1 provides a suboptimal robust controller achieving $\gamma=15.8$, where the control law is on the form (5) with $K_1 = [-0.0944 \ -0.3199]$ and $K_2 = [1.0657 \ -0.3576]$, and $\epsilon = 2.9$. However, by

### Table 1
Calculation of upper bound delay $h_2$ given $h_1$.

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>8</th>
<th>12</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 2 [22] (PDC)</td>
<td>19</td>
<td>23</td>
<td>33</td>
</tr>
<tr>
<td>Theorem 4 (non-delayed non-PDC)</td>
<td>20</td>
<td>24</td>
<td>34</td>
</tr>
<tr>
<td>Theorem 4 (delayed non-PDC)</td>
<td>23</td>
<td>27</td>
<td>37</td>
</tr>
</tbody>
</table>
using Corollary 3, a controller on the form (9) is found with improved robust performance against model uncertainties: the achieved $\gamma$ is enlarged to 21.5. The controller parameters are

$$Y_1 = (-7.7674 \quad -225.0119), \quad Y_2 = (0.4253 \quad -87.4973)$$

$$H_1 = \begin{pmatrix} 7.1897 & 24.4312 \\ 24.2228 & 744.2517 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 6.5495 & 18.6667 \\ 17.9625 & 327.0167 \end{pmatrix}$$

$$Y_{d11} = Y_{d21} = (-0.5120 \quad -1.6108), \quad Y_{d12} = Y_{d22} = (0.7186 \quad 2.0480)$$

and $\sigma = -0.3$. On the other hand, if we consider $Y_{1di} = 0, Y_{2di} = 0, i = 1, 2$ (the version of non-delayed control law in Eq. (9)), then a robust performance $\gamma = 18.9$ is obtained with the following controller parameters:

$$Y_1 = (-7.0496 \quad -251.3783), \quad Y_2 = (0.7572 \quad -122.3891)$$

$$H_1 = \begin{pmatrix} 5.8956 & 17.4757 \\ 21.7379 & 839.4963 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 5.7252 & 15.3532 \\ 14.2958 & 400.9325 \end{pmatrix}$$

Note that Corollaries 3 and 1 involve 48 and 36 decision variables, respectively. As pointed out in Remark 3, the reduction of conservativeness is obtained at expenses of some increment of computational complexity. However, note that in this case, this increment is not too much big with respect to the improvement of robust performance. Therefore, a reasonable trade-off between complexity and robust performance is obtained.

**Simulation results:** Figs. 1 and 2 show the time response of the system states $x_{1k}$ and $x_{2k}$, respectively. The simulations are performed using the control schemes (5) and (9) (non-delayed version) designed above: the dashed line corresponds to the former (Corollary 1), and the solid line corresponds to the latter (Corollary 3). Comparing both time responses, an improvement of the transient can be appreciated for the delayed non-PDC control scheme (9) designed in this example (solid line). The simulation parameters and conditions are the following: $\gamma$ is set at 21.5, the delay $d_k$ and the uncertain matrix $\Delta_k$ vary following the random pattern depicted in Fig. 3.

![Fig. 1](image-url)  
*Fig. 1. Comparison of the transient response of state $x_{1k}$.**
The function $\eta_x(\kappa)$ (initial state condition) is chosen to be $\eta_x(\kappa) = \frac{e^{\kappa h_2}}{h_2}$, $\kappa = -h_2, \ldots, 0$ and the membership functions are $\mu_1(z_k) = \frac{1}{2} \left(1 - \frac{z_k}{m}\right)$, $\mu_2(z_k) = 1 - \mu_1(z_k)$, where $z_k = cx_1k + (1-c)x_1(k-d_k)$ and $z_k \in [-m, m]$.

5. Conclusions

In this paper, the problem of robust stabilization of discrete-time fuzzy uncertain systems with time-varying delays has been addressed under a non-PDC control scheme with delayed terms. The control parameters are designed via LMI and decoupled from the FBDLK functional by
applying Finsler’s lemma. Moreover, it has been shown that the proposed result generalizes some previous results in the literature. Also, a numerical example has been provided to illustrate the achieved improvement. As possible future extensions of this research, it pointed out the improvement of robust performance in the control design with output feedback / observer-based controllers [28], that is, assuming a not-fully measurable state. Also, this work could be extended to networked-control systems. In this respect, phenomena like data-packet dropouts [12] and quantized measurements [28] (due to the limitation of bandwidth resources in the communication links between the plant and the controller) should also be considered in the robust control design.

Acknowledgments

The present research work has been supported by International Campus on Safety and Intermodality in Transportation, the Nord-Pas-de-Calais Region, the European Community, the Regional Delegation for Research and Technology, the Ministry of Higher Education and Research, and the National Center for Scientific Research. The authors gratefully acknowledge the support of these institutions.

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