Topological Spaces and Covering Rough Sets

T. Medhat
Department of Electrical Engineering, Faculty of Engineering, Kafrelsheikh University, 33516, Kafrelsheikh, Egypt.
E-mail: tmedhatm@yahoo.com, tmedhatm@eng.kfs.edu.eg

Abstract
 Rough set theory (RST) is a modern tool for dealing with uncertainty, granularity, and incompleteness of knowledge in information systems. One of the limitations of RST is its dependence on portioning the universe according to equivalence relation on the universe of objects in information systems. The purpose of this paper is to construct connections between generalized rough sets based on covering and the rough sets based on the topology whose sub base is the cover. Firstly, we present basic concepts and properties covering of rough sets. Then we give examples for topologies whose sub base is the cover, relationships between covering approximations and topological approximations are obtained and counter examples for inverse relationships are given. Rough membership function with respect to topology is constructed and compared with its correspondence.

Keywords: Rough Set, Topology, Covering Lower And Upper Approximations, Covering Membership Function.

1. Introduction

The volume and complexity of the collected data in our modern society is growing rapidly. In order to describe and extract the useful information contained in those data, data mining emerges as a research area. It is a non-trivial process of identifying valid, novel, potentially useful, and ultimately understandable patterns in data. Data mining is already widely used in industry and business applications. Since there exists incomplete, vague, and granular information in those data, concept approximation is one of the most fundamental issues in data mining. To solve this problem, scientists in computer science and related areas have proposed various theories such as fuzzy set theory [3,7,9,14], granular computing [17], and rough sets [23].

Rough set theory was first proposed by Pawlak [23] for dealing with vagueness and granularity in information systems. It has been successfully applied to process control, economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis and other fields [2,10,11]. The further investigation into rough set theory and its extension will find new applications and new theories [1,2,5,13].

The classical rough set theory is based on equivalence relations, but this requirement is not satisfied in some situations. Thus classical rough set theory has been extended to similarity relation based rough sets [24], arbitrary binary relation based rough sets [16] and covering-based rough sets [8,12,15,18-22,24,25].

In this paper, we aim to construct connections between covering rough sets and topological structures generated by the cover as a sub base. Approximations with respect the topology are compared with covering approximations and rough membership function is constructed and discussed.

2. Basic concepts

Definition (1) [15,19]
Let \( U \) be a universe of discourse. Let \( C \) be a family of subsets of \( U \). If none subset in \( C \) is empty and \( \bigcup C = U \), then \( C \) is called a covering of \( U \).

It's obvious that the concept of a covering is an extension of the concept of a partition.
Definition (2) [19]
Let $U$ be a non-empty set, $C$ a covering of $U$, then the ordered pair $(U, C)$ is a covering approximation space.

Definition (3) [19]
Let $(U, C)$ be a covering approximation space, $x \in U$ then the set family $Md(x)$ is called the minimal description of $x$:

$$Md(x) = \{ K \in C / x \in K \land (\forall S \in C \land x \in S \land S \subseteq K \Rightarrow K = S ) \}$$

Definition (4) [4,15,19]
Let $X \subseteq U$, then

$$C_*(X) = \{ K \in C / K \subseteq X \}$$

is called the covering lower approximation set family of $X$.

$X_+ = \bigcup C_*(X)$ is called the covering lower approximation of $X$.

$X^- = X - X_+$ is called the covering boundary of $X$.

$$B_n(X) = \{ Md(x) / x \in X_+ \}$$

is called the covering boundary approximation set family of $X$.

$$C^* = C_*(X) \cup B_n(X)$$

is called the covering upper approximation set family of $X$.

$$X^- = \bigcup C^*(X)$$

is called the covering upper approximation of $X$.

-For the pair $(U, \tau)$ is a topological space where $U$ is any set and $\tau$ is a family of subsets of $U$ which satisfies the following

- $U, \emptyset \in \tau$
- The arbitrary union of elements of $\tau$ belongs to $\tau$
- The finite intersection of elements of $\tau$ belongs to $\tau$

Then the pair $(U, \tau)$ is called topological space.

The subsets of $U$ belonging to $\tau$ are called open sets the complement of the subsets of $U$ belonging to $\tau$ are called closed sets in the space. Sometimes we take some subsets of $U$ as subbase for $\tau$, by finite intersection on these subsets we get the base $\beta$ for $\tau$ and by finite union on the base $\beta$ we get $\tau$.

$$\bar{A} = \bigcap \{ F \subseteq U / A \subseteq F \text{ and } F \text{ is closed} \}$$

is called the closure of a subset $A \subseteq U$.

$$A^0 = \bigcup \{ G \subseteq U / G \subseteq A \text{ and } G \text{ is open} \}$$

is called the interior of a subset $A \subseteq U$.

$$A^* = \bar{A} - A^0$$

is called the boundary of subset $A \subseteq U$ [4,5].
3. Relationship between concepts of topology and covering lower and upper approximation

In covering rough set theory the reference space is the covering approximation space. We will consider the covering of the universe as a subbase for topology and we will calculate the closure, the interior and the boundary defined on this topology, then we will compare these concepts with the covering Lower and upper approximation.

**Proposition (1)**  
The covering lower approximation is contained in the interior defined by taking this covering as a subbase for topology.

**Proof**  
Let \( C \) be a covering of the universe \( U \)

let \( X \subseteq U, a \in X \), then \( \exists \ a \) set  
\( K \in C, (X) \ S.T. \ a \in K \)  
since \( K \in C, (X) \)  
\( \Rightarrow k \in C \) and \( K \subseteq X \), therefore \( K \subseteq U \)  
since \( C \) is a subbase for the topology defined on \( U \) then every \( K \in C \) is open.

hence \( a \in \bigcup \{ K \subseteq U : K \text{ is open} , K \subseteq X \} \)  
\( a \in X^O \Rightarrow X_* \subseteq X^O \)

**Corollary (1)**  
the covering lower approximation is proper subset of the interior defined by taking this covering as a subbase for topology.

The following example illustrates this corollary.

**Example 1**  
Let  
\( U = \{a,b,c,d,e,f\} \)  
\( K_1 = \{a,b\} , K_2 = \{a,b,c,d\} , K_3 = \{c,d\} \)  
\( K_4 = \{c,d,e,f\} , K_5 = \{b,e,f\} , K_6 = \{c,e,f\} \)

Let \( X = \{a,b,c,e\} \Rightarrow C_*(X) = \{K_1\} \)  
\( X_* = \{a,b\} \)  
Let \( C = \{\{a,b\}, \{a,b,c,d\}, \{c,d\}, \{c,d,e,f\}, \{b,e,f\}, \{c,e,f\}\} \)  
be the subbase
be the base

\[ \tau = \{ U, \emptyset, \{a,b\}, \{a,b,c,d\}, \{c,d\}, \{c,d,e,f\}, \{b,e,f\}, \{c,e,f\}, \{b\}, \{c\}, \{e,f\} \} \]

\[ X^0 = \{a,b,c\} \]

**Proposition (2)**
The covering upper approximation of a set \( X \) does not necessarily contain the closure of the set with respect to the topology induced by this covering.

**Corollary (2)**
The boundary of a covering rough set does not necessarily contain the boundary of the set with respect to the topology induced by this covering.

The following example will illustrate the last consequences.

**Example 2**
Let \( U=\{a,b,c,d\} \), let \( C=\{K_1,K_2,K_3,K_4,K_5,K_6\} \) be a covering of \( U \) where

\[ K_1 = \{a,b\} \quad K_2 = \{a,b,c\} \quad K_3 = \{b,c,d\} \]

\[ K_4 = \{b,c\} \quad K_5 = \{a,c,d\} \quad K_6 = \{c,d\} \]

1. let \( X=\{a\} \Rightarrow C_* (X) = \emptyset \Rightarrow X_* = \emptyset \)
2. let \( X=\{b,c\} \Rightarrow C_* (X) = \{K_4\}, X_* = \{b,c\} \)

\[ X_* = \emptyset \quad X^* = \{b,c\} \]

\[ B_N (X^*) = X^* - X_* = \emptyset \]

On the other hand, let

\[ C=\{\{a,b\}, \{a,b,c\}, \{b,c,d\}, \{b,c\}, \{a,c,d\}, \{c,d\}\} \]

be a subbase

then the base

\[ \beta=\{ \{a,b\}, \{a,b,c\}, \{b,c,d\}, \{b,c\}, \{a,c,d\}, \{c,d\}, \} \]

\[ \{\{b\}, \{a\}, \{a,c\}, \{c\}, \emptyset, U \} \]
\[ \tau = \left\{ U, \emptyset, \{a, b\}, \{a, b, c\}, \{b, c, d\}, \{b, c\}, \{a, c, d\}, \{c, d\}, \{a, c\}, \{b\}, \{a\}, \{c\} \right\} \]
\[ F = \left\{ U, \emptyset, \{c, d\}, \{d\}, \{a\}, \{a, d\}, \{b\}, \{a, b\} \right\} \]

then \( X^0 = \{b, c\} \quad \bar{X} = \{b, c, d\} \quad X^b = \{d\} \)

then \( X \nsubseteq X^* \quad X^b \nsubseteq B_N(X) \)

4. Covering membership function

Let \( C \) be a covering of \( U \), then the covering rough membership function is defined as

\[ \mu^C_x(x) = \frac{|(\cap K_x) \cap X|}{|\cap K_x|} \quad K_x \in C, x \in U \]

where \( K_x \) is any member of \( C \) containing \( x \)

- (1) If \( \mu^C_x(x) = 1 \) then \( x \) belongs to the positive region
- (2) If \( \mu^C_x(x) = 0 \) then \( x \) belongs to the negative region
- (3) If \( 0 < \mu^C_x(x) < 1 \) then \( x \) belongs to the boundary region

Example 3

Let \( U = \{a, b, c, d, e, f\} \), \( K_1 = \{a, b\}, K_2 = \{a, b, c, d\}, K_3 = \{c, d\}, K_4 = \{c, d, e, f\}, K_5 = \{a, b, e, f\}, K_6 = \{c, e, f\} \),

\( C = \{K_1, K_2, K_3, K_4, K_5, K_6\} \), and \( X = \{b, c, d, e\} \) then:

\[ \mu^C_x(a) = \frac{|\{a, b\} \cap \{b, c, d, e\}|}{|\{a, b\}|} = \frac{1}{2} \]

\[ \mu^C_x(b) = \frac{|\{a, b\} \cap \{b, c, d, e\}|}{|\{a, b\}|} = \frac{1}{2} \]

\[ \mu^C_x(c) = \frac{|\{c\} \cap \{b, c, d, e\}|}{|\{c\}|} = 1 \]

\[ \mu^C_x(d) = \frac{|\{c, d\} \cap \{b, c, d, e\}|}{|\{c, d\}|} = 1 \]

\[ \mu^C_x(e) = \frac{|\{e, f\} \cap \{b, c, d, e\}|}{|\{e, f\}|} = \frac{1}{2} \]

\[ \mu^C_x(f) = \frac{|\{e, f\} \cap \{b, c, d, e\}|}{|\{e, f\}|} = \frac{1}{2} \]
Then the covering lower and upper approximation are:

\[ X_c = \{c, d\} \quad X^* = \{a, b, c, d, e, f\} \]

respectively.

5. Conclusion

This paper studies a new class of rough sets based on coverings. In particular, we investigate the relationship between this type of covering-based rough sets and the generalized rough sets based topological structure generated by covers. We explore the properties of approximations for rough sets and compare them with those of covering rough sets. The important contribution of this paper is that we establish the deviations between this type of covering-based rough sets and a class of generalized rough sets based on topology. The constructed topology is more general and will open the way for applying recent topological concepts in the process of approximations.

6. References


