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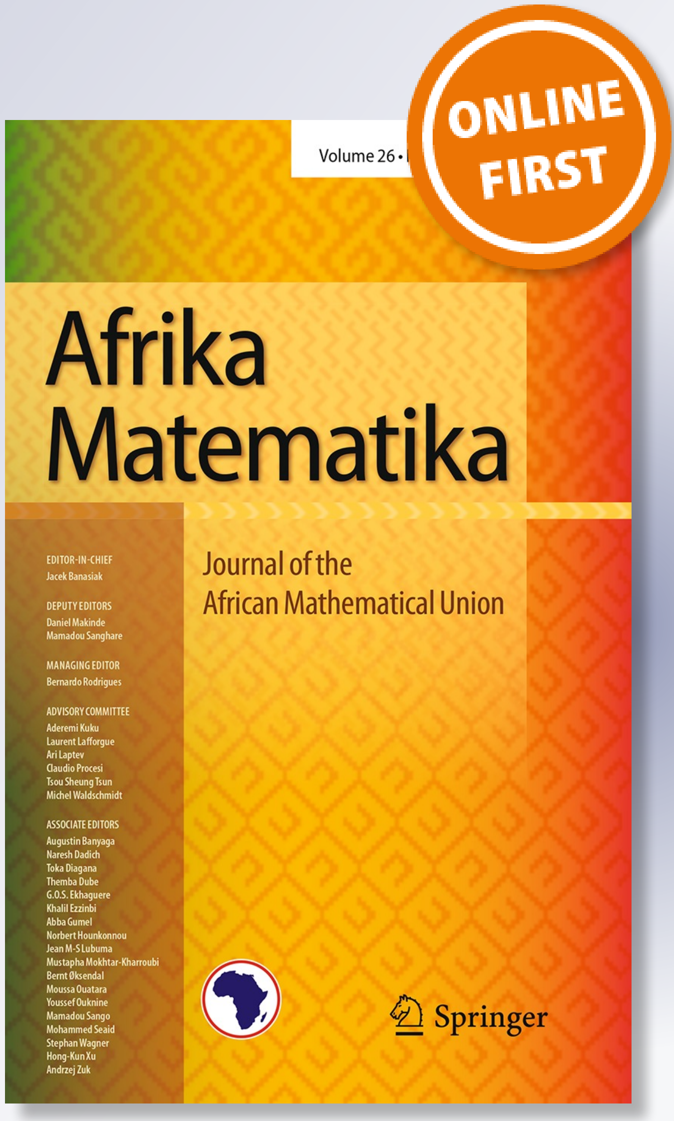
Afrika Matematika

ISSN 1012-9405

Afr. Mat.
DOI 10.1007/s13370-015-0330-y



Springer

Afr. Mat.
DOI 10.1007/s13370-015-0330-y

Triangular norm based fuzzy BG-algebras

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Received: 6 February 2014 / Accepted: 13 April 2015

Abstract In this paper, using t-norm T, the notion of (imaginable) T-fuzzy subalgebras and (imaginable) T-fuzzy closed ideals of BG-algebras are introduced and investigated their related results. Relationship between an imaginable T-fuzzy subalgebra and an imaginable T-fuzzy closed ideal are given. The direct product and T-product of T-fuzzy subalgebras of BG-algebras are discussed.

Keywords BG-algebra · T-norm · T-fuzzy BG-subalgebra · T-fuzzy closed ideal

Mathematics Subject Classification 06F35 · 03G25 · 94D05

1 Introduction

Triangular norms (t-norms for short) were introduced by Schweizer and Sklar in [23,24], following some ideas of Menger in the context of probabilistic metric spaces [17] (as statistical metric spaces were called after 1964). With the development of t-norms in statistical metric spaces, they also play an important role in decision making, in statistics as well as in the theories of cooperative games. In particular, in fuzzy set theory, t-norms have been widely used for fuzzy operations, fuzzy logic and fuzzy relation equations [25]. In recent years, a

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Published online: 01 May 2015

systematic study concerning the properties and related matters of t-norms have been made by Klement et al. [11,12].

BCK-algebras and BCI-algebras are two important classes of logical algebras introduced by Imai and Iseki. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebra. Xi [35] and Jun [7] applied the concept of fuzzy sets to BCK-algebras and BCI-algebras respectively. After that Jun, Meng, Liu and several researchers investigated further properties of fuzzy subalgebras and ideals in BCK/BCI-algebras (see [4,6,8,9,13–16,22,27,34]). Neggers and Kim [19] introduced a new notion, called a B-algebras which is related to several classes of algebras of interest such as BCI/BCK-algebras. Senapati et al. [26] introduced fuzzy dot subalgebras and fuzzy dot ideals of B-algebras. Kim and Kim [10] introduced the notion of BG-algebras, which is a generalization of B-algebras. Ahn and Lee [1] fuzzified BG-algebras. Borumand Saeid [21] introduced fuzzy topological BG-algebras. The author (together with colleagues) presented the concept and basic properties of bipolar fuzzy, fuzzy dot, intuitionistic fuzzy, intuitionistic L-fuzzy and interval-valued intuitionistic fuzzy BG-algebras [2,28–33].

The objective of this paper is to introduce the concept of (imaginable) triangular norm to subalgebras and closed ideals of BG-algebras. We prove that if every T-fuzzy closed ideals has the finite image, then every descending chain of closed ideals terminates at finite step. In addition to it we observe that every ascending chain of closed ideals terminates at finite step if and only if the set of values of any T-fuzzy closed ideals is a well ordered subset of [0, 1].

The remainder of this article is structured as follows: Sect. 2 proceeds with a recapitulation of all required definitions and properties. In Sect. 3, concepts and operations of T-fuzzy subalgebras of BG-algebras are proposed and discussed their properties in details. In Sect. 4, T-fuzzy closed ideals of BG-algebras are introduced are investigated their properties. In Sect. 5, direct product and T-product of T-fuzzy subalgebras of BG-algebras are introduced. Finally, in Sect. 6, conclusion and scope for future research are given.

2 Preliminaries

In this section, some elementary aspects that are necessary for the main part of the paper are included.

Definition 1 [10] (*BG-algebra*) A non-empty set X with a constant 0 and a binary operation $*$ is called a BG-algebra if it satisfies the following axioms

1. $x * x = 0$
2. $x * 0 = x$
3. $(x * y) * (0 * y) = x$, for all $x, y \in X$.

A non-empty subset S of a BG-algebra X is called a BG-subalgebra [10] of X if $x * y \in S$, for all $x, y \in S$. A nonempty subset I of X is called an ideal of X if it satisfies (I_1) $0 \in I$, (I_2) $x * y \in I$ and $y \in I \Rightarrow x \in I$. An ideal I is called closed of X if $0 * x \in I$, for all $x \in I$. A mapping $f : X \to Y$ of BG-algebras is called a BG-homomorphism [10] if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a BG-homomorphism, then $f(0) = 0$. We can define a partial ordering "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

We now review some fuzzy logic concepts as follows:

Let X be the collection of objects denoted generally by x then a fuzzy set [36] A in X is defined as $A = \{< x, \alpha_A(x) >: x \in X\}$ where $\alpha_A(x)$ is called the membership value of x in A and $0 \leq \alpha_A(x) \leq 1$. For any fuzzy sets A and B of a set X, we define $A \cap B = \min\{\alpha_A(x), \alpha_B(x)\}$for all $x \in X$.

Let A be a fuzzy set in X. For $\tilde{s} \in [0, 1]$, the set $U(\alpha_A : \tilde{s}) = \{x \in X : \alpha_A(x) \geq \tilde{s}\}$ is called an upper $\tilde{s}$-level [3] of A.

Let f be a mapping from the set X into the set Y. Let B be a fuzzy set in Y. Then the inverse image [20] of B, denoted by $f^{-1}(B)$ in X and is given by $f^{-1}(\alpha_B)(x) = \alpha_B(f(x))$.

Conversely, let A be a fuzzy set in X with membership function α_A. Then the image [20] of A, denoted by $f(A)$ in Y and is given by

$$\alpha_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \alpha_A(x), & \text{if } f^{-1}(y) \neq \phi \\ 1, & \text{otherwise.} \end{cases}$$

Combined the definitions of BG-subalgebra over crisp set and the idea of fuzzy set Ahn and Lee [1] defined fuzzy BG-subalgebra, which is defined below.

Definition 2 [1] (*Fuzzy BG-subalgebra*) Let α be a fuzzy set in a BG-algebra X. Then α is called a fuzzy BG-subalgebra of X if $\alpha(x * y) \geq \min\{\alpha(x), \alpha(y)\}$ for all $x, y \in X$, where $\alpha(x)$ is the membership value of x in X.

Definition 3 [17] A triangular norm (t-norm) is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies:

(T1) boundary condition: $T(x, 1) = x$;
(T2) commutativity: $T(x, y) = T(y, x)$;
(T3) associativity: $T(x, T(y, z)) = T(T(x, y), z)$;
(T4) monotonicity: $T(x, y) \leq T(x, z)$ whenever $y \leq z$,

for all $x, y, z \in [0, 1]$.

Some example [12] of t-norms are the minimum $T_M(x, y) = \min(x, y)$, the product $T_P(x, y) = x.y$ and the Lukasiewicz t-norm $T_L(x, y) = \max(x + y - 1, 0)$ for all $x, y \in [0, 1]$. Also, it is well known [5,11] that if T is a t-norm, then $T(x, y) \leq \min\{x, y\}$ for all $x, y \in [0, 1]$.

Definition 4 Let P be a t-norm. Denote by $\triangle_P$ the set of elements $x \in [0, 1]$ such that $P(x, x) = x$, that is, $\triangle_P = \{x \in [0, 1] : P(x, x) = x\}$.

A fuzzy set A in X is said to satisfy imaginable property with respect to P if $Im(\alpha_A) \subseteq \triangle_P$.

3 T-fuzzy BG-subalgebras of BG-algebras

In this section, T-fuzzy BG-subalgebras of BG-algebras are defined and some important properties are presented. In what follows, let X denote a BG-algebra unless otherwise specified.

Definition 5 Let A be a fuzzy set in X. Then the set A is a T-fuzzy BG-subalgebra over the binary operator $*$ if it satisfies

$$\alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\} \quad \text{for all } x, y \in X. \quad (F1)$$

Let us illustrate this definition using the following examples.

Example 1 Let $X = \{0, a, b, c, d, e\}$ be a BG-algebra with the following Cayley table:

$*$	0	a	b	c	d	e
0	0	b	a	c	d	e
a	a	0	b	d	e	c
b	b	a	0	e	c	d
c	c	d	e	0	b	a
d	d	e	c	a	0	b
e	e	c	d	b	a	0

Let $T_m : [0, 1] \times [0, 1] \to [0, 1]$ be functions defined by $T_m(x, y) = \max(x + y - 1, 0)$ for all $x, y \in [0, 1]$. Then T_m is a t-norm. Define a fuzzy set A in X by $\alpha_A(0) = 0.75$, $\alpha_A(a) = \alpha_A(b) = 0.64$ and $\alpha_A(x) = 0.38$ for all $x \in X \setminus \{0, a, b\}$. Then A is a T_m-normed fuzzy BG-subalgebra of X.

Definition 6 A T-fuzzy BG-subalgebra A is called an imaginable T-fuzzy BG-subalgebra of X if α_A satisfy the imaginable property with respect to T.

Example 2 Consider T_m be a t-norm and X={0, a, b, c, d, e} be a BG-algebra in Example 3. Define a fuzzy set A in X by $\alpha_A(x) = 1$, if $x \in \{0, a, b\}$ and $\alpha_A(x) = 0$, if $x \in X \setminus \{0, a, b\}$. It is easy to check that $\alpha_A(x * y) \geq T_m\{\alpha_A(x), \alpha_A(y)\}$ for all $x, y \in X$. Also, $Im(\alpha_A) \subseteq \triangle_{T_m}$. Hence, A is an imaginable T_m-fuzzy BG-subalgebra of X.

The following propositions are obvious.

Proposition 1 *If A is an imaginable T-fuzzy BG-subalgebra of X, then $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$.*

Proposition 2 *Let A be an imaginable T-fuzzy BG-subalgebra of X and let $n \in \mathbb{N}$ (the set of natural numbers). Then*

(i) $\alpha_A\left(\prod^n x * x\right) \geq \alpha_A(x)$, *for any odd number n,*

(ii) $\alpha_A\left(\prod^n x * x\right) = \alpha_A(x)$, *for any even number n.*

Proposition 3 *If A is an imaginable T-fuzzy BG-subalgebra of X, then $\alpha_A(0 * x) \geq \alpha_A(x)$ for all $x \in X$.*

Theorem 1 *Let A be an imaginable T-fuzzy BG-subalgebra of X. If there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \alpha_A(x_n) = 1$ then $\alpha_A(0) = 1$.*

Proof By Proposition 1, $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$, therefore, $\alpha_A(0) \geq \alpha_A(x_n)$ for every positive integer n. Consider, $1 \geq \alpha_A(0) \geq \lim_{n\to\infty} \alpha_A(x_n) = 1$. Hence, $\alpha_A(0) = 1$. □

The intersection of any two T-fuzzy BG-subalgebras is also a T-fuzzy BG-subalgebra, which is proved in the following theorem.

Theorem 2 *Let A_1 and A_2 be two T-fuzzy BG-subalgebras of X. Then $A_1 \cap A_2$ is a T-fuzzy BG-subalgebra of X.*

Proof Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2. Now,

$$\begin{aligned}\alpha_{A_1 \cap A_2}(x * y) &= \min\left\{\alpha_{A_1}(x * y), \alpha_{A_2}(x * y)\right\}, \\ &\geq \min\left\{T\left\{\alpha_{A_1}(x), \alpha_{A_1}(y)\right\}, T\left\{\alpha_{A_2}(x), \alpha_{A_2}(y)\right\}\right\} \\ &\geq T\left\{\min\left\{\alpha_{A_1}(x), \alpha_{A_2}(x)\right\}, \min\left\{\alpha_{A_1}(y), \alpha_{A_2}(y)\right\}\right\} \\ &= T\left\{\alpha_{A_1 \cap A_2}(x), \alpha_{A_1 \cap A_2}(y)\right\}\end{aligned}$$

Hence, $A_1 \cap A_2$ is a T-fuzzy BG-subalgebra of X. □

The above theorem can be generalized as follows.

Corollary 1 *Let* $\{A_i : i = 1, 2, 3, 4, \ldots\}$ *be a family of* T*-fuzzy* BG*-subalgebras of* X*. Then* $\bigcap A_i$ *is also a* T*-fuzzy* BG*-subalgebra of* X*, where* $\bigcap A_i = \{< x, \min \alpha_{A_i}(x) >: x \in X\}$.

The set $\{x \in X : \alpha_A(x) = \alpha_A(0)\}$ is denoted by I_{α_A}.

Theorem 3 *Let* A *be an imaginable* T*-fuzzy* BG*-subalgebra of* X*. Then the set* I_{α_A} *is a* BG*-subalgebra of* X*.*

Proof Let $x, y \in I_{\alpha_A}$. Then $\alpha_A(x) = \alpha_A(0) = \alpha_A(y)$ and so, $\alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\} = T\{\alpha_A(0), \alpha_A(0)\} = \alpha_A(0)$. By using Proposition 1, we know that $\alpha_A(x * y) \leq \alpha_A(0)$. Hence, $\alpha_A(x * y) = \alpha_A(0)$ or equivalently $x * y \in I_{\alpha_A}$. Therefore, the set I_{α_A} is a BG-subalgebra of X. □

As is well known, the characteristic function of a set is a special fuzzy set. Suppose A is a non-empty subset of X. By χ_A we denote the characteristic function of A, that is,

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4 *If* A *is a* BG*-subalgebra of* X*, then the characteristic function* χ_A *is a* T*-fuzzy* BG*-subalgebra of* X*.*

Proof Let $x, y \in X$. We consider here four cases:
Case i If $x, y \in A$ then $x * y \in A$ since A is a BG-subalgebra of X. Then

$$\chi_A(x * y) = 1 \geq T\{\chi_A(x), \chi_A(y)\}.$$

Case ii If $x, y \notin A$, then $\chi_A(x) = 0 = \chi_A(y)$. Thus

$$\chi_A(x * y) \geq 0 = \min\{0, 0\} \geq T\{0, 0\} = T\{\chi_A(x), \chi_A(y)\}.$$

Case iii If $x \in A$ and $y \notin A$ then $\chi_A(x) = 1$, $\chi_A(y) = 0$. Thus

$$\chi_A(x * y) \geq 0 = T\{0, 1\} = T\{1, 0\} = T\{\chi_A(x), \chi_A(y)\}.$$

Case iv If $x \notin A$ and $y \in A$ then by the same argument as in Case (iii), we conclude that $\chi_A(x * y) \geq T\{\chi_A(x), \chi_A(y)\}$.

Therefore, the characteristic function χ_A is a T-fuzzy BG-subalgebra of X. □

Theorem 5 *Let* A *be a non-empty subset of* X*. If* χ_A *satisfies* $(F1)$*, then* A *is a* BG*-subalgebra of* X*.*

Proof Suppose that χ_A satisfy $(F1)$. Let $x, y \in A$. Then it follows from $(F1)$ that $\chi_A(x*y) \geq T\{\chi_A(x), \chi_A(y)\} = T\{1, 1\} = 1$ so that $\chi_A(x * y) = 1$, i.e., $x * y \in A$. Hence, A is a BG-subalgebra of X. □

Theorem 6 *Let P be a BG-subalgebra of X and A be a fuzzy set in X defined by*

$$\alpha_A(x) = \begin{cases} \lambda, & \text{if } x \in P \\ \tau, & \text{otherwise} \end{cases}$$

for all $\lambda, \tau \in [0, 1]$ with $\lambda \geq \tau$. Then A is a T_m-fuzzy BG-subalgebra of X, where T_m is the t-norm in Example 3. In particular if $\lambda = 1$ and $\tau = 0$ then A is an imaginable T_m-fuzzy BG-subalgebra of X. Moreover, $I_{\alpha_A} = P$.

Proof Let $x, y \in X$. We consider here three cases:

Case i If $x, y \in P$ then $T_m(\alpha_A(x), \alpha_A(y)) = T_m(\lambda, \lambda) = \max(2\lambda - 1, 0) = \begin{cases} 2\lambda - 1 & \text{if } \lambda \geq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \leq \lambda = \alpha_A(x * y)$.

Case ii If $x \in P$ and $y \notin P$ (or, $x \notin P$ and $y \in P$) then $T_m(\alpha_A(x), \alpha_A(y)) = T_m(\lambda, \tau) = \max(\lambda + \tau - 1, 0) = \begin{cases} \lambda + \tau - 1 & \text{if } \lambda + \tau \geq 1 \\ 0 & \text{otherwise} \end{cases} \leq \tau = \alpha_A(x * y)$.

Case iii If $x, y \notin P$ then $T_m(\alpha_A(x), \alpha_A(y)) = T_m(\tau, \tau) = \max(2\tau - 1, 0) = \begin{cases} 2\tau - 1 & \text{if } \tau \geq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \leq \tau = \alpha_A(x * y)$.

Hence, A is a T_m-fuzzy BG-subalgebra of X.

Assume that $\lambda = 1$ and $\tau = 0$. Then $T_m(\lambda, \lambda) = \max(\lambda + \lambda - 1, 0) = 1 = \lambda$ and $T_m(\tau, \tau) = \max(\tau + \tau - 1, 0) = 0 = \tau$. Thus $\lambda, \tau \in \triangle_{T_m}$ i.e., $Im(\alpha_A) \subseteq \triangle_{T_m}$. So, A is an imaginable T_m-fuzzy BG-subalgebra of X.

Also, $I_{\alpha_A} = \{x \in X, \alpha_A(x) = \alpha_A(0)\} = \{x \in X, \alpha_A(x) = 1\} = P$. Therefore, $I_{\alpha_A} = P$. □

Theorem 7 *Let A be a T-fuzzy BG-subalgebra of X and $\tilde{s} \in [0, 1]$. Then if $\tilde{s} = 1$, the upper level set $U(\alpha_A : \tilde{s})$ is either empty or a BG-subalgebra of X.*

Proof Let $\tilde{s} = 1$ and $x, y \in U(\alpha_A : \tilde{s})$. Then $\alpha_A(x) \geq \tilde{s} = 1$ and $\alpha_A(y) \geq \tilde{s} = 1$. It follows that $\alpha_A(x * y) \geq T(\alpha_A(x), \alpha_A(y)) \geq T(1, 1) = 1$ so that $x * y \in U(\alpha_A : \tilde{s})$. Hence, $U(\alpha_A : \tilde{s})$ is a BG-subalgebra of X when $s = 1$. □

Theorem 8 *If A is an imaginable T-fuzzy BG-subalgebra of X, then the upper $\tilde{s}$-level of A is BG-subalgebra of X.*

Proof Assume that $x, y \in U(\alpha_A : \tilde{s})$. Then $\alpha_A(x) \geq \tilde{s}$ and $\alpha_A(y) \geq \tilde{s}$. It follows that $\alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\} \geq T(\tilde{s}, \tilde{s}) = \tilde{s}$ so that $x * y \in U(\alpha_A : \tilde{s})$. Hence, $U(\alpha_A : \tilde{s})$ is a BG-subalgebra of X. □

Theorem 9 *Let A be a fuzzy set in X such that the set $U(\alpha_A : \tilde{s})$ is a BG-subalgebra of X for every $\tilde{s} \in [0, 1]$. Then A is a T-fuzzy BG-subalgebra of X.*

Proof Let for every $\tilde{s} \in [0, 1]$, $U(\alpha_A : \tilde{s})$ is subalgebra of X. In contrary, let $x_0, y_0 \in X$ be such that $\alpha_A(x_0 * y_0) < T\{\alpha_A(x_0), \alpha_A(y_0)\}$. Let us consider,

$$\tilde{s}_0 = \frac{1}{2}\Big[\alpha_A(x_0 * y_0) + T\{\alpha_A(x_0), \alpha_A(y_0)\}\Big].$$

Then $\alpha_A(x_0 * y_0) < \tilde{s}_0 \leq T\{\alpha_A(x_0), \alpha_A(y_0)\} \leq \min\{\alpha_A(x_0), \alpha_A(y_0)\}$ and so $x_0 * y_0 \notin U(\alpha_A : \tilde{s})$ but $x_0, y_0 \in U(\alpha_A : \tilde{s})$. This is a contradiction and hence α_A satisfies the inequality $\alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\}$ for all $x, y \in X$. □

Theorem 10 *Let $f : X \to Y$ be a homomorphism of BG-algebras. If $B = \{< x, \alpha_B(x) >: x \in Y\}$ is a T-fuzzy BG-subalgebra of Y, then the pre-image $f^{-1}(B) = \{< x, f^{-1}(\alpha_B)(x) >: x \in X\}$ of B under f is a T-fuzzy BG-subalgebra of X.*

Proof Assume that B is a T-fuzzy BG-subalgebra of Y and let $x, y \in X$. Then $f^{-1}(\alpha_B)(x * y) = \alpha_B(f(x * y)) = \alpha_B(f(x) * f(y)) \geq T\{\alpha_B(f(x), \alpha_B(f(y))\} = T\{f^{-1}(\alpha_B)(x), f^{-1}(\alpha_B)(y)\}$. Therefore, $f^{-1}(B)$ is a T-fuzzy BG-subalgebra of X. □

Theorem 11 [21] *Let $f : X \to Y$ be a homomorphism from a BG-algebra X onto a BG-algebra Y. If $A = \{< x, \alpha_A(x) >: x \in X\}$ is a fuzzy BG-subalgebra of X, then the image $f(A) = \{< x, f_{\sup}(\alpha_A)(x) >: x \in Y\}$ of A under f is a fuzzy BG-subalgebra of Y.*

Theorem 12 *Let $f : X \to Y$ be a homomorphism from a BG-algebra X onto a BG-algebra Y. If A is an imaginable T-fuzzy BG-subalgebra of X, then the image $f(A)$ of A under f is a T-fuzzy BG-subalgebra of Y.*

Proof Let A be an imaginable T-fuzzy BG-subalgebra of X. By Theorem 8, $U(\alpha_A : \tilde{s})$ is a BG-subalgebra of X for every $\tilde{s} \in [0, 1]$. Therefore, by Theorem 11, $f(U(\alpha_A : \tilde{s}))$ is a BG-subalgebra of Y. But $f(U(\alpha_A : \tilde{s})) = U(f(\alpha_A) : \tilde{s})$. Hence, $U(f(\alpha_A) : \tilde{s})$ is a BG-subalgebra of Y for every $\tilde{s} \in [0, 1]$. By Theorem 9, $f(A)$ is a T-fuzzy BG-subalgebra of Y. □

4 T-fuzzy closed ideals of BG-algebras

In this section, T-fuzzy closed ideals of BG-algebras are defined and some propositions and theorems are presented.

Definition 7 A fuzzy set A in X is called a fuzzy closed ideal of X under a t-norm T (briefly, T-fuzzy closed ideal of X) if it satisfies

$$\alpha_A(x) \geq T\{\alpha_A(x * y), \alpha_A(y)\} \quad \text{for all } x, y \in X. \quad (F2)$$

$$\alpha_A(0 * x) \geq \alpha_A(x) \quad \text{for all } x \in X. \quad (F3)$$

A T-fuzzy closed ideals of X is called an imaginable T-fuzzy ideals of X if satisfy the imaginable property with respect to T.

Let us illustrate this definition using the following examples.

Example 3 Let $X = \{0, a, b, c\}$ be a BG-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	b	a	c
a	a	0	c	b
b	b	c	0	a
c	c	a	b	0

Let $T_m : [0, 1] \times [0, 1] \to [0, 1]$ be functions defined by $T_m(x, y) = \max(x + y - 1, 0)$ for all $x, y \in [0, 1]$. Then T_m is a t-norm. Define a fuzzy set A in X by $\alpha_A(0) = \alpha_A(c) = 0.7$ and $\alpha_A(a) = \alpha_A(b) = 0.4$. Then A is a T_m-normed fuzzy closed ideal of X.

Let $B = \{< x, \alpha_B(x) >: x \in X\}$ be a fuzzy set in X defined by $\alpha_B(0) = \alpha_B(c) = 1$ and $\alpha_B(x) = 0$, for all $x \in X/\{0, c\}$. Then B is an imaginable T_m-normed fuzzy closed ideal of X.

Theorem 13 *Every imaginable T-fuzzy subalgebra satisfying $(F3)$ $\alpha_A(0*x) \geq \alpha_A(x)$, for all $x \in X$, is an imaginable T-fuzzy closed ideal of X.*

Proof Using Proposition 3, it is straightforward. □

Theorem 14 *Every T-fuzzy closed ideal of X is a T-fuzzy subalgebra of X.*

Proof Let A be a T-fuzzy closed ideal of X and $x, y \in X$. Then we have $\alpha_A(0*x) \geq \alpha_A(x)$. Now

$$\begin{aligned}\alpha_A(x*y) &\geq T\{\alpha_A((x*y)*(0*y)), \alpha_A(0*y)\}, \text{ by (F2)}\\ &= T\{\alpha_A(x), \alpha_A(0*y)\}\\ &\geq T\{\alpha_A(x), \alpha_A(y)\}, \text{ by (F3)}.\end{aligned}$$

Therefore, A is a T-fuzzy subalgebra of X. □

Proposition 4 *Let A be a T-fuzzy closed ideal of X. If $x*y \leq z$, then $\alpha_A(x) \geq T\{\alpha_A(y), \alpha_A(y)\}$ for all $x, y, z \in X$.*

Proof Let $x, y, z \in X$ be such that $x*y \leq z$. Then $(x*y)*z = 0$ and thus $\alpha_A(x) \geq T\{\alpha_A(x*y), \alpha_A(y)\} \geq T\{T\{\alpha_A((x*y)*z), \alpha_A(z)\}, \alpha_A(y)\} = T\{T\{\alpha_A(0), \alpha_A(z)\}, \alpha_A(y)\} = T\{\alpha_A(z), \alpha_A(y)\} = T\{\alpha_A(y), \alpha_A(z)\}$. □

Theorem 15 *A fuzzy set A is a T-fuzzy closed ideal of X if and only if the upper $\tilde{s}$-level set $U(\alpha_A : \tilde{s})$ is a closed ideal of X for every $\tilde{s} \in [0, 1]$.*

Proof Suppose that A is a T-fuzzy closed ideal of X. Obviously, $0*x \in U(\alpha_A : \tilde{s})$, where $x \in X$. Let $x, y \in X$ be such that $x*y \in U(\alpha_A : \tilde{s})$ and $y \in U(\alpha_A : \tilde{s})$. Then $\alpha_A(x) \geq \min\{\alpha_A(x*y), \alpha_A(y)\} \geq \tilde{s}$ i.e., $x \in U(\alpha_A : \tilde{s})$. Hence, $U(\alpha_A : \tilde{s})$ is a closed ideal of X.

Conversely, assume that each non-empty level subset $U(\alpha_A : \tilde{s})$ is a closed ideals of X. For any $x \in X$, let $\alpha_A(x) = \tilde{s}$. Then $x \in U(\alpha_A : \tilde{s})$. Since $0*x \in U(\alpha_A : \tilde{s})$, it follows that $\alpha_A(0*x) \geq \tilde{s} = \alpha_A(x)$, for all $x \in X$.

If there exist $\lambda, \kappa \in X$ such that $\alpha_A(\lambda) < T\{\alpha_A(\lambda*\kappa), \alpha_A(\kappa)\}$, then by taking $s' = \frac{1}{2}\Big[\alpha_A(\lambda*\kappa) + T\{\alpha_A(\lambda), \alpha_A(\kappa)\}\Big]$, it follows that $\lambda*\kappa \in U(\alpha_A : s')$ and $\kappa \in U(\alpha_A : s')$, but $\lambda \notin U(\alpha_A : s')$, which is a contradiction. Therefore, $U(\alpha_A : s')$ is not a closed ideal of X. Hence, A is a T-fuzzy closed ideal of X. □

Theorem 16 *Let $S_1 \supseteq S_2 \supseteq S_3 \cdots$ be a descending chain of closed ideals of X which terminates at finite step. For a T-fuzzy closed ideal A of X, if a sequence of elements of $Im(\alpha_A)$ is strictly increasing, then A is finite valued.*

Proof Assume that A is infinite valued. Let $\{\psi_n\}$ be a strictly increasing sequence of elements of $Im(\alpha_A)$. Then $0 \leq \psi_1 < \psi_2 < \cdots < 1$. Note that $U(\alpha_A : \psi_t)$ is a closed ideal of X for $t = 1, 2, 3, \ldots$ Let $x \in U(\alpha_A : \psi_t)$ for $t = 2, 3, \ldots$ Then $\alpha_A(x) \geq \psi_t > \psi_{t-1}$, which implies that $x \in U(\alpha_A : \psi_{t-1})$. Hence $U(\alpha_A : \psi_t) \subseteq U(\alpha_A : \psi_{t-1})$ for $t = 2, 3, \ldots$ Since $\psi_{t-1} \in Im(\alpha_A)$ there exists x_{t-1} such that $\alpha_A(x_{t-1}) = \psi_{t-1}$. It follows that $x_{t-1} \in U(\alpha_A : \psi_{t-1})$, but $x_{t-1} \notin U(\alpha_A : \psi_t)$. Thus $U(\alpha_A : \psi_t) \subsetneq U(\alpha_A : \psi_{t-1})$, and so we obtain a strictly descending chain $U(\alpha_A : \psi_1) \supsetneq U(\alpha_A : \psi_2) \supsetneq \cdots$ of closed ideals of X which is not terminating. This is impossible. Therefore, A is finite valued. □

Now we consider the converse of Theorem 16.

Theorem 17 *If every T-fuzzy closed ideal A of X has the finite image, then every descending chain of closed ideals of X terminates at finite step.*

Proof Suppose there exists a strictly descending chain $S_0 \supsetneq S_1 \supsetneq S_2 \cdots$ of closed ideals of X which does not terminate at finite step. Define a fuzzy set A in X by

$$\alpha_A(x) = \begin{cases} \frac{n}{n+1} & \text{if} \quad x \in S_n \backslash S_{n+1} \\ 1 & \text{if} \quad x \in \bigcap\limits_{n=0}^{\infty} S_n \end{cases}$$

where $n = 0, 1, 2, \ldots$ and S_0 stands for X. Clearly, $\alpha_A(0 * x) \geq \alpha_A(x)$ for all $x \in X$. Let $x, y \in X$. Assume that $x * y \in S_n \backslash S_{n+1}$ and $y \in S_k \backslash S_{k+1}$ for $n = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots$ Without loss of generality, we may assume that $n \leq k$. Then obviously $x * y$ and $y \in S_n$, so $x \in S_n$ because S_n is a closed ideal of X. Hence,

$$\alpha_A(x) \geq \frac{n}{n+1} = T\{\alpha_A(x * y), \alpha_A(y)\}.$$

If $x * y, y \in \bigcap_{n=0}^{\infty} S_n$, then $x \in \bigcap_{n=0}^{\infty} S_n$. Thus

$$\alpha_A(x) = 1 = T\left\{\alpha_A(x * y), \alpha_A(y)\right\}.$$

If $x * y \notin \bigcap_{n=0}^{\infty} S_n$ and $y \in \bigcap_{n=0}^{\infty} S_n$, then there exists a positive integer r such that $x * y \in S_r \setminus S_{r+1}$. It follows that $x \in S_r$ so that

$$\alpha_A(x) \geq \frac{r}{r+1} = T\{\alpha_A(x * y), \alpha_A(y)\}.$$

Finally suppose that $x * y \in \bigcap_{n=0}^{\infty} S_n$ and $y \notin \bigcap_{n=0}^{\infty} S_n$. Then $y \in S_s \backslash S_{s+1}$ for some positive integer s. It follows that $x \in S_s$, and hence

$$\alpha_A(x) \geq \frac{s}{s+1} = T\{\alpha_A(x * y), \alpha_A(y)\}.$$

This proves that A is a T-fuzzy closed ideal with an infinite number of different values, which is a contradiction. This completes the proof. □

Theorem 18 *Every ascending chain of closed ideals of X terminates at finite step if and only if the set of values of any T-fuzzy closed ideals is a well ordered subset of* $[0, 1]$.

Proof Let A be a T-fuzzy closed ideals of X. Suppose that the set of values of A is not a well-ordered subset of $[0, 1]$. Then there exist a strictly decreasing sequence $\{\gamma_n\}$ such that $\alpha_A(x_n) = \gamma_n$. It follows that $U(\alpha_A : \gamma_1) \subsetneq U(\alpha_A : \gamma_2) \subsetneq U(\alpha_A : \gamma_3) \subsetneq \cdots$ is a strictly ascending chain of closed ideals of X which is not terminating. This is impossible.

To prove the converse suppose that there exist a strictly ascending chain

$$S_1 \subsetneq S_2 \subsetneq S_3 \subsetneq \cdots \tag{1}$$

of closed ideal of X which does not terminate at finite step. Note that $S = \bigcup_{n \in \mathbb{N}} S_n$ is a closed ideal of X. Define a fuzzy set A in X by

$$\alpha_A(x) = \begin{cases} \frac{1}{k} & \text{where} \quad k = \min\{n \in \mathbb{N} | x \in S_n\} \\ 0 & \text{if} \quad x \notin S_n \end{cases}$$

We claim that A is a T-fuzzy closed ideal of X. Let $x \in X$. If $x \notin S_n$, then obviously $\alpha_A(0 * x) \geq 0 = \alpha_A(x)$. If $x \in S_n \backslash S_{n-1}$ for $n = 2, 3, \ldots$, then $0 * x \in S_n$. Hence,

$\alpha_A(0*x) \geq \frac{1}{n} = \alpha_A(x)$. Let $x, y \in X$. If $x*y \in S_n \backslash S_{n-1}$ and $y \in S_n \backslash S_{n-1}$ for $n = 2, 3, \ldots$ then $x \in S_n$. It follows that

$$\alpha_A(x) \geq \frac{1}{n} = T\{\alpha_A(x*y), \alpha_A(y)\}.$$

Suppose that $x*y \in S_n$ and $y \in S_n \backslash S_p$ for all $p < n$. Since A is a closed ideal of X, then $x \in S_n$, and so $\alpha_A(x) \geq \frac{1}{n} \geq \frac{1}{p+1} \geq \alpha_A(y)$. Hence $\alpha_A(x) \geq T\{\alpha_A(x*y), \alpha_A(y)\}$.

Similarly, for the case $x*y \in S_n \backslash S_p$ and $y \in S_n$, we have $\alpha_A(x) \geq T\{\alpha_A(x*y), \alpha_A(y)\}$.

Therefore, A is a T-fuzzy closed ideal of X. Since the chain (1) is not terminating, A has a strictly descending sequence of values. This contradicts that the value set of any T-fuzzy closed ideal is well-ordered. This completes the proof. □

5 Product of T-fuzzy BG-subalgebras

In this section, the direct product and T-normed product of fuzzy BG-subalgebras of BG-algebras with respect to t-norm are presented and several properties are studied. Before going into the product of fuzzy BG-subalgebras of BG-algebras, we first define some kind of product of fuzzy subsets.

Definition 8 Let $A_1 = \{< x, \alpha_{A_1}(x) >: x \in X\}$ and $A_2 = \{< x, \alpha_{A_2}(x) >: x \in X\}$ be fuzzy subsets of X. Then the T-product of A_1 and A_2 denoted by $[A_1.A_2]_T = \{< x, [\alpha_{A_1}.\alpha_{A_2}]_T(x) >: x \in X\}$ and is defined by $[\alpha_{A_1}.\alpha_{A_2}]_T(x) = T(\alpha_{A_1}(x), \alpha_{A_2}(x))$ for all $x \in X$.

Theorem 19 *Let A_1 and A_2 be two T-fuzzy BG-subalgebras of X. If T^* is a t-norm which dominates T, i.e., $T^*(T(a,b), T(c,d)) \geq T(T^*(a,c), T^*(b,d))$ for all a, b, c and $d \in [0,1]$, then the T^*-product of A_1 and A_2, $[A_1.A_2]_{T^*}$ is a T-fuzzy BG-subalgebra of X.*

Proof For any $x, y \in X$, we have

$$\begin{aligned}
[\alpha_{A_1}.\alpha_{A_2}]_{T^*}(x*y) &= T^*(\alpha_{A_1}(x*y), \alpha_{A_2}(x*y)) \\
&\geq T^*\left(T\left(\alpha_{A_1}(x), \alpha_{A_1}(y)\right), T\left(\alpha_{A_2}(x), \alpha_{A_2}(y)\right)\right) \\
&\geq T\left(T^*\left(\alpha_{A_1}(x), \alpha_{A_2}(x)\right), T^*\left(\alpha_{A_1}(y), \alpha_{A_2}(y)\right)\right) \\
&= T\left(\left[\alpha_{A_1}.\alpha_{A_2}\right]_{T^*}(x), \left[\alpha_{A_1}.\alpha_{A_2}\right]_{T^*}(y)\right).
\end{aligned}$$

Hence, $[A_1.A_2]_{T^*}$ is a T-fuzzy BG-subalgebra of X. □

Let $f : X \to Y$ be an epimorphism of BG-algebras. Let T, T^* be t-norms such that T^* dominates T. If A_1 and A_2 be two T-fuzzy BG-subalgebras of Y, then the T^*-product of A_1 and A_2, $[A_1.A_2]_{T^*}$ is a T-fuzzy BG-subalgebra of Y. Since every epimorphic pre-image of a T-fuzzy BG-subalgebra is a T-fuzzy BG-subalgebra, the pre-images $f^{-1}(A_1)$, $f^{-1}(A_2)$ and $f^{-1}([A_1.A_2]_{T^*})$ are T-fuzzy BG-subalgebras of X. The next theorem provides the relation between $f^{-1}([A_1.A_2]_{T^*})$ and the T^*-product $[f^{-1}(A_1).f^{-1}(A_2)]_{T^*}$ of $f^{-1}(A_1)$ and $f^{-1}(A_2)$.

Theorem 20 *Let $f : X \to Y$ be an epimorphism of BG-algebras. Let T, T^* be t-norms such that T^* dominates T. Let A_1 and A_2 be two T-fuzzy BG-subalgebras of Y. If $[A_1.A_2]_{T^*}$ is the T^*-product of A_1 and A_2 and $[f^{-1}(A_1).f^{-1}(A_2)]_{T^*}$ is the T^*-product of $f^{-1}(A_1)$ and $f^{-1}(A_2)$, then*

$$f^{-1}\left(\left[\alpha_{A_1}.\alpha_{A_2}\right]_{T^*}\right) = \left[f^{-1}\left(\alpha_{A_1}\right).f^{-1}\left(\alpha_{A_2}\right)\right]_{T^*}.$$

Proof For any $x \in X$, we get, $f^{-1}([\alpha_{A_1}.\alpha_{A_2}]_{T^*})(x) = [\alpha_{A_1}.\alpha_{A_2}]_{T^*}(f(x)) = T^*(\alpha_{A_1}(f(x)), \alpha_{A_2}(f(x))) = T^*([f^{-1}(\alpha_{A_1})](x), [f^{-1}(\alpha_{A_2})](x)) = [f^{-1}(\alpha_{A_1}).\ f^{-1}(\alpha_{A_2})]_{T^*}(x)$. □

Lemma 1 [5] *Let T be t-norm. Then $T(T(x, y), T(z, t)) = T(T(x, z), T(y, t))$ for all x, y, z and $t \in [0, 1]$.*

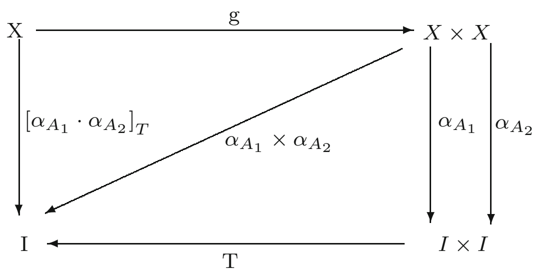
Theorem 21 *Let $X = X_1 \times X_2$ be the direct product BG-algebra of BG-algebras X_1 and X_2. If $A_1 = \{< x, \alpha_{A_1}(x) >: x \in X\}$ and $A_2 = \{< x, \alpha_{A_2}(x) >: x \in X\}$ be two T-fuzzy BG-subalgebras of X_1 and X_2 respectively, then $A = \{< x, \alpha_A(x) >: x \in X\}$ is a T-fuzzy BG-subalgebra of X defined by $\alpha_A = \alpha_{A_1} \times \alpha_{A_2}$ such that $\alpha_A(x_1, x_2) = (\alpha_{A_1} \times \alpha_{A_2})(x_1, x_2) = T(\alpha_{A_1}(x_1), \alpha_{A_2}(x_2))$ for all $(x_1, x_2) \in X_1 \times X_2$.*

Proof Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any two elements of X. Since X is a BG-algebra, we have,

$$\begin{aligned}
\alpha_A(x * y) &= \alpha_A((x_1, x_2) * (y_1, y_2)) = \alpha_A(x_1 * y_1, x_2 * y_2)\\
&= (\alpha_{A_1} \times \alpha_{A_2})(x_1 * y_1, x_2 * y_2)\\
&= T(\alpha_{A_1}(x_1 * y_1), \alpha_{A_2}(x_2 * y_2))\\
&\geq T(T(\alpha_{A_1}(x_1), \alpha_{A_1}(y_1)), T(\alpha_{A_2}(x_2), \alpha_{A_2}(y_2)))\\
&= T(T(\alpha_{A_1}(x_1), \alpha_{A_2}(x_2)), T(\alpha_{A_1}(y_1), \alpha_{A_2}(y_2)))\\
&= T((\alpha_{A_1} \times \alpha_{A_2})(x_1, x_2), (\alpha_{A_1} \times \alpha_{A_2})(y_1, y_2))\\
&= T(\alpha_A(x), \alpha_A(y)).
\end{aligned}$$

Hence, $A = \{< x, \alpha_A(x) >: x \in X\}$ is a T-fuzzy BG-subalgebra of X. □

The relationship between T-fuzzy BG-subalgebras $A_1 \times A_2$ and $[A_1 \cdot A_2]_T$ can be viewed via the following diagram where $I = [0, 1]$ and $g : X \to X \times X$



is defined by $g(x) = (x, x)$. It is not difficult to see that $[A_1 \cdot A_2]_T$ is the preimage of $A_1 \times A_2$ under g.

6 Conclusions and future work

In this paper, notion of T-fuzzy BG-subalgebras and T-fuzzy closed ideals of BG-algebras are introduced and investigated some of their useful properties. Using imaginable property, imaginable T-fuzzy BG-subalgebras and T-fuzzy closed ideals of BG-algebras has been constructed. Finally, direct products and T-products of T-fuzzy BG-subalgebras has been introduced and some important properties of it are studied.

It is our hope that this work would other foundations for further study of the theory of BG-algebras. In our future study of fuzzy structure of BG-algebra, may be the following topics should be considered: (a) to find interval-valued T-fuzzy BG-subalgebras of BG-algebras, (b) to find interval-valued T-fuzzy closed ideals of BG-algebras, (c) to find intuitionistic (T, S)-fuzzy BG-subalgebras of BG-algebras, (d) to find intuitionistic (T, S)-fuzzy closed ideals of BG-algebras, where S is a given t-conorm.

Acknowledgments The authors would like to express sincere appreciation to the reviewers for their valuable suggestions and comments helpful in improving this paper.

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