The Solution of KdV and mKdV Equations Using Adomian Padé Approximation

Tamer A. Abassy

Department of Basic Science, Benha Higher Institute of Technology, Benha, 13512, EGYPT, Email: tamerabassy@yahoo.com

Magdy A. El-Tawil*

Department of Engineering Mathematics and Physics, Faculty of Engineering, Cairo University, Giza-EGYPT, Email: magdyeltawil@yahoo.com

Hassan K. Saleh

Department of Engineering Mathematics and Physics, Faculty of Engineering, Cairo University, Giza-EGYPT.

Abstract

Adomian Decomposition method (ADM) is an approximate method, which can be adapted to solve nonlinear partial differential equations. In this paper, we solve the KdV and modified KdV (mKdV) equations using ADM-Padé technique, which gives the approximate solution with fast convergence rate and high accuracy in the case of solitary wave solution and closed form solution in the case of rational polynomial solution.

Keywords: KdV equation, mKdV equation, Adomian decomposition method (ADM), Padé approximants, ADM-Padé technique, Solitary wave, Mathematica.

1 Introduction

The Korteweg-de Vries (KdV) equation

\[ u_t - 6uu_x + u_{xxx} = 0, \quad x \in \mathbb{R}, \quad u(x,0) = f(x), \]  

is one of the most popular soliton equations and was originally derived by Korteweg and de Vries [19] in the 19th century as water waves equations. It is a useful approximation in many studies when one wishes to include a simple nonlinearity and a simple dispersive effect. Some of these studies are ion-acoustic and magnetohydro dynamic waves in plasma, longitudinal dispersive waves in elastic rods, pressure waves in liquid-gas bubble mixtures, rotating flow down a tube and thermally excited photon packets in low temperature nonlinear crystals [24].

A great deal of research work has been invested during the past decades for the study of the KdV equation. The main goal of these studies was directed towards its analytical and numerical solutions. Zabusky and Kruskal [30] solved the KdV equation using a finite difference explicit method with periodic boundary conditions and showed the existence of solitons, which propagated with their own velocities, exerting essentially no influence on each other. Ablowitz et al [2-4] implemented the inverse scattering transform method to handle the nonlinear equations of physical significance.

* Corresponding author.
where soliton solutions and rational solutions where developed. Taha and Ablowitz [25,26] have done excellent comparisons between different known schemes and their scheme for KdV equation, which is developed using notations of the inverse scattering transformation. Hirota [15-17] constructed the solution of the evolution equation by reducing it to the bilinear form. Freeman [10,11] and Nemmo and Freeman [22,23] introduced an alternative formulation of the N-Soliton solution in terms of some function of the Wronskian determinant and inverse scattering. Lax [20,21] discussed the case when the potential $u(x,t)$, instead of tending to 0 as $x \to \infty$ is periodic in $x$. El-Zoheiry et. al [9] solved the KdV equation numerically by using the quintic spline approximation. Moreover, Wazwaz [30] construct the solution of (1) in the form of truncated Taylor series by using Adomian decomposition method.

The modified Korteweg-de Vries (mKdV) equation

\[ u_t + 6u^2u_x + u_{xxx} = 0, \quad x \in R, \quad u(x,0) = f(x), \]  

appears in electric circuits and multicomponent plasmas. The modified KdV equation (2) has a pulse travelling solution. Wadati and Ohkuma [27] have used the inverse scattering method to investigate the multiple pole solution of the modified KdV equation. Wazwaz [29] constructed the solution of (2) in the form of Taylor series by using Adomian decomposition method.

In this paper, The ADM method is used to obtain an approximate truncated series solution in a very narrow domain around the initial condition point ($t = 0$). Although the series can be rapidly convergent in a very small region, it has very slow convergence rate in the wider region we examine, and the truncated series solution is an inaccurate solution in that region, which means that we can’t rely on this solution globally. To extend the domain of solution and obtain a better accuracy and better convergence, we use the ADM method together with Padé approximants (ADM-Padé technique). This link is used before with ordinary differential equations (see [18] [28]). The new in this paper is that we will use ADM-Padé technique with KdV and mKdV equations, which gives very good results with respect to the use of ADM alone. Accordingly, we introduce it as an efficient technique for solving not only KdV and mKdV equations (in this paper) but also as an efficient technique in solving partial differential equation. Some other modified Adomian method is suggested by He in his review article [12], and the comparison of Adomian method and variational iteration method [13,14] is also illustrated in [13]. Mathematica is used powerfully in obtaining the different approximations. Different case studies and figures are illustrated to examine the method of analysis.

2. Analysis

Following the analysis of Adomian [5,6] and Wazwaz [29,30] equation (1) can be re-written in an operator form as the following:

\[ Lu + Ru + 6N(u) = 0, \quad x \in R, \quad u(x,0) = f(x), \]  

where the differential operator $L$ and $R$ are:

\[ L = \frac{\partial}{\partial t}, \]  
\[ R = \frac{\partial^3}{\partial x^3}, \]  

and the nonlinear term $N(u)$ equals $(-uu_x)$ in the case of KdV equation and $(u^2u_x)$ in the case of mKdV equation.

The unknown function $u(x,t)$ can be expressed by an infinite series of the form

\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \]  

and the nonlinear term $N(u)$ can be decomposed by an infinite series of polynomials given by

\[ N(u) = \sum_{n=0}^{\infty} A_n, \]  

where the components $u_n(x,t)$ will be determined recurrently and $A_n$ are the so-called Adomian
polynomials of \(u_0\), \(u_1\), \ldots \(u_n\) defined by
\[
A_n = \frac{1}{n!} \left[ \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right)^n \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda = 0}, \quad n = 0, 1, 2, 3, \ldots
\] (8)

These polynomials can be constructed for all nonlinearity according to algorithms set by Adomian [5,6]. The inverse operator \(L^{-1}\) is an integral operator given by
\[
L^{-1}(\cdot) = \int_0^t \cdot dt.
\] (9)

Applying \(L^{-1}\) on equation (3) and using the initial condition we find
\[
u(x,t) = f(x) - L^{-1}(6N(u) + Ru).
\] (10)

Substituting equation (6), (7) into equation (10) gives
\[
\sum_{n=0}^{\infty} u_n(x,t) = f(x) - L^{-1}(6 \sum_{n=0}^{\infty} A_n + R \sum_{n=0}^{\infty} u_n)),
\] (11)

where \(A_n\) are Adomian polynomials that represent the nonlinear term, in the case of \(N(u) = -uu_x\), \(A_n\) given by:
\[
\begin{align*}
A_0 &= -u_0 u_0, \\
A_1 &= -u_0 u_1 - u_1 u_0, \\
A_2 &= -u_0 u_2 - u_1 u_1 - u_2 u_0, \\
A_3 &= -u_0 u_3 - u_1 u_2 - u_2 u_1 - u_3 u_0.
\end{align*}
\] (13)

and in the case of \(N(u) = u^2 u_x\), \(A_n\) given by:
\[
\begin{align*}
A_0 &= u_0 u_0^2, \\
A_1 &= 2u_0 u_1 u_0 + u_1 u_0^2, \\
A_2 &= u_0 u_2 + 2u_0 u_1 u_1 + u_0 (2u_2 u_0 + u_0 u_2), \\
A_3 &= 2(u_1 u_2 + u_0 u_3) u_0 + (u_1^2 + 2u_0 u_2) u_1 + 2u_0 u_1 u_2 + u_3 u_0^2.
\end{align*}
\] (14)

Other polynomials can be generated in a like manner.

The component of \(u_n(x,t)\) follows immediately upon setting
\[
u_0(x,t) = f(x),
\]
\[
u_1(x,t) = -L^{-1}(6A_0 + Ru_0),
\]
\[
u_2(x,t) = -L^{-1}(6A_1 + Ru_1),
\]
\[
\vdots
\]
\[
u_{n+1}(x,t) = -L^{-1}(6A_n + Ru_n), \quad n \geq 0.
\] (15)

In some papers, see Wazwaz [29,30], it is said that we can get better approximations by adding new terms and the exact solution is got asymptotically. This is correct in a very narrow region but we can’t rely on this globally. Practically the convergence region is very small and we need to extend this reliable region. This is not reached by adding new terms and in fact this drawback is the major criticism against ADM method [1,18].

In this paper, we introduce the use of Padé approximants [7] in improving this major criticism in KdV equation.

Let us first introduce the Padé approximants. To obtain Padé \([L/M](x,t)\) we must have a truncated series solution for \(u(x,t)\) of order at least \((L+M)\) in \(t\). For example if we have
\[
u(x,t) \approx c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4.
\] (16)

The Padé approximants [2/2] is
\[
a_0 = c_0, \quad a_1 = c_1 + \frac{c_0 (-c_2 c_3 + c_2 c_4)}{c_0^2 - c_1 c_3}, \quad a_2 = \frac{c_2^2 - c_1 c_3}{c_2^2 - c_1 c_3}, \quad a_3 = -c_2 c_3 + c_2 c_4, \quad a_4 = -c_0 c_2 c_4,
\] (17)
\[ b_2 = \frac{c_1^2 - c_2 c_4}{c_2^2 - c_4 c_3}, \]

where \( c_0, c_1, \ldots, c_4 \) are functions of \( x \) or constants.

To obtain diagonal Padé approximants of different order like, \([4/4]\) or \([6/6]\) we can use Mathematica.

2.1 The used Algorithm

ADM method is used first to obtain the approximate truncated series solution \( u(x,t) \). Then the Padé approximants is used to obtain an equivalent rational function approximation or the closed form solution (see Figure 1.).

![Fig.1 The algorithm of ADM-Padé technique.](image-url)

3. Case-Studies

Four test case studies are solved to illustrate the efficiency of the proposed method.

3.1. Case-Study 1

Consider equation (1) with the following initial condition \([8]\)

\[ u(x,0) = -\frac{k^2}{2} \text{sech}^2 \left[ \frac{k}{2} x \right]. \]  

Substituting equation (18) in equation (12), we obtain the following resulting components

\[ u_0(x,t) = -\frac{k^2}{2} \text{sech}^2 \left[ \frac{k}{2} x \right]. \]

\[ u_1(x,t) = -\frac{k^5}{2} \text{sech}^2 \left[ \frac{k}{2} x \right] \tanh \left[ \frac{k}{2} x \right] t, \]

\[ u_2(x,t) = -\frac{k^8}{8} \text{sech}^4 \left[ \frac{k}{2} x \right] (-2 + \cosh[kx]) t^2, \]

\[ u_3(x,t) = -\frac{k^{11}}{48} \text{sech}^5 \left[ \frac{k}{2} x \right] (-11 \sinh \left[ \frac{k}{2} x \right] + \sinh \left[ \frac{3k}{2} x \right]) t^3, \]

\[ u_4(x,t) = -\frac{k^{14}}{384} \text{sech}^6 \left[ \frac{k}{2} x \right] (33 - 26 \cosh[kx]) t^4, \]

\[ u_5(x,t) = -\frac{k^{17}}{3840} \text{sech}^7 \left[ \frac{k}{2} x \right] (302 \sinh \left[ \frac{k}{2} x \right] - 57 \sinh \left[ \frac{3k}{2} x \right] + \sinh \left[ \frac{5k}{2} x \right]) t^5, \]

\[ u_6(x,t) = -\frac{k^{20}}{46080} \text{sech}^8 \left[ \frac{k}{2} x \right] (-1208 + 1191 \cosh[kx]) \]

\[-120 \cosh[2kx] + \cosh[3kx]) t^6, \]

\[ u_7(x,t) = -\frac{k^{23}}{645120} \text{sech}^9 \left[ \frac{k}{2} x \right] (-15619 \sinh \left[ \frac{k}{2} x \right] + 4293 \sinh \left[ \frac{3k}{2} x \right] - 247 \sinh \left[ \frac{5k}{2} x \right]) t^7, \]

\[ \]

\[ u_8(x,t) = -\frac{k^{26}}{10321920} \text{sech}^{10} \left[ \frac{k}{2} x \right] (78095 - 88234 \cosh[kx] + 14608 \cosh[2kx]) \]

\[-502 \cosh[3kx] + \cosh[4kx]) t^8. \]

Considering these components, the solution can be approximated as:

\[ \phi_n(x,t) = \sum_{m=0}^{\infty} u_m(x,t). \]  

It is known that the exact solution of this problem is \([8]\)

\[ u(x,t) = -\frac{k^2}{2} \text{sech}^2 \left[ \frac{k}{2} (x-k^2 t) \right]. \]  

For \( k = 1 \), ADM truncated series solution...
(equation (20) with \( n = 8 \)) gives good approximation in the interval of convergence \([-2,2]\) [25]. This is illustrated in figure 2 which shows ADM truncated series solution \( \phi_8(x,t) \) compared with the exact solution at \( x = 0 \). The error between the exact solution and the ADM truncated series solution \( \phi_8(x,t) \) at \( x = 0 \) is shown in figure 3, where we can notice high errors beyond the interval \([-2,2]\).

Using ADM-Padé approximation at \( x = 0 \), the rational approximation \([2/2]\) and \([4/4]\) take the form of equations (22) and (23) respectively

\[
[2/2](0,t) = \frac{1}{1 + \frac{t^2}{6}} + \frac{t^2}{2}, \tag{22}
\]

\[
[4/4](0,t) = \frac{1}{1 + \frac{t^2}{63} + \frac{13t^4}{15120}} - \frac{1}{1 + \frac{t^2}{2048} + \frac{13t^4}{30240}}. \tag{23}
\]

The results are shown in figure 4 which illustrate how much the results are improved at \( x = 0 \). The error between the exact solution and \([2/2](x,t)\) and \([4/4](x,t)\) at \( x = 0 \) is also illustrated in figure 5 where we can notice that \([4/4](x,t)\) is better than \([2/2](x,t)\) and both are more reliable than ADM alone.
Repeating the previous calculations at $x = 1$, we obtain the following rational approximation

$$[2/2](1,t) = \frac{(-0.393224 - 0.0813819t + 0.0121087t^2)}{(1 - 0.255156t + 0.1769555t^2)},$$

$$[4/4](1,t) = \frac{(-0.393224 - 0.042292t + 0.012834r^2 + 0.001644r^3 - 0.000096731r^4)}{1 - 0.354565t + 0.221048r^2 - 0.0334872r^3 + 0.0092745r^4}.$$  (24)

(25)

Figures 6-9 show how much the results improved after using ADM-Padé approximation.

Also, figures 10 and 11 show the 3-dimensional plot of $\phi_k(x,t)$ and the Padé $[4/4](x,t)$ in the interval indicated in the figures. Figure 10 shows the deterioration in the ADM solution beyond the interval of convergence. Figure 11 shows the improvement in results after using ADM-Padé approximation. It is clear that the interval of convergence has increased. Figure 12 shows the approximate solution $-[4/4](x,t)$ for KdV equation at $k = 1$ and different time level.
Fig. 10 The surface generated from the truncated series solution $-\phi_8(x,t)$ for KdV equation in the interval $-3 < t < 3$ and $-4 < x < 4$ at $k = 1$.

Fig. 11 The surface generated from the approximate solution $-\left[\frac{4}{4}\right](x,t)$ for KdV equation in the interval $-5 < t < 5$ and $-10 < x < 10$ at $k = 1$.

Fig. 12 Plot of the approximate solution $-\left[\frac{4}{4}\right](x,t)$ for KdV equation at $k = 1$ and different time level.
3.2. Case-Study 2

Consider equation (1) with the following initial condition [8]

\[ u(x,0) = \frac{6}{x^2}. \]  

(26)

Substituting equation (26) in equation (12) we obtain the resulting components

\[
\begin{align*}
 u_0(x,t) &= \frac{6}{x^2}, \\
 u_1(x,t) &= -\frac{288}{x^3}t, \\
 u_2(x,t) &= \frac{6048}{x^8}t^2, \\
 u_3(x,t) &= -\frac{103680}{x^{11}}t^3, \\
 u_4(x,t) &= \frac{1617408}{x^{14}}t^4, \\
\end{align*}
\]

(27)

and so on (see [29]). Consequently, the series solution will be in the form given by

\[
\begin{align*}
 u(x,t) &= \frac{6}{x^2} - \frac{288}{x^3}t + \frac{6048}{x^8}t^2 - \frac{103680}{x^{11}}t^3 + \frac{1617408}{x^{14}}t^4 + \ldots
\end{align*}
\]

(28)

The new here is that we can obtain the solution directly by using the ADM-Padé technique. By substituting in equations (16) and (17) using equation (28), we get

\[
\begin{align*}
 u(x,t) &= \frac{6(x^3 - 24t)}{(x^3 + 12t)^2}, \\
\end{align*}
\]

(29)

which is the exact solution of this problem [8]. The proposed ADM-Padé technique succeeds to get the exact solution whenever a closed solution exists and it is rational polynomial.

3.3 Case-Study 3

Consider equation (2) with the following initial condition [8]

\[ u(x,0) = k \text{ sech}[k \ x] \]

(30)

Following the same procedure as we do in case-study 1 we obtain the following results

\[
\begin{align*}
 u_0(x,t) &= k^4 \sec h[k \ x] \tanh[k \ x]t, \\
 u_1(x,t) &= \frac{k^7}{4} \sec h^3[k \ x](-3 + \cosh[2k]t^2), \\
 u_2(x,t) &= \frac{k^{10}}{24} \sec h^5[k \ x](-23 \sinh[k \ x] + \sinh[3k]t^3), \\
 u_3(x,t) &= \frac{k^{13}}{192} \sec h^7[k \ x][115 - 76 \cosh[2k]] + \cosh[4k]t^4, \\
 u_4(x,t) &= \frac{k^{16}}{1920} \sec h^9[k \ x][1682 \sinh[k \ x] - 237 \sinh[3k] + \sinh[5k]t^5], \\
 u_5(x,t) &= \frac{-k^{20}}{46080} \sec h^{11}[k \ x](-1208 + 1191 \cosh[2k]) - 120 \cosh[2k] + \cosh[3k]t^6, \\
 u_6(x,t) &= \frac{-k^{23}}{645120} \sec h^{13}[k \ x][15619 \sinh[k \ x] + 4293 \sinh[3k] - 247 \sinh[5k]t^7] + \sinh[7k/2]t^7, \\
 u_7(x,t) &= \frac{-k^{26}}{10321920} \sec h^{15}[k \ x][78095 - 88234 \cosh[k \ x] + 14608 \cosh[2k] - 502 \cosh[3k] + \cosh[4k]t^8],
\end{align*}
\]

(31)

It is known that the exact solution of this problem is [8]

\[ u(x,t) = \frac{-k^2}{2} \sec h^2\left[\frac{k}{2}(x - k^2t)\right]. \]

(32)

Taking \( k = 1 \) and using ADM-Padé approximation at \( x = 0 \), the rational approximation [2/2] and [4/4] take the form of equations (33) and (34) respectively
Repeating the previous calculations at $x = 1$, we obtain the following rational approximation

\[ \frac{2}{2} \left( 1, t \right) = \frac{1 - 0.08333333r^2}{1 + 0.416667r^2}, \]  

\[ \frac{4}{4} \left( 1, t \right) = \frac{(1 - 0.043560r^2 + 0.000859788r^4) / (1 + 0.456349r^2 + 0.020701r^4)}{1 - 0.554633r + 0.311587r^2} \] (33)

\[ \frac{4}{4} \left( 0, t \right) = \frac{(0.64805 + 0.069699t - 0.0212r^2 - 0.002709r^3 + 0.0001594t^4) / (1 - 0.654042t + 0.385451r^2 - 0.0524795r^3 + 0.0151255r^4)}{1 - 0.456349r + 0.0207011r^2} \] (34)

Fig. 13 The ADM-Padé rational approximation $\frac{2}{2}(x,t)$ and $\frac{4}{4}(x,t)$ for mKdV equation at $k = 1$ and $x = 0$.

Fig. 14 The error in ADM-Padé between the exact solution $u(x,t)$ and the rational approximation $\frac{2}{2}(x,t)$ and $\frac{4}{4}(x,t)$ for mKdV equation at $k = 1$ and $x = 0$.

Fig. 15 The ADM-Padé rational approximation $\frac{2}{2}(x,t)$ and $\frac{4}{4}(x,t)$ for mKdV equation at $k = 1$ and $x = 1$.

Fig. 16 The error in ADM-Padé between the exact solution $u(x,t)$ and the rational approximation $\frac{2}{2}(x,t)$ and $\frac{4}{4}(x,t)$ for mKdV equation at $k = 1$ and $x = 1$.\]
Fig. 17 The surface generated from the truncated series solution $\phi_8(x,t)$ for mKdV equation in the interval $-3 < t < 3$ and $-4 < x < 4$ at $k = 1$.

Fig. 18 Shows the surface generated from the approximate solution $[4/4](x,t)$ for mKdV equation in the interval $-5 < t < 5$ and $-10 < x < 10$ at $k = 1$.

Fig. 19 Plot of the approximate solution $[4/4](x,t)$ for mKdV equation at $k = 1$ and different time level.
3.4. Case-Study 4

Consider equation (2) with the following initial condition [8]

\[ u(x,0) = c - \frac{4c}{4c^2 - 1}, \quad (37) \]

where \( c \) is constant.

Following the same procedure as we do in case-study 2 we obtain the resulting components

\[
\begin{align*}
 u_0(x,t) & = c - \frac{4c}{4c^2 - 1}, \\
 u_1(x,t) & = -\frac{192c^2x}{(1+4c^2x^2)^2}t, \\
 u_2(x,t) & = \frac{576c^7(1-12c^2x^2)}{(1+4c^2x^2)^3}t^2, \\
 u_3(x,t) & = \frac{55296c^{11}x(1-4c^2x^2)}{(1+4c^2x^2)^4}t^3, \\
 u_4(x,t) & = \frac{-82944c^{13}(1-40c^2x^2 + 80c^4x^4)}{(1+4c^2x^2)^5}t^4, \\
 \end{align*}
\]

and so on. Consequently, the truncated series solution will be in the form given by

\[
 u(x,t) = c - \frac{4c}{4c^2 - 1} + \frac{-192c^2x}{(1+4c^2x^2)^2}t^2 + \frac{576c^7(1-12c^2x^2)}{(1+4c^2x^2)^3}t^3 + \frac{55296c^{11}x(1-4c^2x^2)}{(1+4c^2x^2)^4}t^4 + \ldots \quad (39)
\]

Applying the \([L/M]\) Padé approximants on equation (39) where \( L > 0 \) and \( M > 2 \), we get

\[
 u(x,t) = c - \frac{4c}{4c^2 - 1}, \quad (40)
\]

which is the exact solution of this problem[8].

4. Conclusion

Using Adomian decomposition method in solving non-linear partial differential equations not generally gives accurate results, specially, in the global solution interval and accordingly the use of another enhancing technique is almost required. In this paper, the ADM and Padé approximants link shows a good enhancement. The convergence interval becomes wider and the error becomes smaller. Also, this link gives the closed form solution in the cases of rational polynomials solutions and other cases as well. The use of Mathematica facilitates the calculations of the ADM-Padé technique.

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