Origami Axioms and Circle Extension

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1. INTRODUCTION

Origami, i.e. paper folding, is a powerful tool for geometrical constructions. In 1989, Humiaki Huzita introduced six folding operations based on aligning one or more combinations of points and lines [6]. Jacques Justin, in his paper of the same proceedings, also presented a list of seven distinct operations [9]. Justin’s work was written in French, and was somehow unknown among researchers. This led Hatori [5] to ‘discover’ the same seventh operation in 2001. Alperin and Lang in 2006 [1] showed, by exhaustive enumeration of combinations of superpositions of points and lines involved, that the seven operations are complete combinations of the alignments. Huzita did not call his list of operations axioms. However, over years, the term Huzita axioms, or Huzita-Justin or Huzita-Hatori axioms, has been widely used in origami community. From logical point of view, it is not accurate to call Huzita’s original statements of folding operations as axioms, because they are not always true in plane Euclidean geometry. In this paper, we present precise statements of the folding operations, by which naming them ‘axioms’ is logically valid, and we make some notes about the work of Huzita and Justin.

One of the interesting features about origami folding is the ability to find, by hand, certain tangents of parabolas without the need to construct the parabolas themselves. Aligning points and lines, with sliding, is enough to achieve this feature. In further exploiting the power of origami, we became interested in studying possible extensions of origami with other geometrical tools. One such tool is the compass. We investigated the possible folding operations that can be performed if we allow origamists to draw circles on origami paper, and fold to align points, lines, and circles together. We will show that the introduction of circles into origami allows finding common tangents to different conic sections, and present three new folding axioms with circles. We also show that the new extended set of axioms includes all possible alignments of points, lines, and circles, which specify a folding crease. Using this set of axioms, we give new construction method for interesting geometrical problem: trisecting an arbitrary angle. In the paper, we discuss the mathematical and operational aspects of this extension, and also describe an implementation of the new axiom set in a computational origami system [8].

2. FOLDING OPERATIONS AND AXIOMS

The following are the basic six origami operations presented by Huzita [6], with statement (7) defined by Hatori [5], and described in a consistent way with the rest:

(1) Given two distinct points, you can fold making the crease pass through both points (ruler operation).

(2) Given two distinct points, you can fold superposing one point onto the other point (perpendicular bisector).

(3) Given two distinct (straight) lines, you can fold superposing one line onto another (bisector of the angle).

(4) Given one point and two lines, you can fold to place the point onto one line, while the crease is perpendicular to the other line.

2.1 Discussion

We first note the following about the seven operations:
Impossibility of Fold

The validity of some of the statements does not hold. For example, the ability to perform the fold operation according to (5), (6), and (7) depends on the relative positions of the concerned points and lines. In (5), by superposing point $P$ onto line $m$ and making the fold line $x$ pass through $Q$, we have a parabola $\gamma$ defined by the focus $P$ and the directrix $m$, for which $x$ is a tangent. If the point $Q$ is inside the parabola $\gamma$, such a construction is impossible since $x$ can never pass through a point inside the parabola. The same observation applies to (6). Each pair of point and line defines one parabola, and the desired fold line should be a common tangent of both parabolas. There could be 1, 2, or 3 common tangents. For example, if one of the parabolas lies completely inside the other, then the fold is impossible. Also in (7), if the two given lines are parallel, there is no fold line that satisfies the description, unless the point is already on the desired line, and in that case we have infinite number of fold lines.

Infinite number of fold lines

Huzita did not explicitly mention the cases where there arise infinite number of fold lines satisfying the description of the operations. We believe he tacitly excluded such cases. Otherwise this would immediately lead to the questions of undecidability. Huzita made distinction restrictions to exclude such cases in his operations. However, one case was skipped in operation (6) which allows infinite folding possibilities as shown in Fig. 1 (a). The same is possible in Hatori’s operation (7) as mentioned above.

Superfluous conditions

The condition given in (5) is actually stricter than necessary. If one of the points, or even both, is on the given line, the fold operation is always defined as shown in Fig. 2. While cases (a) and (b) in the figure can be folded by simpler operations ((4) and (1)), the condition also excludes case (c) which can not be folded by another operation.

Figure 2: Possible fold lines of (5) even if one or both points are on the line

Huzita’s (6) is:

$$\{P, Q\} \not\subset m \cap n \land (P, m) \neq (Q, n) \land (P \not\in m \lor Q \not\in n \lor m \cap n \neq \emptyset)$$

The condition $P \not\in m \lor Q \not\in n \lor m \cap n \neq \emptyset$ eliminates the case where $m$ and $n$ are parallel, and $P$ and $Q$ are on $m$ and $n$ (Fig. 1 (a)). The condition $\{P, Q\} \not\subset m \cap n$ eliminates the cases shown in Fig. 1 (c) and (d), and $(P, m) \neq (Q, n)$ handles the case in Fig. 1 (b).

Operational origami

From operational point of view, we would like to replace the perpendicularity notion of two lines, with the principle of superposing a line with itself by folding along another line. After all, we have no means in origami to decide if two lines are perpendicular, except for folding along one of them, and see if the other one is superposed with itself. Justin’s list of operations used the same notion of superposing a line with itself. By noting that in Justin’s operation $\otimes$ (corresponds to Huzita’s operation (4)), two possible fold lines are allowed, we concluded that Justin permitted superposing a given line with itself by folding along it. We noted a mistake in the number of possible folds in Justin’s operation $\otimes$, which corresponds to Hatori’s operation (7). Justin says there are only 0 or 1 possible fold lines, where actually it is possible to have up to 2 fold lines. This case happens when the line to be superposed with itself is a tangent to the parabola formed by the point and the other line.

In our paper, we will use line superposition to itself instead of the term ‘perpendicular’. However, we do not consider the given line itself as a possible fold line since it means that we are constructing a line that already exists.

2.2 The Proposed Axiom Set

From the previous discussion, we see a need to specify good combination of points and lines, such that the definition of fold line is either unique, or the number of possible fold lines is finite so that we can choose one among them. We call the cases where the configurations of points and lines give rise to infinite possible fold lines as degenerate cases. When these configurations allow the usage of simpler axioms, we call them special cases. We modify the original folding statements to obtain an axiom set that eliminates both degenerate and special cases. We will use (Oi) to distinguish the new axioms from the original set of operations (i). Having a set of points and lines on origami, the axioms are:

(O1) Given two distinct points $P$ and $Q$, we can fold along a unique line that passes through $P$ and $Q$.

(O2) Given two distinct points $P$ and $Q$, we can fold along a unique line to superpose $P$ and $Q$.  

1There is a mistake in the original paper which shows $m \cap n = \emptyset$ instead of $m \cap n \neq \emptyset$. 

Figure 1: Infinite fold lines in (6) to bring $P$ to $m$ and $Q$ to $n$
3. ORIGAMI AND COMPASS

In the same paper [9], Justin hinted that it is possible to fold a point onto a circle, which should allow solving equations of degree 4. However, he did not include a thorough discussion for his proposition. In 2001, Edwards and Shurman showed how to solve reduced quartic equations by origami folds that involved bringing a point onto a circle [2]. Their work was focused on solving certain equations by finding the common tangents of a parabola and a circle, rather than defining origami axioms with circles. According to our knowledge, this is the first formal discussion to introduce compass into origami and formalize folding axioms involving circles. By introducing compass into origami operations, we allow the construction of the circle \( D \), centered at point \( C \) and having a radius \( r \), which equals the distance between two points on origami.

### 3.1 Possible Alignments

An origami operation \( o \) is possible if and only if it does not reduce the DOF of the fold line. To find the fold line, we need to specify conditions that constrain these two DOF. These constraints are the superposition relations among the points and lines of origami, or what Alperin and Lang call alignments in their proof of the enumeration completeness of the seven origami operations [1, 10]. Once we have circles on origami paper, we can think of the following additional alignments:

**Aligning a point and a circle**

The only possible alignment of a point and a circle is \( P \leftrightarrow D \), where the bidirectional arrow denotes a symmetric superposition relation. Aligning a point and a circle means to fold to bring the point to another point on the circumference of the circle, or vice versa. Bringing a point inside or outside the circle does not reduce the DOF of the fold line. To understand this, let the fold line pass through another fixed point to limit its y-intercept degree of freedom. The only freedom left for the line is its slope. There are infinite slope values that can bring a point inside or outside the circle. However, to bring the point onto the circumference of the circle, only finite number of slope values can do this (actually only 2).

**Aligning a line and a circle**

A line can either intersect a circle at two points, be a tangent to it, or has no intersection at all. The only alignment between a line and a circle which reduces the DOF of the fold line is being tangent to it. From practical point of view, it is arguable whether such alignment can be accurately performed by hands or not. We believe there is no precise method to observe, while folding, when a given line or the fold line itself is tangent to a given circle. By comparing such alignment with the capability of observing when a point is on a line or on a circle, we decide that such alignment in origami folding is discarded.

**Aligning two circles together**

Two circles can either intersect at two points, be tangent to each other, coincide with each other, or have no intersection at all. The first and last cases are not considered as alignments since they do not reduce the DOF of the fold line. We also discard the second case since it is imprecise to observe when two circles are tangent to each other. Furthermore, coinciding two circles is only possible when they have the same radius. In that case, it is clear that this alignment is equivalent to aligning the two center points of the circles together, which is already a known alignment in origami axioms without the use of circles.

**Fold lines**

From the previous discussion, we conclude that the only allowed alignment with the introduction of circles is bringing a point onto a circle.

**Theorem 1.** The fold line that brings a point onto a circle is a tangent to a conic section, which is an ellipse when the point is inside the circle, and a hyperbola when the point is outside.
**Figure 4: Fold lines that bring a point onto a circle are tangents to an ellipse or a hyperbola**

**Proof.** In Fig. 4(a), we have a circle at center C with radius \( r > 0 \), and a point \( P \) inside the circle. We want to superpose \( P \) and another arbitrary point \( Q \) on the circle. The fold line that does this is the perpendicular bisector of the segment \( PQ \). Let line \( \overline{CQ} \) intersect the fold line at point \( E \). Such point always exists since angle \( \angle PQC \) is always smaller than 90°. We have \( |EP| = |EQ| \) because \( E \) belongs to the fold line. Since \( P \) in inside the circle, \( E \) always lies between \( C \) and \( Q \), and therefore \( |EP| + |EC| = |EQ| + |EC| = r \). Wherever \( Q \) is on the circle, \( |EP| + |EC| \) is constant, and thus \( E \) fits the definition of an ellipse with \( P \) and \( C \) as foci. To prove that the fold line is tangent to that ellipse, we can draw a perpendicular to the fold line at \( E \), and easily show that it bisects angle \( \angle PEC \) equally, which is a known property of ellipse tangents.

Similarly in Fig. 4(b), we have the point \( P \) outside the circle and we distinguish two cases. First, when \( \overline{CQ} \) intersects the fold line at point \( E \), we have \( |EP| = |EQ| \). Since \( P \) is outside the circle, \( C \) always lies between \( Q \) and \( E \), and therefore \( |EP| + |EC| = |EQ| + |EC| = r \). Wherever \( Q \) is on the circle, \( |EP| + |EC| \) is constant, and thus \( E \) fits the definition of a hyperbola with \( P \) and \( C \) as foci. It is clear that the fold line is a bisector of the angle \( \angle PEC \), and therefore it bisects the angle formed with the two foci \( P \) and \( C \), which is a known property of hyperbola tangents. The second case is when the fold line does not intersect \( \overline{CQ} \). In that case, the fold line is one of the hyperbola asymptotes, which are tangent lines at infinity.

**Remark:** If the point is already on the circle, then all lines passing through it, in addition to all lines passing through the circle’s center, keep the point on the circle.

**Corollary 1.** The general equation of the conic section is:

\[
\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1
\]

where \( 2c = |PC| \) and \( 2a = r \) the radius of the circle.

**Proof.** The proof can be easily deduced if we choose a Cartesian coordinate system centered at the middle point of segment \( PC \), and make the X-axis pass through \( P \).

### 3.2 Additional Axioms

In addition to the following ones, the previous origami axioms constitute Origami-and-Compass Axioms (OCA):

- **(O8)** Given two points \( P \) and \( Q \) and a circle \( D \), where \( P \) and \( Q \) are not on \( D \) and \( m \) respectively, either we can fold along finite number of lines to superpose \( P \) and \( D \), and \( Q \) and \( m \) simultaneously, or such fold is impossible.

- **(O9)** Given two points \( P \) and \( Q \), a line \( m \) and a circle \( D \), where \( P \) and \( Q \) are not on \( D \) and \( m \) respectively, either we can fold along finite number of lines to superpose \( P \) and \( D \), and \( Q \) and \( m \) simultaneously, or such fold is impossible.

- **(O10)** Given a point \( P \), a line \( m \) and a circle \( D \), where \( P \) is not on \( D \), either we can fold along finite number of lines to superpose \( P \) and \( D \), and \( m \) onto itself simultaneously, or such fold is impossible.

For each of the new axioms, Fig. 5 shows the maximum number of possible fold lines, and distinguishes the two positions of point \( P \) inside or outside the circle in each axiom.

The fold line in (O8) is a tangent to the conic section (ellipse or hyperbola) that passes through \( Q \). The fold is impossible if \( Q \) is inside the ellipse or within one branch of the hyperbola. The maximum number of possible fold lines is 2. The fold line in (O9) is a common tangent to the conic section and the parabola formed by \( Q \) and \( m \). If the parabola lies completely within one branch of the hyperbola, for example, then the fold is impossible. The maximum number of common tangent lines is 4. In (O10), the fold line is a tangent to the conic section that is perpendicular to \( m \). If \( P \) is inside the circle, the fold is always possible. The fold is impossible if \( P \) is outside the circle and \( m \) is parallel to the conjugate axis of the hyperbola. The maximum number of fold lines is 2.

**Figure 5: Possible fold lines of (O8), (O9), and (O10)**

The conditions stated in (O8) and (O9) eliminate the degenerate cases shown in Fig. 6. The special cases where each axiom can be reduced to other ones are also eliminated by the given conditions. If \( P \) is on \( D \), (O8) can be reduced to (O1) to obtain the fold lines passing through \( Q \) and \( P \), and \( Q \) and \( C \).
3.3 Complete Enumeration of Combinations

To find a fold line in origami, we need to specify conditions that constrain its two degrees of freedom. This can be done either by superposing the fold line with existing points and lines, or by folding to superpose existing points and lines together in a way that exactly specifies the fold line. A superposition, or alignment, relation between points and lines is one of the following: a point with another point, a line with itself, or by folding to superpose existing points and lines together. Each of (O1) ↔ (O2) and (O3) ↔ (O4) can be expressed as (O5) to obtain the lines passing through P and C and bring Q onto m (Fig. 8 (a)). If Q is on m, (O9) can be expressed as (O8) to obtain the lines passing through P and Q and bring P onto D, and as (O10) to obtain the lines that superpose m with itself and bring P onto D (Fig. 8 (b)).

As discussed by Lang [10] and Alperin [1], there are alignments which produce single equation, and alignments which produce two equations. To determine a fold line we need exactly two equations corresponding to the two DOF of the fold line. Each of P1 ↔ P2 and L1 ↔ L2 by itself imposes two equations, and is enough to specify the fold line when P1 ≠ P2 and L1 ≠ L2, which correspond to axioms (O2) and (O3) respectively. Aligning a line with itself, L ↔ L, imposes a single equation, similarly to aligning a point with itself, P ↔ P.

We follow the same way and add the alignments resulting from introducing a circle, that is aligning a point with a circle, P ↔ C. This alignment generates only one equation, and it is the only alignment we allow with circles as discussed earlier.

Therefore, we will take pairs of single-equation alignments to get the necessary two equations to fully specify the fold line. Table 1 shows a table of all possible pairs.

<table>
<thead>
<tr>
<th></th>
<th>L1 ↔ L1</th>
<th>P1 ↔ L1</th>
<th>P1 ↔ P1</th>
<th>P1 ↔ D1</th>
</tr>
</thead>
<tbody>
<tr>
<td>L2 ↔ L2</td>
<td>N/A</td>
<td>O7</td>
<td>O4</td>
<td>O10</td>
</tr>
<tr>
<td>P2 ↔ L2</td>
<td>O7</td>
<td>O6</td>
<td>O5</td>
<td>O9</td>
</tr>
<tr>
<td>P2 ↔ P2</td>
<td>O4</td>
<td>O5</td>
<td>O1</td>
<td>O8</td>
</tr>
<tr>
<td>P2 ↔ D2</td>
<td>O10</td>
<td>O9</td>
<td>O8</td>
<td>N/A</td>
</tr>
</tbody>
</table>

Table 1: All possible pairs of single-equation alignments

We note that the combination that folds two different lines onto themselves cannot be part of a valid axiom, since it will result either in infinite number of fold lines if the two lines are parallel, or impossible fold operation otherwise.

As we can see form the table, all alignments to specify the fold line using points, lines, and circles are included in the extended origami axiom set OCA.

4. APPLICATIONS

Origami axioms are capable of solving cubic equations, and we know from algebra how to reduce equation of degree 4 to equations of lower degree. However, translating the algebraic reduction into origami folding operation is unappealing and geometrically tedious. One application of introducing circles to origami axioms is an elegant way of solving degree 4 equations that is presented by Edwards and Shurman [2].

Another nice application of using circles in origami is a simple method for trisecting an arbitrary angle, which is one of the famous impossibilities in classical geometry.

4.1 Angle Trisection

Tsune Abe [3] presented a method to trisect an acute angle using origami, by constructing intermediate parallel and perpendicular lines to one of the sides of the angle, and then performing (O6) fold to obtain the trisectors. The method requires four folding steps. Justin [9] proposed another method for trisecting an arbitrary angle (acute or obtuse) using (O6). The method requires constructing intermediate point and line, and one trisector can be achieved in three steps.

Alperin and Lang introduced the multi-fold method that can be used to solve cubic (or higher degree) equations, and remarked a method for trisecting an angle by two-fold operations [1]. However, it is arguable whether paper folding alone more than one fold line simultaneously is accurately achievable by hands.
With origami and compass

We present here another method for angle trisection which involves using a circle. In Fig. 9, we want to trisect the angle $\angle PCX$. We can do so by drawing a circle centered at $C$ and passing through $P$, with a radius $r = |PC|$. We extend $CX$ and apply (O9) to bring its circumference, and the center $C$ to the constructed circle, simultaneously. The fold line is a common tangent to the parabola formed by $P$ and $CX$, and the circle centered at $C$ with radius $r/2$.

**Theorem 2.** Angle $\angle PP'C$ equals one third of $\angle PCX$

**Proof.** By the symmetry of fold operation, we have $|PC| = |PC'| = r$ and $\angle CPD = \angle C'P'D$. We also have $|CD| = |CC'| = r \Rightarrow \triangle PCD$ is isosceles, and therefore $\angle CDP = \angle CPD = \angle C'P'D$. From this we can conclude that $\overline{CD} \parallel \overline{CP'}$, and that $\angle CDP'C$ is a rhombus. Therefore $|DC| = |DP'|$ and $\angle CP'D = \angle C'P'C$. This means that: $\angle PP'C = 1/2\angle PP'C = 1/2\angle CP'P''$.

On the other hand, from the triangle $\triangle PP'$ we have: $180^\circ = \angle PCC' + (\angle CPP' + \angle PP'C) = \angle PCC' + 3\angle PP'C$. We know that $\angle PCX = 180^\circ - \angle PCX$, and therefore prove that $\angle PCX = 3\angle PP'C$.

Compared with Abe’s method which does not use compass, the proposed method is simpler and does not require intermediate folding steps. However, Abe’s method has the nice feature of obtaining the two trisectors of the angle inside the angle itself. In our method, we obtain the trisection on one extended side of the given angle, and we need further folding steps if we want to construct the trisector inside the angle (two (O4) folds for example).

### 4.2 Implementation

The authors have participated in developing the computational origami system Eos (E-origami System) [8], which has capabilities of symbolic and numeric constraint solving, visualization of origami constructions, and assists in proving origami geometric theorems using algebraic theorem proving methods. Folding is the main operation in origami construction programs of Eos. Folding methods, i.e. origami axioms, are expressed as formulae using elements of a many-sorted first order logic language [4, 7]. An algebraic interpretation of these formulae gives rise to polynomial relations. The problem of finding particular fold lines of a folding method centers around solving these polynomial relations. The language was extended to include circle type as well as additional predicates required for describing the properties of the new axioms. The automated proof of the proposed angle trisection method was successfully checked using Eos.

Eos system also has a web interface, called WraEos, that allows users to dynamically construct origamis using a web browser, and several interesting origami examples are available for the users to fold step by step. The system provides interactive interface to experiment with the new axioms, and angle trisection examples using Abe’s method and the circle method are also available. WraEos can be accessed at the following URL webeos.score.cs.tsukuba.ac.jp.

### 5. CONCLUSION AND FUTURE WORK

We have presented a review of the origami folding operations, with axioms extension that allows the use of compass in origami folding. Introducing circles into origami allows solving equations of degree 4, and finding the 4 common tangents to pairs of conic sections. Since degree 4 equations can be reduced to cubic ones, this extension does not increase origami construction power outside the field of constructible numbers using the famous Huzita-Justin operations.

However, this extension is useful in finding elegant construction steps for interesting problems. We think it can be used to find nicer ways of constructing regular polygons using origami and compass, since we know that all the vertices of regular polygons lie on their inscribing circles.

### 6. REFERENCES


