TREES, TIGHT-SPANS AND POINT CONFIGURATIONS

SVEN HERRMANN AND VINCENT MOULTON

Abstract. Tight-spans of metrics were first introduced by Isbell in 1964 and rediscovered and studied by others, most notably by Dress, who gave them this name. Subsequently, it was found that tight-spans could be defined for more general maps, such as directed metrics and distances, and more recently for diversities. In this paper, we show that all of these tight-spans can be defined in terms of point configurations. This provides a useful way in which to study these objects in a unified and systematic way. We also show that by using point configurations we can recover results concerning one-dimensional tight-spans for all of the maps we consider, as well as extend these and other results to more general maps such as symmetric and unsymmetric maps.

1. Introduction

Let $V$ be a real vector space with standard scalar product $\langle \cdot, \cdot \rangle$ with respect to some fixed basis. Given a point configuration $\mathcal{A}$ in $V$, that is, a finite subset of $V$, (For technical reasons, the affine hull of $\mathcal{A}$ is assumed to have codimension 1.) and a function $w : \mathcal{A} \to \mathbb{R}$, we define the envelope of $\mathcal{A}$ with respect to $w$ to be the polyhedron

$$E_w(\mathcal{A}) := \{x \in V \mid \langle a, x \rangle \geq -w \text{ for all } a \in \mathcal{A}\},$$

and the tight-span $T_w(\mathcal{A})$ of $\mathcal{A}$ to be the union of the bounded faces of $E_w(\mathcal{A})$. Tight-spans of point configurations were introduced in [15] for vertex sets of polytopes, as a tool for studying subdivisions of polytopes. Even so, they first appeared several years ago in a somewhat different guise.

More specifically, let $X$ be a finite set, $V = \mathbb{R}^X$ be the vector space of functions $X \to \mathbb{R}$ and, for $x \in X$, $e_x$ denote the elementary function assigning 1 to $x$ and 0 to all other $y \in X$. In addition, let $D$ be a metric on $X$, that is, a symmetric map on $X \times X$ that vanishes on the diagonal and satisfies the triangle inequality. Then, as first remarked by Sturmfels and Yu [24], by setting $w(e_x + e_y) = -D(x, y)$, the tight-span $T_w(\mathcal{A})$ of $\mathcal{A}(X) := \{e_x + e_y \mid x \neq y \in X\}$ is nothing other than the injective hull of $D$, that was first introduced by Isbell [20] and subsequently studied by Dress [8] (who called it the tight-span of $D$), and Chrobak and Lamore [4, 5].

Since tight-spans were introduced by Isbell, tight-span theory has been intensively studied (see, e.g., [11] for an overview). For example, Hirai and Koichi [19] studied tight-spans of directed metrics and directed distances, and more recently Bryant and Tupper [3] introduced the concept of the tight-span of so-called diversities. As we shall see in this paper, tight-spans for all such maps can also be defined in terms of point configurations, providing a useful way to systematically study these objects.

In particular, after presenting some preliminary results concerning point configurations in Sections 2 and 3 in Section 4, we shall show that the tight-span of a distance on $X$ can be defined

\begin{flushright}
\textit{Date: 8 April 2011.}
\end{flushright}

\textit{Key words and phrases.} tight-span, polytopal subdivision, metric, diversity, point configuration, injective hull.

The first author was supported by a fellowship within the Postdoc-Programme of the German Academic Exchange Service (DAAD) and thanks the UEA School of Computing Sciences for hosting him during the writing of this paper.
in terms of the configuration $\mathcal{A}(X) := \tilde{\mathcal{A}}(X) \cup \{2e_x \mid x \in X\} = \{e_x + e_y \mid x, y \in X\}$ (Proposition 1.1). Also, for $Y$ a finite set with $X \cap Y = \emptyset$, let $\tilde{\mathcal{B}}(X, Y) \subset \mathbb{R}^{X \cup Y}$ be the configuration of all points $e_x + e_y$, with $x \in X$, $y \in Y$ and $\mathcal{B}(X, Y) := \tilde{\mathcal{B}}(X, Y) \cup \{2e_x \mid x \in X \cup Y\}$. We show that the tight-span of a directed metric (distance) can be defined in terms of $\tilde{\mathcal{B}}(X) := \tilde{\mathcal{B}}(X, Y)$ or $\mathcal{B}(X) := \mathcal{B}(X, Y)$, where we consider $Y$ as a disjoint copy of $X$ (Proposition 5.1). Using these point configurations, we will also extend this analysis to include arbitrary symmetric and even unsymmetric maps (Section 5).

Furthermore, given a diversity on a set $Y$, that is, a function $\delta : \mathcal{P}(Y) \rightarrow \mathbb{R}$ defined on the powerset $\mathcal{P}(Y)$ of $Y$ that satisfies

(D1) $\delta(A \cup B) + \delta(B \cup C) \geq \delta(A \cup C)$ for all $A, C \in \mathcal{P}(Y)$ and $B \in \mathcal{P}(Y) - \{\emptyset\}$, and

(D2) $\delta(A) = 0$ for all $A \in \mathcal{P}(Y)$ with $|A| \leq 1$,

we derive a relationship between metrics and diversities in Section 6, which we then use in Section 7 to show that the tight-span of $\delta$ as defined by Bryant and Tupper can be given in terms of the point configuration $C(X) := \{\sum_{e_i} e_i \mid A \in \mathcal{P}(X)\}$ (the vertices of a cube). Intriguingly, we also show that an alternative tight-span for $\delta$ can be defined by considering the point configuration $\mathcal{A}(\mathcal{P}(X) \setminus \{\emptyset\})$, and that for a special class of diversities (split system diversities) the tight-spans associated to $\mathcal{A}(\mathcal{P}(X) \setminus \{\emptyset\})$ and $C(X)$ are in fact the same (Theorem 7.4).

In addition to providing some new insights on tight-spans using point configurations, we will also focus on one-dimensional tight-spans. These are important since, for example, they provide ways to generate phylogenetic trees and networks (see, e.g., [9, 10]). To see why this is the case, note that a one-dimensional tight-span associated to a point configuration $A$ and weight function $w$ can also be regarded as a graph, with vertex set equal to that of $\mathcal{E}_w(A)$ and edge set consisting of precisely those pairs of vertices that both lie in a one-dimensional face of $\mathcal{E}_w(A)$. Since the union of bounded faces of an unbounded polyhedron is contractible (see, e.g., [17, Lemma 4.5]) it follows that in this case the tight-span is, in fact, a tree.

The archetypal characterisation for one-dimensional tight-spans was first observed by Dress (for metrics [8]):

**Theorem 1.1** (Tree Metric Theorem). The tight-span of a metric $D$ on a finite set $X$ is a tree if and only if $D$ satisfies

\[ D(x, y) + D(u, v) \leq \max\{D(x, u) + D(y, v), D(x, v) + D(y, u)\} \]

for any $x, y, u, v \in X$.

As we shall see, this theorem can be extended in a natural way using point configurations to all of the various maps which we consider (Theorems 4.5, 5.5 and 7.3). This allows us to recover and extend various theorems concerning tight-spans and trees that arise in the literature. We conclude the paper with a discussion on some possible future directions.

2. **Tight-Spans and Splits of Point Configurations**

In this section, we will recall some definitions and results about tight-spans and splits of general point configurations as well as give some elementary properties of these that we will use later. For details, we refer the reader to [15] and [14, Section 2]. First we give a characterisation of the tight-span as the set of minimal elements of the envelope of a configuration if the configuration satisfies certain conditions. These conditions are fulfilled by all of the configurations that we will consider. When tight-spans (of metric spaces, but also of diversities) are considered and thought of in a non-polyhedral way, this characterisation is normally used as definition instead.
Lemma 2.1. Let set of minimal elements of equals the set of minimal elements of if there exists some \( M' \) such that \( v' \in A \). Let \( \{x \in V \mid \varphi(x) \geq b\} \), an element \( x \in P \) is contained in a bounded face of \( P \) if and only if there does not exist some \( r \in \{x \in V \mid \varphi(x) \geq 0\} \) (a ray of \( P \)) and some \( \lambda \in \mathbb{R}_{>0} \) with \( x - \lambda r \in P \). Note that it is easily seen that \( P \) is bounded from below if and only if all rays of \( P \) are positive. We now give an alternative characterisation for the tight-span.

Lemma 2.2. Let \( A \subset V \) be a point configuration, \( w : A \rightarrow \mathbb{R} \) a weight function, \( v \in V \), and \( w' := w + \langle \cdot, v \rangle \). Then \( \mathcal{T}_w(A) = \mathcal{T}_{w'}(A) + v \).

Proof. For all \( x \in V \), we have
\[
\langle a, x \rangle \geq -w'(a) = -(w(a) + \langle a, v \rangle) \iff \langle a, x + v \rangle \geq -w \text{ for all } a \in A.
\]

Hence
\[
\mathcal{E}_{w'}(A) = \{x \in V \mid \langle a, x + v \rangle \geq -w \text{ for all } a \in A\}
= \{y - v \in V \mid \langle a, y \rangle \geq -w \text{ for all } a \in A\} = \mathcal{E}_w(A) - v.
\]

Obviously, this equation carries over to the unions of the bounded faces, that is, the tight-spans.

Tight-spans of a point configuration with certain weight functions are closely associated to other objects defined by these weight functions, so-called regular subdivisions: For a point configuration \( A \) we call \( F \subseteq A \) a face of \( A \) if there exists a supporting hyperplane \( H \) of \( \text{conv} \ A \) such that \( F = A \cap H \). The convex hull of \( A \) is denoted by \( \text{conv} \ A \) and the relative interior of a set \( A \subset V \) is denoted by \( \text{relint} A \). A subdivision \( \Sigma \) of a point configuration \( A \) is a collection of subconfigurations of \( A \) satisfying the following three conditions (see [4] Section 2.3):

\begin{itemize}
  \item [(SD1)] If \( F \in \Sigma \) and \( \bar{F} \) is a face of \( F \), then \( \bar{F} \in \Sigma \).
  \item [(SD2)] \( \text{conv} \ A = \bigcup_{F \in \Sigma} \text{conv} F \).
  \item [(SD3)] If \( F, \bar{F} \in \Sigma \), then \( \text{relint} (\text{conv} F) \cap \text{relint} (\text{conv} \bar{F}) = \emptyset \).
\end{itemize}

A common way to define such a subdivision is the following: Given a weight function \( w : A \rightarrow \mathbb{R} \) we consider the lifted polyhedron
\[
\mathcal{L}_w(A) := \text{conv} \{ (w(a), a) \mid a \in A \} + \mathbb{R}_{\geq 0}(1, 0) \subset \mathbb{R} \times V.
\]

The regular subdivision \( \Sigma_w(A) \) of \( A \) with respect to \( w \) is obtained by taking the configurations \( \{ b \in A \mid (w(b), b) \in F \} \) for all lower faces \( F \) of \( \mathcal{L}_w(A) \) (with respect to the first coordinate; by
definition, these are exactly the bounded faces). So the elements of \( \Sigma_w(\mathcal{A}) \) are the projections of the bounded faces of \( \mathcal{L}_w(\mathcal{A}) \) to the last \( d \) coordinates.

We can now state the relationship between tight-spans and regular subdivisions of point configurations:

**Proposition 2.3** (Proposition 2.1 in [13]). The polyhedron \( \mathcal{E}_w(\mathcal{A}) \) is affinely equivalent to the polar dual of the polyhedron \( \mathcal{L}_w(\mathcal{A}) \). Moreover, the face poset of \( \mathcal{I}_w(\mathcal{A}) \) is anti-isomorphic to the face poset of the interior lower faces (with respect to the first coordinate) of \( \mathcal{L}_w(\mathcal{A}) \).

We shall not define all the notions of this proposition, but note that, as a consequence, the (inclusion) maximal faces of the tight-span \( \mathcal{I}_w(\mathcal{A}) \) correspond to the (inclusion) minimal interior faces of \( \Sigma_w(\mathcal{A}) \). Here, a face of \( \Sigma_w(\mathcal{A}) \) is an interior face if it is not entirely contained in the boundary of \( \operatorname{conv} \mathcal{A} \). In particular, the structure of \( \mathcal{I}_w(\mathcal{A}) \) determines the structure of \( \Sigma_w(\mathcal{A}) \) and vice versa.

We now consider splits of point configurations (see [15] for details on splits of polytopes and [14] for generalisations to point configurations): A split \( T \) of a point configuration \( \mathcal{A} \) is a subdivision of \( \mathcal{A} \) which has exactly two maximal faces denoted by \( T_+ \) and \( T_- \). The affine hull of \( T_+ \cap T_- \) is a hyperplane \( H_T \) (in the affine hull of \( \mathcal{A} \)), the split hyperplane of \( T \) with respect to \( \mathcal{A} \). Conversely, it is easily seen that a hyperplane defines a split of \( \mathcal{A} \) if and only if its intersection with the (relative) interior of \( \mathcal{A} \) is nontrivial and it does not separate any edge of \( \mathcal{A} \). A set \( \mathcal{T} \) of splits of \( \mathcal{A} \) is called compatible if for all \( T_1, T_2 \in \mathcal{T} \) the intersection of \( H_{T_1} \cap H_{T_2} \) with the relative interior of \( \operatorname{conv} \mathcal{A} \) is empty.

The following observation, which is a slight generalisation of [15, Proposition 4.6], characterises when the tight-span of a point configuration is a tree and will be the key to some of our results.

**Proposition 2.4.** Let \( \mathcal{A} \) be a point configuration and \( w : \mathcal{A} \to \mathbb{R} \) a weight function. Then the tight-span \( \mathcal{I}_w(\mathcal{A}) \) is a tree if and only if the subdivision \( \Sigma_w(\mathcal{A}) \) is a common refinement of compatible splits of \( \mathcal{A} \).

An important theorem concerning splits of point configurations is the Split Decomposition Theorem; see [15] Theorem 3.10 and [18] Theorem 2.2]. It states that each weight function \( w \) inducing a subdivision \( \Sigma_w(\mathcal{A}) \) of a point configuration \( \mathcal{A} \) can be uniquely decomposed in a certain coherent way into a split prime weight function and a sum of split weight functions. Here, we only need the following easily proven corollary of this fact:

**Corollary 2.5.** Let \( \mathcal{A} \) be a point configuration and \( w : \mathcal{A} \to \mathbb{R} \) a weight function such that \( \Sigma_w(\mathcal{A}) \) is a common refinement of a set of \( \mathcal{T} \) compatible splits of \( \mathcal{A} \). Then there exists a function \( \alpha : \mathcal{T} \to \mathbb{R}_{>0} \) such that

\[
    w = \sum_{T \in \mathcal{T}} \alpha(T)w_T.
\]

### 3. Splits of Sets and Point Configurations

When relating the tight-span of metrics on \( X \) and the point configuration \( \mathcal{A}(X) \) (the set of vertices of the second hypersimplex \( \Delta(2, X) \)), it is a key observation that a split of the set \( X \) corresponds to a split of the point configuration \( \mathcal{A}(X) \) and vice versa. We now explain how this fact leads to some further relationships between splits of \( X \) and splits of \( \mathcal{A}(X) \).

Let \( X \) be a finite set and \( A, B \subset X \) two non-empty subsets with \( A \cap B = \emptyset \). The collection \( \{A, B\} \) is called a partial split of \( X \). The pair \( (A, B) \) is called a directed partial split of \( X \). If in addition \( A \cup B = X \), we call \( \{A, B\} \) a split of \( X \) and \( (A, B) \) a directed split of \( X \). For a subset
$C \subset X$ a (partial) split $\{A, B\}$ of $X$ is said to *split* $C$ if neither of the intersections $A \cap C$ or $B \cap C$ is empty.

Two partial splits $\{A, B\}$, $\{C, D\}$ of $X$ are called *compatible* if one of the following four conditions is satisfied:

\begin{align*}
A \subseteq C & \quad \text{and} \quad B \supseteq D, \\
A \subseteq D & \quad \text{and} \quad B \supseteq C, \\
A \supseteq C & \quad \text{and} \quad B \subseteq D, \\
A \supseteq D & \quad \text{and} \quad B \subseteq C.
\end{align*}

(3.1)

Note that this condition is equivalent to saying that there exits $E \in \{A, B\}$ and $F \in \{C, D\}$ such that $E \cap F = \emptyset$, a characterisation usually taken as the definition of compatibility for splits of $X$ (see, e.g., [23]).

Two directed partial splits $(A, B)$, $(C, D)$ of $X$ are called *compatible* if one of the following four conditions is satisfied:

\begin{align*}
A \subseteq C & \quad \text{and} \quad B \supseteq D, \\
A \subseteq X \setminus C & \quad \text{and} \quad B \supseteq X \setminus D, \\
A \supseteq C & \quad \text{and} \quad B \subseteq D, \\
X \setminus A \supseteq C & \quad \text{and} \quad B \subseteq X \setminus D.
\end{align*}

(3.2)

Note that Conditions (3.1) and (3.2) are equivalent for (directed) splits. A set $S$ of partial (directed) splits of $X$ is called *compatible* if each two elements of $S$ are compatible. Furthermore, a set $S$ of directed splits of $X$ is called *strongly* compatible if there exists an ordering $(A_1, B_1), \ldots, (A_l, B_l)$ of the elements of $S$ such that $A_i \subseteq A_{i+1}$ and $B_i \supseteq B_{i+1}$ for all $1 \leq i < l$.

We now investigate the relation of these different kinds of compatibility with compatibility of splits of the point configurations defined in the introduction.

First, we consider the point configuration $\mathcal{A}(X)$. Splits of this point configuration were first studied by Hirai [17, 18].

**Proposition 3.1** (Proposition 4.4 in [18]). Let $X$ be a finite set. For a partial split $\{A, B\}$ of $X$ the hyperplane given by the equation

$$\sum_{i \in A} f(i) = \sum_{j \in B} f(j)$$

defines a split of the point configuration $\mathcal{A}(X)$. Moreover, all splits of $\mathcal{A}(X)$ arise in this way.

The compatibility can be characterised as follows:

**Proposition 3.2** (Theorem 2.3 in [17]). A set $S$ of partial splits of $X$ is compatible if and only if $\{T_S \mid S \in S\}$ is a compatible set of splits of $\mathcal{A}(X)$.

Now we consider the point configurations $\mathcal{B}(X, Y)$, first describing what their splits are.

**Proposition 3.3.** Let $X$ and $Y$ be two disjoint finite sets and $A \subseteq X$, $B \subseteq Y$ non-empty. Then the hyperplane given by

$$\sum_{i \in A} f(i) = \sum_{j \in B} f(j)$$

(3.3)

defines a split of the point configuration $\mathcal{B}(X, Y)$. Moreover, all splits of $\mathcal{B}(X, Y)$ arise in this way.

Note that taking the complements of $A$ and $B$ simultaneously yields the same split of $\mathcal{B}(X, Y)$ (but there are no other choices).
Proof. First we remark that for all non-empty $A \subseteq X$, $B \subseteq Y$ the function $f \in \mathbb{R}^{X \cup Y}$ defined by

$$f(i) = \begin{cases} \frac{1}{|A|}, & \text{if } i \in A, \\ \frac{1}{|X \setminus A|}, & \text{if } i \in X \setminus A, \\ \frac{1}{2|X|}, & \text{if } i \in B, \\ \frac{1}{2|Y|}, & \text{if } i \in Y \setminus B, \end{cases}$$

is in the interior of $P$ since $0 < f(i) < 1$ for all $i \in X \cup Y$ and $\sum_{i \in X \cup Y} f(i) = 1$. It is also in the hyperplane defined by Equation (3.3), since $\sum_{i \in A} f(i) = 1/2$, $\sum_{i \in B} f(i) = 1/2$. Hence all those hyperplanes meet the interior of $\mathcal{B}(X,Y)$.

It is easily seen that two vertices $u = (f_1, f_2), v = (g_1, g_2) \in \mathbb{R}^X \times \mathbb{R}^Y$ of $\mathcal{B}(X,Y)$ are connected by an edge if and only if $f_1 = g_1$ or $g_2 = g_2$. So by going from $u$ to $v$ along an edge, the value on at most one side of Equation (3.3) changes by at most 1. Since all values that occur are integers, this implies that the corresponding hyperplane does not cut an edge of $\mathcal{B}(X,Y)$ and hence defines a split of $\mathcal{B}(X,Y)$.

Now, let $H = \{f \in \mathbb{R}^{X \cup Y} \mid \sum_{i \in X \cup Y} \alpha_i f(i) = 0\}$ define a split of $\mathcal{B}(X,Y)$ for some $\alpha_i \in \mathbb{R}$. We can assume that the first non-zero $\alpha_i$ is equal to 1. However, since the matrix of vertices of a product of simplices is totally unimodular (i.e., all the determinants of all square submatrices are in $\{0, 1, -1\}$), all other non-zero $\alpha_j$ have to be equal to $\pm 1$. Also, note that the product of simplices $\text{conv} \mathcal{B}(X,Y)$ has $|V|$ facets that are isomorphic to $|X|$-dimensional simplices. The hyperplane $H$ has to meet at least one of these facets non-trivially. Since simplices have no splits, $H$ has to have a face of this facet $F$. So we can conclude that we cannot have $\alpha_i = -\alpha_j$ for $i, j \in [k]$ since $H$ would then meet the interior of $F$. The only remaining possibility for $H$ is therefore Equation (3.3) for arbitrary $A$ and $B$. Since for $A = \emptyset$, $B = \emptyset$, $A = X$, or $B = Y$ this hyperplane would not intersect the relative interior of $\text{conv} \mathcal{B}(X,Y)$, the proof is complete. □

Thus, a split of the point configuration $\mathcal{B}(X,Y)$ is defined by two sets $\emptyset \neq A \subseteq X$ and $\emptyset \neq B \subseteq Y$ and this representation is unique up to simultaneously taking the complements of $A$ and $B$. Compatibility can now be characterised as follows

**Proposition 3.4.** Let $T$ be a split of $\mathcal{B}(X,Y)$ defined by $A \subseteq X$ and $B \subseteq Y$, and $U$ a split of $\mathcal{B}(X,Y)$ defined by $C \subseteq X$ and $D \subseteq Y$. Then $T$ and $U$ are compatible if and only if one of the following conditions is satisfied:

\begin{equation}
A \subseteq C \quad \text{and} \quad B \supseteq D, \\
A \subseteq X \setminus C \quad \text{and} \quad B \supseteq Y \setminus D, \\
A \supseteq C \quad \text{and} \quad B \subseteq D, \quad \text{or} \\
X \setminus A \supseteq C \quad \text{and} \quad B \subseteq Y \setminus D.
\end{equation}

Proof. We define the sets

$$A_1 = A \setminus C, \quad A_2 = C \setminus A, \quad A_3 = A \cap C, \quad A_4 = X \setminus (A \cup C),$$

$$B_1 = B \setminus D, \quad B_2 = D \setminus B, \quad B_3 = B \cap D, \quad B_4 = Y \setminus (B \cup D);$$

and set

$$X_i = \sum_{i \in A_i} f(i), \quad Y_i = \sum_{i \in B_i} f(i).$$

Then the hyperplanes for $S$ and $T$ are defined by

\begin{equation}
X_1 + X_3 = Y_1 + Y_3 \quad \text{and} \quad X_2 + X_3 = Y_2 + Y_3,
\end{equation}
respectively. If we subtract these two equations, we get
\begin{equation}
X_1 - X_2 = Y_1 - Y_2.
\end{equation}

We first prove that Condition (3.4) is sufficient for compatibility of $T$ and $U$. So suppose $S$ and $T$ were not compatible, $A \subseteq C$, and $D \subseteq B$. This implies that for any $x \in H_S \cap H_T$ we have $X_1 = Y_2 = 0$. From this and Equation (3.6) we can further conclude that $X_2 = Y_1 = 0$. So $x_i = 0$ for all $i \in A_2 \cup B_1$. Since $x$ should not be in the boundary of $\text{conv } \mathcal{B}(X, Y)$, this implies that $A_2$ and $B_1$ are empty, and so $S = T$. The second case follows similarly.

For the necessity, assume that (3.4) does not hold. This is equivalent to
\begin{equation}
A_1 \not= \emptyset \text{ or } B_2 \not= \emptyset, A_2 \not= \emptyset \text{ or } B_1 \not= \emptyset, A_3 \not= \emptyset \text{ or } B_3 \not= \emptyset, \text{ and } A_4 \not= \emptyset \text{ or } B_4 \not= \emptyset.
\end{equation}

We will now distinguish several cases, depending on the number of sets $A_j$, $B_j$ are empty. In each case we will give a point $x \in \text{relint}(\text{conv } \mathcal{B}(X, Y)) \cap H_S \cap H_T$. This will be done by assigning values in the interval $[0, 1)$ to all $X_j, Y_j$ for which $A_j, B_j$, respectively, are non-empty such that Equation (3.6) holds and $\sum X_j = \sum Y_j = 1$. The explicit values of $f \in \mathbb{R}^{X \cup Y}$ are then obtained by setting $f(i) = \frac{x_i}{|A_j|}$, $f(i) = \frac{y_i}{|B_j|}$, for $i \in A_j$, $i \in B_j$, respectively.

**Case 1:** None of the sets $A_j, B_j$ are empty. Then we simply set $X_j, Y_j = \frac{1}{4}$ for all $j \in \{1, 2, 3, 4\}$.

**Case 2:** One of the sets $A_j, B_j$ is empty. We assume without loss of generality that $A_1 = \emptyset$. Then we set $X_3 = \frac{3}{4}$, $X_4 = Y_2 = \frac{1}{4}$, $Y_1 = Y_3 = \frac{1}{4}$, and $X_2 = Y_4 = \frac{1}{4}$.

**Case 3:** Two of the sets $A_j, B_j$ are empty. As in Case 2, we assume that one of these sets is $A_1$. Using (3.7), and taking into account that neither $A, B, C, D$ nor their complements (in $X$ and $Y$, respectively) can be empty, we get the following possibilities:

\begin{itemize}
  \item $A_1 = A_2 = \emptyset$: Set $X_3 = X_4 = \frac{3}{4}$ and $Y_1 = \frac{1}{4}$ for all $i \in \{1, 2, 3, 4\}$.
  \item $A_1 = B_1 = \emptyset$: Set $X_3 = Y_1 = \frac{1}{4}$ and $X_2 = X_4 = Y_2 = Y_4 = \frac{1}{4}$.
  \item $A_1 = B_3 = \emptyset$: Set $X_4 = Y_2 = \frac{1}{4}$ and $X_2 = X_3 = Y_1 = Y_4 = \frac{1}{4}$.
  \item $A_1 = B_4 = \emptyset$: Set $X_3 = Y_2 = \frac{1}{4}$ and $X_2 = X_4 = Y_1 = Y_3 = \frac{1}{4}$.
\end{itemize}

**Case 4:** Three of the sets $A_j, B_j$ are empty. We again assume that $A_1$ is one of the sets. There remain three possibilities:

\begin{itemize}
  \item $A_1 = A_2 = B_3 = \emptyset$: Set $X_4 = \frac{2}{3}$ and $X_3 = Y_1 = Y_2 = Y_4 = \frac{1}{3}$.
  \item $A_1 = A_2 = B_4 = \emptyset$: Set $X_3 = \frac{1}{3}$ and $X_4 = Y_1 = Y_2 = Y_3 = \frac{1}{3}$.
  \item $A_1 = B_3 = B_4 = \emptyset$: Set $Y_2 = \frac{2}{3}$ and $X_2 = X_3 = X_4 = Y_1 = \frac{1}{3}$.
\end{itemize}

**Case 5:** Four of the sets $A_j, B_j$ are empty. By assuming that $A_1$ is one of them, this yields $A_1 = A_2 = B_3 = B_4 = \emptyset$. Set $X_3 = X_4 = Y_1 = Y_2 = \frac{1}{3}$.

Note that in the special case where $Y$ is a disjoint copy of $X$ a partial directed split $S = (A, B)$ of $X$ gives rise to a split $T_S$ of $\mathcal{B}(X)$, namely the one defined by $A$ and $B$. In general, not all splits of $\mathcal{B}(X)$ arise in this way, since we assume that $A$ and $B$ are disjoint. Proposition 3.4 then gives us the following.

**Corollary 3.5.** A collection $S$ of directed partial splits of $X$ is compatible if and only if $\{T_S | S \in S\}$ is a compatible system of splits for $\mathcal{B}(X)$.

Note that a characterisation of the splits of the point configuration $C(X)$ (the vertices of the cube) and their compatibility is given in [13] Propositions 3.15 and 3.16; we refrain from stating or proving it here since we will not use it later.
4. Tight-Spans of Symmetric Maps, Distances and Metrics

A symmetric map $D : X \times X \to \mathbb{R}$ with $D(x, x) = 0$ for all $x \in X$ and $D(x, y) \geq 0$ for all $x, y \in X$ is called a distance (or dissimilarity map) on $X$. It is called a metric on $X$ if it additionally satisfies $D(x, y) + D(y, z) \geq D(x, z)$ for all $x, y, z \in X$ (triangle inequality).

Now, consider the vector space $\mathbb{R}^X := \{f : X \to \mathbb{R}\}$ of all functions $X \to \mathbb{R}$ with the natural basis $\{e_x | x \in X\}$ where $e_x \in \mathbb{R}^X$ denotes the function sending $x$ to 1 and all other elements of $X$ to 0. Then the tight-span $T_D$ of a symmetric map $D : X \times X \to \mathbb{R}$ is defined to be the set of all minimal elements of the polyhedron

$$ P_D := \left\{ f \in \mathbb{R}^X \mid f(x) + f(y) \geq D(x, y) \text{ for all } x, y \in X \right\}. $$

Note that if $D$ is a distance, this is exactly the definition of $P_D$ and $T_D$ given by Hirai [17, Section 2.3]. Also, if $D$ is a metric, $T_D$ is easily seen to correspond to the tight-span of the metric $D$ as defined by Isbell [20] and Dress [8].

Now, given a symmetric map $D : X \times X \to \mathbb{R}$ we define a weight function $w_D : \mathcal{A}(X) \to \mathbb{R}$ on the point configuration $\mathcal{A}(X)$ via $w_D(e_x + e_y) = -D(x, y)$. The following proposition is the key to deriving the relation between tight-spans of symmetric functions and tight-spans of the point configuration $\mathcal{A}(X)$. In the special case where $D$ is a metric, this was the observation of Sturmfels and Yu [24] mentioned in the introduction.

**Proposition 4.1.** Let $D : X \times X \to \mathbb{R}$ be a symmetric function. Then we have:

(a) $P_D = \mathcal{E}_{w_D}(\mathcal{A}(X))$, and
(b) $T_D = \mathcal{T}_{w_D}(\mathcal{A}(X))$.

**Proof.** (a) We have

$$ P_D = \left\{ f : X \to \mathbb{R} \mid f(x) + f(y) \geq D(x, y) \text{ for all } x, y \in X \right\} $$

$$ = \left\{ f : X \to \mathbb{R} \mid \langle e_x + e_y, f \rangle \geq D(x, y) \text{ for all } x, y \in X \right\} $$

$$ = \left\{ f : X \to \mathbb{R} \mid \langle a, f \rangle \geq w_D(a) \text{ for all } a \in \mathcal{A}(X) \right\} $$

$$ = \mathcal{E}_{w_D}(\mathcal{A}(X)). $$

(b) Obviously, $\mathcal{A}(X)$ is positive and $\mathcal{E}_{w_D}(\mathcal{A}(X))$ is bounded from below since we have the inequalities $2e_x \geq -D(x, x)$ for all $x \in X$. Thus, the claim follows from (a) and Lemma 2.2.

We will now see that tight-spans of general symmetric maps and tight-spans of distances are essentially the same in the sense that they only differ by a simple shift.

**Proposition 4.2.** Let $D : X \times X \to \mathbb{R}$ be a symmetric function, $D'$ defined via

$$ D'(x, y) := D(x, y) - \frac{1}{2} (D(x, x) + D(y, y)), $$

and $v : X \to \mathbb{R}$ defined by $v(x) = \frac{1}{2} D(x, x)$. Then $T_D = T_{D'} + v$.

**Proof.** Our definitions imply that for $a = e_x + e_y \in \mathcal{A}(X)$, we have

$$ w_{D'}(a) = -D(x, y) + \frac{1}{2} (D(x, x) + D(y, y)) = w_D(a) + \langle a, v \rangle. $$

Hence the claim follows from Lemma 2.2.

Obviously, for an arbitrary symmetric map $D : X \times X \to \mathbb{R}$ we have $D'(x, x) = 0$ for all $x \in X$. However, $D'$ is not necessarily a distance, since $D'$ need not to be positive, even if $D$ is. Even
so, the following lemma shows that the negative values of $D$ can be ignored when looking at the tight-span:

**Lemma 4.3.** Let $D : X \times Y \to \mathbb{R}$ be a symmetric function with $D(x,x) = 0$ for all $x \in X$, and $D_+$ be defined by $D_+(x,y) := \max(0,D(x,y))$ for all $x,y \in X$. Then $T_D = T_{D_+}$.

**Proof.** For two symmetric maps $E, F : X \to \mathbb{R}$ with $E \geq F$ (pointwise) one directly sees that $P_E \subseteq P_F$, so $P_{D_+} \subseteq P_D$. On the other hand, for all $f \in P_D$ and $x \in X$, we have $2f(x) \geq D(x,x) = 0$, hence $f(x) + f(y) \geq 0 \geq \max(0,D(x,y)) = D_+(x,y)$ for all $x,y \in X$. This implies $P_D \subseteq P_{D_+}$. Altogether, we have $P_D = P_{D_+}$ and hence $T_D = T_{D_+}$.

So, by Proposition 4.2 and Lemma 4.3 we get:

**Corollary 4.4.** For any symmetric map $D : X \times X \to \mathbb{R}$ there exists a distance map (namely $(D'_+)_+ : X \times X \to \mathbb{R}$) and some $v \in \mathbb{R}^X$ such that $T_D = T_{(D'_+)_+} + v$.

We now turn to trees: Given a tree $T = (V,E)$, an edge-length function $\alpha : E \to \mathbb{R}$ and a family $\mathcal{F} = \{F_x | x \in X\}$ of subtrees of $T$, we can define a distance $D$ on $X$ by setting $D(x,y) = \min\{D_T(x,y) | x \in F_x, y \in F_y\}$. A distance arising in that way is called a distance between subtrees of a tree. Those distances can also be characterised in terms of partial splits: Given a partial split $P = \{A,B\}$ of $X$, a corresponding distance on $X$ is the defined by

$$d_p(i,j) = \begin{cases} 1, & \text{if } i \in A, j \in B \text{ or } i \in B, j \in A, \\ 0, & \text{else}, \end{cases}$$

for all $i,j \in X$. Furthermore, for a set $\mathcal{P}$ of partial splits of $X$ and a function $\alpha : \mathcal{P} \to \mathbb{R}_{>0}$, we define $d_{(\mathcal{P},\alpha)} = \sum_{P \in \mathcal{P}} \alpha(P)d_P$.

Using the relation between splits of the point configuration $\mathcal{A}(X)$ and partial splits of $X$ (Propositions 3.1 and 3.2), the following result easily follows, or it can alternatively be inferred from Hirai [17, Theorem 2.3] together with Proposition 4.4.

**Theorem 4.5.** Let $D$ be a distance. Then the following are equivalent:

(a) The tight-span $T_{\alpha}(\mathcal{A}(X))$ is a tree.

(b) $D$ is a distance of subtree weights of a tree.

(c) There exists a compatible set $\mathcal{P}$ of partial splits of $X$ and $\alpha : \mathcal{P} \to \mathbb{R}_{>0}$ such that $D = d_{(\mathcal{P},\alpha)}$.

5. Tight-Spans of Non-Symmetric Maps

Hirai and Koichi [19] introduced the concept of tight-spans of directed distances in order to study certain multicommodity flow problems. In this section, we will show that these tight-spans can also be considered as tight-spans of point configurations and we will generalise their concept to general non-symmetric maps.

Let $X$ and $Y$ be finite sets with $X \cap Y = \emptyset$ and $D : X \times Y \to \mathbb{R}$ an (arbitrary) map. We define the polyhedra

$$\Pi_D := \{f : X \cup Y \to \mathbb{R} | f(x) + f(y) \geq D(x,y) \text{ for all } x \in X, y \in Y\},$$

and

$$P_D := \Pi_D \cap \mathbb{R}_{\geq 0}^{X \cup Y}.$$

In addition, the sets of minimal elements of $\Pi_D$ and $P_D$ are called $\Theta_D$ and $T_D$, respectively. The set $T_D$ is called the tight-span of $D$.

Recall that $\mathcal{B}(X,Y) \subseteq \mathbb{R}_{\geq 0}^{X \cup Y}$ is defined as the configuration of all points $e_x + e_y$ with $x \in X, y \in Y$ and also that $\mathcal{B}(X,Y) := \mathcal{B}(X,Y) \cup \{2e_x | x \in X \cup Y\}$. Note that $\mathcal{B}(X,Y), \bar{\mathcal{B}}(X,Y) \subseteq \mathcal{B}(X \cup Y)$
and that \( \text{conv } \mathcal{B}(X, Y) \) is the simplex \( \text{conv}\{2e_x | x \in X \cup Y\} \), whereas all elements of \( \tilde{\mathcal{B}}(X, Y) \) are vertices of \( \text{conv } \tilde{\mathcal{B}}(X, Y) \), which is the product of a \((|X| - 1)\) - and a \((|Y| - 1)\)-dimensional simplex.

To the map \( D : X \times Y \to \mathbb{R} \) we associate weight functions \( w^D : \mathcal{B}(X, Y) \to \mathbb{R} \) by \( \tilde{w}^D(e_x + e_y) = -D(x, y), x \in X, y \in Y \) and \( w^D : \mathcal{B}(X, Y) \to \mathbb{R} \) by

\[
    w^D(a) = \begin{cases} 
        \tilde{w}^D(a), & \text{if } a = e_x + e_y, x \in X, y \in Y, \\
        0, & \text{else}.
    \end{cases}
\]

The following can be proven in a similar way to Proposition 4.1.

**Proposition 5.1.** Let \( D : X \times Y \to \mathbb{R} \), then we have

1. \( \Pi_D = \mathcal{E}^{\mathcal{B}}(\tilde{\mathcal{B}}(X, Y)) \)
2. \( \Theta_D = T^{\mathcal{B}}(\tilde{\mathcal{B}}(X, Y)) \)
3. \( P_D = \mathcal{E}^{\mathcal{B}}(\mathcal{B}(X, Y)) \), and
4. \( T_D = T^{\mathcal{B}}(\mathcal{B}(X, Y)) \).

It was shown in [7, Lemma 22] that \( \Theta_D \) is piecewise-linear isomorphic to the tropical polytope (see, e.g., Devlin and Sturmfels [4]) with vertex set \( \{(D(x, y))_{x \in X} | x \in X\} \). Similarly, using a proof like that for Lemma 4.3, we have:

**Lemma 5.2.** Let \( D : X \times Y \to \mathbb{R} \) and \( D_* \) defined by \( D_*(x, y) := \max(0, D(x, y)) \) for all \( x \in X, y \in Y \). Then \( T_D = T_{D_*} \).

Of particular interest is the case where \( Y \) is a disjoint copy of \( X \), that is, \( D \) is a (not necessarily symmetric) function from \( X \times X \) to \( \mathbb{R} \). We denote the two distinct copies of \( X \) by \( X_l \) and \( X_r \) and set \( X_d := X_l \cup X_r \). If in this case \( D \geq 0 \) and \( D(x, x) = 0 \) for all \( x \in X \) then \( D \) is called a directed distance; and if \( D \) also satisfies the triangle inequality, it is called a directed metric. These were considered by Hirai and Koichi [13]. In fact, our definitions of \( \Pi_d, \Theta_d, P_d \) and \( T_d \) are generalisations of their definitions.

As in the case of symmetric maps, it can be deduced from Lemma 2.2 that a map \( D : X \times X \to \mathbb{R} \) can be transformed into a map with \( D(x, x) = 0 \) for all \( x \in X \) by shifting it by a vector \( v \in \mathbb{R}^{X_d} \) defined by \( v(x) = 1/2D(x, x) \) for \( x \in X_d \) (here \( x \) is identified with its copy in \( X \)). Together with Lemma 5.2, this leads to the following corollary.

**Corollary 5.3.** For each map \( D : X \times X \to \mathbb{R} \) there exists a directed distance \( D' \) on \( X \) and some \( v \in \mathbb{R}^{X_d} \) such that \( T_D = T_{D'} + v \).

To a directed distance \( D \) on \( X \) we associate a (symmetric) distance \( D^\# : X_d \times X_d \to \mathbb{R}_{\geq 0} \) by setting

\[
    D^\#(x, y) = \begin{cases} 
        D(x, y), & \text{if } x \in X_l \text{ and } y \in X_r \text{ or } x \in X_r \text{ and } y \in X_l, \\
        0, & \text{else}.
    \end{cases}
\]

It follows that \( P^\#_{D^\#} = P_D \) and \( T^\#_{D^\#} = T_D \). So tight-spans of undirected distances are just tight-spans of special directed distances.

In [7], Hirai and Koichi give explicit conditions on \( D \) for when \( T_D \) and \( \Theta_D \) are trees. We now give new and we feel somewhat conceptually simpler proofs of these results using point configurations.

We begin by recalling some basic definitions from [4, Section 3]. An oriented tree \( \Gamma = (V(\Gamma), E(\Gamma)) \) is a directed graph whose underlying undirected graph is a tree. For an oriented tree \( \Gamma \) and an edge-length function \( \alpha : E(\Gamma) \to \mathbb{R}_{\geq 0} \), we define a directed distance \( D_{\Gamma, \alpha} \) on \( V(\Gamma) \) by setting \( D_{\Gamma, \alpha}(x, y) = \sum_{e \in \Pi(x, y)} \alpha(e) \) for all \( x, y \in V(\Gamma) \), where \( \Pi(x, y) \) is the set of all edges on the unique (undirected) path from \( x \) to \( y \) that are directed from \( x \) to \( y \). For \( A, B \subseteq V(\Gamma) \) we
set $D_{\Gamma,\alpha}(A,B) := \min\{D_{\Gamma,\alpha}(a,b)\mid a \in A, b \in B\}$ For an undirected distance $D : X \times X \to \mathbb{R}$ and a family $\mathcal{F} = \{F_x \mid x \in X\}$ of subtrees of $\Gamma$, we say that $(\Gamma, \alpha, \mathcal{F})$ is an oriented tree realisation of $D$ if

$$D(x,y) = D_{\Gamma,\alpha}(F_x, F_y) \text{ for all } x, y \in X.$$ 

Now, let $S = (A,B)$ be a partial directed split of $X$. We can define a directed distance $D_S$ on $X$ by setting

$$D_S(x,y) = \begin{cases} 1, & \text{if } x \in A \text{ and } y \in B, \\ 0, & \text{else.} \end{cases}$$

Note that $D_S$ is a directed metric if and only if $S$ is a directed split. Also note that the oriented tree $\Gamma$ consisting of a single edge $(v,w)$ with weight $\lambda$ and

$$F_x = \begin{cases} \{v\} & \text{if } x \in A, \\ \{w\} & \text{if } x \in B, \\ \Gamma & \text{else,} \end{cases}$$

for all $x \in X$, is an oriented tree realisation of $D_S$.

Now let $D$ be an arbitrary directed distance with oriented tree realisation $(\Gamma, \alpha, \mathcal{F})$. For each $e = (a,b) \in E(\Gamma)$ we define a directed partial split $S_e = (A_e, B_e)$ of $X$, where $A_e$ is the set of all $x \in X$ whose subtree $F_x \in \mathcal{F}$ is entirely contained in the same connected component of $\Gamma \setminus e$ as $a$ and $B_e$ the same for $b$. (Here $\Gamma \setminus e$ denotes the forest obtained from $\Gamma$ by deleting the edge $e$.) It is now easily seen, that $D = \sum_{e \in E(\Gamma)} \alpha(e)D_{S_e}$.

We can now show the following:

**Proposition 5.4.** Let $S$ be a set of partial directed splits, $\alpha : S \to \mathbb{R}_{>0}$ a function, and $D$ the directed distance defined by

$$D = \sum_{S \in \mathcal{S}} \alpha(S)D_S.$$  

Then we have:

(a) $D$ has an oriented tree realisation $(\Gamma, \alpha, \mathcal{F})$ where each $F \in \mathcal{F}$ is a directed path if and only if $S$ is compatible.

(b) $D$ has an oriented tree realisation $(\Gamma, \alpha, \mathcal{F})$ such that $\Gamma$ is a directed path if and only if $S$ is strongly compatible.

**Proof.**

(a) Let $e$ and $f$ be two edges of $\Gamma$ and $S_e$ and $S_f$ the associated directed partial splits. We have to show that $S_e$ and $S_f$ are compatible. Suppose first that in any undirected path in $\Gamma$ containing $e$ and $f$ these two edges are directed in the same direction. (Since $\Gamma$ is a tree, this is the case if there exists some undirected path with this property.) Then (possibly after exchanging $e$ and $f$) we have $A_e \subseteq A_f$ and $B_f \subseteq B_e$, which implies that $S_e$ and $S_f$ are compatible by the first condition of $(5,2)$.

Now, suppose that in any undirected path in $\Gamma$ containing $e$ and $f$ these two edges are directed in the different directions. Because each $F \in \mathcal{F}$ is a directed path, this implies that there does not exist an $F \in \mathcal{F}$ that contains $e$ as well as $f$. Hence, we get (again after a possible exchange of $e$ and $f$) $X \setminus A_e \subseteq A_f$ and $B_e \subseteq X \setminus B_f$, implying that $S_e$ and $S_f$ are compatible by the third condition of $(5,2)$. On the other hand, given a compatible set of directed partial splits of $X$, it is easily seen that we can construct a corresponding tree $\Gamma$ with a family of subtrees $\mathcal{F}$ that are directed paths such that $(\Gamma, \alpha, \mathcal{F})$ is an oriented tree realisation of $D$.

(b) Follows directly from the definition. \(\square\)
Theorem 5.5 (Theorems 3.1 and 3.2 in [4]). Let $D$ be a directed distance on $X$. Then we have:

(a) $\Theta_D$ is a tree if and only if $D$ has an oriented tree realisation $(\Gamma, \alpha, \mathcal{F})$ such that each $F \in \mathcal{F}$ is a directed path.

(b) $T_D$ is a tree if and only if $D$ has an oriented tree realisation $(\Gamma, \alpha, \mathcal{F})$ such that $\Gamma$ is a directed path.

Proof. 

(a) Let $D$ be a directed distance on $X$ such that $D$ has an oriented tree realisation $(\Gamma, \alpha, \mathcal{F})$ such that each $F \in \mathcal{F}$ is a directed path. By Proposition 5.4 (a), this implies that there exists a set $S$ of partial directed splits of $X$ such that

$$D = \sum_{S \in S} \alpha(S)D_S.$$ 

For each $S \in S$ and $a \in \tilde{B}(X)$ we have

$$\bar{w}^{D_S}(a) = \begin{cases} -1, & \text{if } a = e_x + e_y \text{ and } x \in A \text{ and } y \in B, \\ 0, & \text{else,} \end{cases}$$

and so it is immediately seen that $\bar{w}^{D_S}$ defines the split $T_S$ of $\tilde{B}(X)$. By Corollary 3.5, the set $\{T_S \mid S \in S\}$ is a compatible set of splits for $\tilde{B}(X)$, and so the subdivision $\Sigma_{\bar{w}^{D}}(\tilde{B}(X))$ is a common refinement of compatible splits. Proposition 2.4 now implies that $\Theta_D = \Sigma_{\bar{w}^{D}}(\tilde{B}(X))$ is a tree.

Conversely, let $D$ be a directed distance on $X$ such that $\Theta_D$ is a tree. By Proposition 2.4 there exists a compatible set $\mathcal{T}$ of splits of $\tilde{B}(X)$ such that $\Sigma_{\bar{w}^{D}}(\tilde{B}(X))$ is the common refinement of all splits in $\mathcal{T}$. Since $D$ is a directed distance, we have $w^D(e_x + e_y) = 0$ for all $x \in X$, which implies that each $T \in \mathcal{T}$ is defined by two disjoint subsets of $X$. Hence there exists a partial directed split $S$ with $T = T_S$. Let $S$ be the set of all such splits, which is compatible by Corollary 3.5. By Corollary 2.5 there exists $\alpha_S \in \mathbb{R}_{>0}$ such that $\bar{w}^D = \sum_{T \in \mathcal{T}} \alpha_T \bar{w}^{D_T}$, so $D = \sum_{S \in S} \alpha_S D_S$ and the claim now follows from Proposition 5.4 (a).

(b) The splits of the point configuration $\mathcal{B}(X)$ are given by partial splits of the set $X_d$. However, given a directed distance $D$, by definition, the value $w^D(e_x + e_y)$ can only be non-zero if $x \in X_d$ and $y \in X_d$. This implies that the only possible partial splits $(A, B)$ of $X_d$ that may occur are those with $A \subseteq X_d$ and $B \subseteq X_d$ (or vice versa) and $A \cap B = \emptyset$ (where $A$ and $B$ are considered as subsets of $X$). The splits of this type are in bijection with partial directed splits $(A, B)$ of $X$.

Now, given two such partial splits $(A, B)$, $(C, D)$ of $X$, the corresponding splits of $\mathcal{B}(X)$ are compatible if and only if

$$A \subseteq C \text{ and } B \supseteq D, \text{ or } A \supseteq C \text{ and } B \subseteq D.$$ 

Proposition 5.4 (a) now implies the theorem in a similar way as Proposition 5.4 (a) implies Theorem 5.5.}

In the case where $D$ is a directed metric, that is, satisfies the triangle inequality, it is obvious that in each directed tree realisation $(\Gamma, \alpha, \mathcal{F})$ all $F_x$ are single vertices. In this special case, given a directed tree $\Gamma$, an edge length function $\alpha$ and a map $\varphi : X \rightarrow V(\Gamma)$, we also call $(\Gamma, \alpha, \varphi)$ an oriented tree realisation of $D$ if and only if $(\Gamma, \alpha, \{\varphi(x) \mid x \in X\})$ is an oriented tree realisation of $D$. 

\[\Box\]
Corollary 5.6. Let $D$ be a directed metric on $X$. Then we have:

(a) $\Theta_D$ is a tree if and only if $D$ has an oriented tree realisation.
(b) $T_D$ is a tree if and only if $D$ has an oriented tree realisation $(\Gamma, \alpha, \varphi)$ such that $\Gamma$ is a directed path.

6. Diversities as Distances

Diversities can be regarded as a generalisation of metrics that assign values to all subsets of a given set $Y$ as opposed to pairs in $Y$. They have been studied for example in [12, 21, 22] and, more recently, by Bryant and Tupper [3]. In this section, we will explain how such a diversity can be translated into a distance on the powerset of $Y$. This will allow us to apply the results from Section 4 when we study tight-spans of diversities in the next section.

For a finite set $Y$, we denote by $\mathcal{P}^*(Y) = \mathcal{P}(Y) \setminus \{\emptyset\}$ the set of non-empty subsets of $Y$. As mentioned in the introduction, diversity on a set $Y$ is a function $\delta : \mathcal{P}(Y) \to \mathbb{R}$ satisfying:

(D1) $\delta(A \cup B) + \delta(B \cup C) \geq \delta(A \cup C)$ for all $A, C \in \mathcal{P}(Y)$ and $B \in \mathcal{P}^*(Y)$,

(D2) $\delta(A) = 0$ for all $A \in \mathcal{P}(Y)$ with $|A| \leq 1$.

We first note two trivial properties of such maps; for a proof see [3, Proposition 2.1].

Lemma 6.1. Let $\delta$ be a diversity on $Y$ and $A, B \subset Y$.

(a) If $A \cap B \neq \emptyset$, then $\delta(A) + \delta(B) \geq \delta(A \cup B)$.
(b) If $A \subseteq B$, then $\delta(A) \leq \delta(B)$.

Now, for an arbitrary symmetric map $D : \mathcal{P}^*(Y) \times \mathcal{P}^*(Y) \to \mathbb{R}$, we define the following properties:

(A1) $D(A, B) + D(B, C) \geq D(A, C)$ for all $A, B, C \in \mathcal{P}^*(Y)$, (triangle inequality)

(A2) $D([x], [x]) = 0$ for all $x \in Y$, and

(A3) $D(A, B) = \frac{1}{2}D(A \cup B, A \cup B)$ for all $A \neq B \in \mathcal{P}^*(Y)$.

Given a map $\delta : \mathcal{P}(Y) \to \mathbb{R}$, let $D_\delta : \mathcal{P}^*(Y) \times \mathcal{P}^*(Y) \to \mathbb{R}$ be given by

$$D_\delta(A, B) = \begin{cases} 2\delta(A), & \text{if } A = B, \\
\delta(A \cup B), & \text{else.} \end{cases}$$

Obviously, $D_\delta$ is symmetric. Conversely, given a symmetric map $D : \mathcal{P}^*(Y) \times \mathcal{P}^*(Y) \to \mathbb{R}$, we define the function $\delta(D) : \mathcal{P}(Y) \to \mathbb{R}$ by setting $\delta(D)(A) = \frac{1}{8}D(A, A)$ for $A \in \mathcal{P}^*(Y)$ and $\delta(D)(\emptyset) = 0$.

Proposition 6.2. (a) Given an arbitrary function $\delta : \mathcal{P}(Y) \to \mathbb{R}$, the map $D_\delta$ satisfies (A3). Moreover, if $\delta$ is a diversity, then $D_\delta$ also satisfies (A1) and (A2).
(b) If $D : \mathcal{P}^*(Y) \times \mathcal{P}^*(Y) \to \mathbb{R}$ is a symmetric map fulfilling (A1)--(A3), then $D_\delta$ is a diversity.
(c) Let $\delta : \mathcal{P}(Y) \to \mathbb{R}$. Then $\delta(D_\delta) = \delta$.
(d) Let $D : \mathcal{P}^*(Y) \times \mathcal{P}^*(Y) \to \mathbb{R}$ be a symmetric map. Then $D_{\delta(D)} = D$ if and only if (A3) holds.

Proof. (a) For all $A \neq B \in \mathcal{P}^*(Y)$ we have $D_\delta(A, B) = \frac{1}{2}D(A \cup B, A \cup B)$, that is, (A3) holds. If in addition $\delta$ is a diversity, then for all $A, B, C \in \mathcal{P}^*(Y)$ with $A \neq C$ we have

$$D_\delta(A, B) + D_\delta(B, C) \geq \delta(A \cup B) + \delta(B \cup C) \geq \delta(A \cap C) = D_\delta(A, C).$$

So, if $A = C$, using Lemma 6.1 we get

$$D_\delta(A, B) + D_\delta(B, C) = 2D_\delta(A, B) = 2\delta(A \cup B) \geq 2\delta(A) = D_\delta(A, C).$$
Hence (A1) holds. Furthermore, \( D_\delta((\{x\}, \{x\})) = 2\delta(\{x\}) = 0 \) for all \( x \in Y \) by (D2), so (A1) and (A2) also hold.

(b) We have

\[
\delta(D(A \cup B) + \delta(D)(B \cup C) = \frac{1}{2}D(A \cup B, A \cup B) + \frac{1}{2}D(B \cup C, B \cup C) \\
\overset{(A3)}{=} D(A, B) + D(B, C) \overset{(A1)}{\geq} D(A, C) \\
= \frac{1}{2}D(A \cup C, A \cup C) = \delta(D)(A \cup C),
\]

which is (D1). From (A2), we conclude that \( \delta(D)((\{x\}) = D((\{x\}, \{x\}) = 0 \) for all \( x \in Y \), hence (D2) holds.

(c) By definition, \( \delta(D_\delta)(A) = \frac{1}{2}D_\delta(A, A) = \frac{2}{3}\delta(A) = \delta(A) \) for all \( A \in \mathcal{P}(Y) \).

(d) For all \( A \neq B \in \mathcal{P}^*(Y) \) we have

\[
D_{\delta(D)}(A, B) = \delta(D)(A \cup B) = \frac{1}{2}D(A \cup B, A \cup B)
\]

and \( D_{\delta(D)}(A, A) = 2\delta(D)(A) = D(A, A) \). Hence \( D_{\delta(D)} = D \) if and only if condition (A3) is satisfied.

\[\square\]

**Corollary 6.3.** Diversities on \( Y \) are in bijective correspondence with symmetric maps \( \mathcal{P}^*(Y) \times \mathcal{P}^*(Y) \to \mathbb{R} \) satisfying (A1)–(A3).

To not only obtain a symmetric map, but even a distance on \( \mathcal{P}^*(Y) \), one can use the process described in Section 4 to arrive at the distance map \( d_\delta : \mathcal{P}^*(Y) \times \mathcal{P}^*(Y) \to \mathbb{R} \) defined by \( d_\delta = ((D_\delta')^*)_\star \), or, equivalently, define

\[
(6.1) \quad d_\delta(A, B) := \begin{cases} 
\max(0, \delta(A \cup B) - (\delta(A) + \delta(B))), & \text{if } A \neq B, \\
0, & \text{if } A = B.
\end{cases}
\]

**Lemma 6.4.** The mapping \( \alpha \) from the set of diversities on \( Y \) to the set of all distances on \( \mathcal{P}^*(Y) \) defined by \( \delta \mapsto d_\delta \) is injective.

**Proof.** Let \( \delta, \delta' \) be diversities on \( Y \) with \( d_\delta(A, B) = d_{\delta'}(A, B) \) for all \( A, B \in \mathcal{P}^*(Y) \). We will show that \( \delta(A) = \delta'(A) \) implies \( \delta(A \cup \{i\}) = \delta'(A \cup \{i\}) \) for all \( i \in Y \), which will prove the claim by induction. Since we have

\[
\max(0, \delta(A \cup \{i\}) - \delta(A) - \delta(\{i\})) = d_\delta(A, \{i\}) = d_{\delta'}(A, \{i\}) = \max(0, \delta'(A \cup \{i\}) - \delta'(A) - \delta'(\{i\}))
\]

by Lemma 6.1 we also have \( \delta(A \cup \{i\}) - \delta(A) = \delta'(A \cup \{i\}) - \delta'(A) \), which completes the proof. \[\square\]

We conclude with one final observation that will be useful later:

**Lemma 6.5.** Let \( Y \) be a finite set, \( \delta \) a diversity on \( Y \) and \( d_\delta \) the associated distance on \( \mathcal{P}^*(Y) \). Then \( d_\delta(A, B) = 0 \) for all \( A, B \in \mathcal{P}^*(Y) \) with \( A \cap B \neq \emptyset \).

**Proof.** By Lemma 6.1 we have \( \delta(A \cup B) - \delta(A) - \delta(B) \leq 0 \) which implies \( d_\delta(A, B) = 0 \) by the definition of \( d_\delta \). \[\square\]
7. Tight-Spans of Diversities

There are at least two natural ways to define the tight-span of a diversity which, as we shall see, correspond to the tight-span of two types of point configurations. We can either define it to be the set \( T_\delta \) of all minimal elements of the polyhedron

\[
P_\delta := \left\{ f \in \mathbb{R}^{P(Y)} \mid f(\emptyset) = 0 \text{ and } f(A) + f(B) \geq \delta(A \cup B) \text{ for all } A, B \in P(Y) \right\},
\]

or to be the set \( \hat{T}_\delta \) of all minimal elements of the polyhedron

\[
\hat{P}_\delta := \left\{ f \in \mathbb{R}^{P(Y)} \mid f(\emptyset) = 0 \text{ and } \sum_{A \in \mathcal{A}} f(A) \geq \delta \left( \bigcup \mathcal{A} \right) \text{ for all } \mathcal{A} \subseteq P(Y) \right\}.
\]

The latter definition is given by Bryant and Tupper \cite{BT}, whereas the first one appeared in a preliminary version of their paper \cite{BT} Version 1.

As we shall now prove, the first definition arises from the point configuration \( \mathcal{A}(P^*(Y)) \), the configuration of vertices of a cube. Let \( w^\delta : C(P^*(Y)) \to \mathbb{R} \) be given by \( w^\delta(\sum_{i \in \mathcal{I}} e_i) := \delta(\cup \mathcal{A}) \) for all \( \mathcal{A} \subseteq P^*(Y) \).

**Proposition 7.1.** Let \( \delta : P(Y) \to \mathbb{R} \). Then we have:

(a) \( P_\delta = \{0\} \times E_{w_\delta}(\mathcal{A}(P(Y))) \), and \( T_\delta = \{0\} \times \sum_{\mathcal{A}}(\mathcal{A}(P(Y))) \).

(b) \( \hat{P}_\delta = \{0\} \times E_{w^\delta}(C(P(Y))) \), and \( \hat{T}_\delta = \{0\} \times \sum_{\mathcal{A}}(C(P(Y))) \).

**Proof.** (a) We have

\[
P_\delta = \left\{ f \in \mathbb{R}^{P(Y)} \mid f(\emptyset) = 0 \text{ and } f(A) + f(B) \geq \delta(A \cup B) \text{ for all } A, B \in P(Y) \right\}
\]

and

\[
= \left\{ f \in \mathbb{R}^{P(Y)} \mid f(\emptyset) = 0, f(A) + f(\emptyset) = f(A) \geq \delta(A) \text{ for all } A \in P^*(Y) \text{ and } f(A) + f(B) \geq \delta(A \cup B) \text{ for all } A, B \in P^*(Y) \right\}
\]

\[
= \{0\} \times \left\{ f \in \mathbb{R}^{P^*(Y)} \mid f(A) + f(A) \geq 2\delta(A) \text{ for all } A \in P^*(Y), f(A) + f(A) \geq \delta(A \cup A) \text{ for all } A \in P^*, \text{ and } f(A) + f(B) \geq \delta(A \cup B) \text{ for all } A \neq B \in P^* \right\}
\]

\[
= \{0\} \times P_{D_\delta}.
\]

The statement now follows from Proposition 7.1.

(b) We have

\[
\hat{P}_\delta = \left\{ f \in \mathbb{R}^{P(Y)} \mid f(\emptyset) = 0 \text{ and } \sum_{A \in \mathcal{A}} f(A) \geq \delta \left( \bigcup \mathcal{A} \right) \text{ for all } \mathcal{A} \subseteq P(Y) \right\}
\]

\[
= \{0\} \times \left\{ f \in \mathbb{R}^{P^*(Y)} \mid \sum_{A \in \mathcal{A}} e_A, f \geq \delta \left( \bigcup \mathcal{A} \right) \text{ for all } \mathcal{A} \subseteq P(Y) \right\}
\]

\[
= \{0\} \times \left\{ f \in \mathbb{R}^{P^*(Y)} \mid \langle a, f \rangle \geq w^\delta(a) \text{ for all } a \in C(P^*(Y)) \right\}
\]

\[
= \{0\} \times E_{w^\delta}(C(P^*(Y))).
\]

The statement now follows from Lemma 7.1 since \( C(P^*(Y)) \) is positive and \( E_{w^\delta}(C(P^*(Y))) \) is bounded from below by the inequalities \( e_A \geq \delta(A) = 0 \) for all \( A \in P^*(Y) \).

\[\square\]

Note that, by considering the distance map \( d_\delta \) defined in \( 6.3.1 \), we get

\[
T_\delta = \{0\} \times T_{d_\delta} + (\delta(A))_{A \in P(Y)}.
\]
which, in particular, implies the following:

**Corollary 7.2.** The tight-span $T_\delta$ of the diversity $\delta$ is a tree if and only if the tight-span $T_{d_\delta}$ of the distance $d_\delta$ is a tree.

A special kind of diversity arising from a phylogenetic tree is given as follows: Let $T$ be a weighted tree with leaf set $Y$. The phylogenetic diversity $\delta_T$ associated to $T$ is defined by mapping any subset $A \subset Y$ to the length $\delta_T(A)$ of the smallest subtree of $T$ connecting taxa in $A$. These are precisely the diversities whose tight-spans are trees:

**Theorem 7.3.** Let $\delta$ be diversity on $Y$. Then the following are equivalent:

(a) $\delta$ is a phylogenetic diversity.

(b) The tight-span $T_\delta$ is a tree.

(c) The tight-span $T_{d_\delta}$ is a tree.

That (a) implies (b) follows from [3, Theorem 5.8], since the tight-span of a phylogenetic diversity is isomorphic to the tight-span of the metric space associated to the same tree. To show that (b) is equivalent (c), we will prove that $T_\delta$ and $T_{d_\delta}$ are equal for a much larger class of diversities, the so-called split-system diversities (Theorem 7.4 below). The proof that (c) implies (a) will then take up the remainder of this section.

Let $S$ be a split of $Y$. We define the split diversity $\delta_S : \mathcal{P}(Y) \to \mathbb{R}$ of $S$ as

$$\delta_S(A) = \begin{cases} 1, & \text{if } S \text{ splits } A, \\ 0, & \text{else}. \end{cases}$$

Given a set $S$ of splits of $Y$ and a function $\alpha : S \to \mathbb{R}_{\geq 0}$ assigning weights to the splits in $S$, the split system diversity $\delta_{(S, \alpha)}$ of $(S, \alpha)$ is defined as

$$\delta_{(S, \alpha)}(A) = \sum_{S \in S} \alpha(S) \delta_S(A) = \sum_{S \in S, S \text{ splits } A} \alpha(S).$$

A phylogenetic diversity is a special case of a split system diversity where the set $S$ is compatible. We now show that in case $\delta$ is a split diversity, the tight-spans $T_\delta$ and $T_{d_\delta}$ are equal:

**Theorem 7.4.** Let $S$ be a split system, $\alpha : S \to \mathbb{R}_{\geq 0}$ and $\delta_{(S, \alpha)}$ the associated split system diversity. Then

$$\hat{P}_{\delta_S} = P_{\delta_{(S, \alpha)}} \quad \text{and} \quad \hat{T}_{\delta_S} = T_{\delta_{(S, \alpha)}}.$$

**Proof.** We will show that $\hat{P}_{\delta_{(S, \alpha)}} = P_{\delta_{(S, \alpha)}}$ which obviously implies $\hat{T}_{\delta_{(S, \alpha)}} = T_{\delta_{(S, \alpha)}}$. First note that, by definition, for any diversity $\delta$, one has $\hat{P}_\delta \subseteq P_\delta$ and furthermore, for all $f \in P_\delta$ and $A \in \mathcal{P}(Y)$, one has $f(A) + f(\emptyset) = f(A) \geq \delta(A)$. It now suffices to show that for any $\mathcal{A} \subset \mathcal{P}(Y)$ with $|\mathcal{A}| \geq 2$ and $f \in P_{\delta_{(S, \alpha)}}$ the system of inequalities

$$f(A) + f(B) \geq \delta_{(S, \alpha)}(A \cup B) \quad \text{for all distinct } A, B \in \mathcal{A},$$

implies the inequality

$$\sum_{A \in \mathcal{A}} f(A) \geq \delta_{(S, \alpha)} \left( \bigcup_{A \in \mathcal{A}} \mathcal{A} \right).$$

Summing up all the Inequalities (7.1) we get

$$\sum_{A \neq B \in \mathcal{A}} (f(A) + f(B)) \geq \sum_{A \neq B \in \mathcal{A}} \delta_{(S, \alpha)}(A \cup B) \quad \iff \quad (|\mathcal{A}| - 1) \sum_{A \in \mathcal{A}} f(A) \geq \sum_{A \neq B \in \mathcal{A}} \sum_{S \subset S, S \text{ splits } A \cup B} \alpha(S) = \sum_{S \in S} \alpha(S) \text{SP}(S, \mathcal{A}, 2).$$
where \(SP(S, \mathcal{A}, 2)\) denotes the number of unordered pairs of distinct \(A, B \in \mathcal{A}\) such that \(S\) splits \(A \cup B\). We now show that \(SP(S, \mathcal{A}, 2) \geq |\mathcal{A}| - 1\) for all \(S \in S\) that split \(\cup \mathcal{A}\). Dividing the above inequality by \(|\mathcal{A}| - 1\) then gives Inequality (7.2) as desired.

Let \(S \in S\) be a split that splits \(\cup \mathcal{A}\). First suppose that there exists some \(A \in \mathcal{A}\) that is split by \(S\). Then \(SP(S, \mathcal{A}, 2) \geq |\mathcal{A}| - 1\) since obviously for all \(B \in \mathcal{A}\) distinct from \(A\) the set \(A \cup B\) is also split by \(S\). So we can assume that, for \(S = (C, D)\) and all \(A \in \mathcal{A}\), we have either \(A \subset C\) or \(A \subset D\). Let \(s\) be the number of \(A \in \mathcal{A}\) with \(A \subset C\). Since \(S\) splits \(\cup \mathcal{A}\), both \(s\) and \(|\mathcal{A}| - 1\) have to be at least 1, so we get \(SP(S, \mathcal{A}, 2) = s(\mathcal{A} - s) \geq \mathcal{A} - 1\), which finishes the proof. \(\square\)

In preparation to complete the proof of Theorem 7.3, we first examine the distance \(d_\delta\) in case \(\delta\) is a split diversity. So, let \(S = \{C, D\}\) be a split of \(Y\). For each \(A, B \subset Y\) we have

\[
\delta_\delta(A \cup B) - \delta_\delta(A) - \delta_\delta(B) = \begin{cases} 
1, & \text{if } C \subseteq A \text{ and } D \subseteq B \text{ or } C \subseteq B \text{ and } D \subseteq A \\
-1, & \text{if } A \cap C \neq \emptyset, A \cap D \neq \emptyset, B \cap C \neq \emptyset, \text{ and } B \cap D \neq \emptyset, \\
0, & \text{else},
\end{cases}
\]

so that

\[
d_\delta(A, B) = \begin{cases} 
1, & \text{if } C \subseteq A \text{ and } D \subseteq B \text{ or } C \subseteq B \text{ and } D \subseteq A, \\
0, & \text{else}.
\end{cases}
\]

Hence \(d_\delta = d_P\), where \(P\) is the partial split \([P^*(A), P^*(B)]\) of \(P^*(Y)\). By Proposition 3.1, the corresponding weight function defines a split of \(\mathcal{A}(P^*(Y))\). So as to show that a diversity whose tight-span is a tree comes from a split system, we consider these steps in reverse order. The following lemmas will be the key to our proof.

**Lemma 7.5.** Let \(Y\) be a finite set, \(\{\mathcal{A}, \mathcal{B}\}, \{\mathcal{C}, \mathcal{D}\}\) two compatible partial splits of \(P(Y)\) and \(A, B \subseteq Y\), \(C \subset A\) with \(A \in \mathcal{A}, B \in \mathcal{B}, C \in \mathcal{C}\) and \(B \in \mathcal{D}\). Then either \(C \in \mathcal{A}\) or \(A \in \mathcal{C}\).

**Proof.** Since \(\{\mathcal{A}, \mathcal{B}\}, \{\mathcal{C}, \mathcal{D}\}\) are compatible and \(B \in \mathcal{B} \cap \mathcal{D}\), by definition, we must have \(\mathcal{A} \subseteq \mathcal{C}\) or \(\mathcal{C} \subseteq \mathcal{A}\). \(\square\)

**Lemma 7.6.** Let \(P\) be a compatible set of partial splits of \(P^*(Y)\), \(\alpha : P \rightarrow \mathbb{R}_{\geq 0}\) and \(\delta\) be a diversity on \(Y\) such that \(d_\delta = d_{P, \alpha}\). Then we have:

(a) If \(A \in \mathcal{A}\) for some \(\{\mathcal{A}, \mathcal{B}\} \in P\), then \([i] \in \mathcal{A}\) for all \(i \in A\).

(b) If \(A \in \mathcal{A}\) for some \(\{\mathcal{A}, \mathcal{B}\} \in P\), then \(d_\delta(A, B) = 0\).

(c) If \(i \in A \in P(Y)\) with \([i], A \in \mathcal{A}\) for some \(\{\mathcal{A}, \mathcal{B}\} \in P\), then \([i] \cup A \in \mathcal{A}\).

**Proof.** (a) Let \(B \in \mathcal{B}\). By (D1), we get

\[
\delta(A \cup [i]) + \delta([i] \cup B) \geq \delta(A \cup B).
\]

This implies that

\[
\delta([i] \cup B) - \delta(B) \geq \delta(A \cup B) - \delta(A) - \delta(B).
\]

By definition of \(d_\delta\) (and (D2)), this is equivalent to \(d_\delta([i], B) \geq d_\delta(A, B)\). The claim no follows from Lemma 7.5 and the definition of \(d_{P, \alpha}\).

(b) If \(d_\delta(A, B) \neq 0\) this would imply that there exists a partial split \([C, D] \in P\) with \(A \in C\) and \(B \in D\) (or vice versa). However, this split could not be compatible with \(\{\mathcal{A}, \mathcal{B}\}\).
(c) By \( [17] \), we have \( d_\delta(|i|, A) = 0 \) which implies \( \delta(A \cup \{i\}) = \delta(A) \) by the definition of \( d_\delta \). Let \( B \in B \). Since \( d_\delta(A, B) > 0 \) by assumption, we get

\[
d_\delta(A, B) = \delta(A \cup B) - \delta(A) - \delta(B)
\]

\[
= \delta(A \cup B) - \delta(A \cup \{i\}) - \delta(B)
\]

\[
\leq \delta(A \cup \{i\} \cup B) - \delta(A \cup \{i\}) - \delta(B)
\]

\[
= d_\delta(A \cup \{i\}, B).
\]

The claim now follows from Lemma \( 7.5 \) and the definition of \( d_{(\mathcal{P}, a)} = d_\delta \).

\[\Box\]

**Corollary 7.7.** Let \( \mathcal{P} \) be a compatible set of partial splits of \( \mathcal{P}^*(Y) \), \( \alpha : \mathcal{P} \to \mathbb{R}_{\geq 0} \) and \( \delta \) a diversity on \( Y \) such that \( d_\delta = d_{(\mathcal{P}, a)} \). Then for all \( P = \{A, B\} \in \mathcal{P} \) there exists a partial split \( p(P) = \{A, B\} \) of \( Y \) such that \( \mathcal{A} = \mathcal{P}^*(A) \) and \( \mathcal{B} = \mathcal{P}^*(B) \). Furthermore the set \( \{p(P) \mid P \in \mathcal{P}\} \) of partial splits of \( Y \) is compatible.

**Proof.** The existence of \( p(P) \) follows by iteratively applying Lemma \( 7.6 \) \( (a) \) and \( (c) \). The compatibility follows from the compatibility of \( \mathcal{P} \) by Proposition \( 3.12 \). \( \Box \)

We can now finish the proof of the main theorem in this section.

**Proof of Theorem 7.3** We only have to show that \( (a) \) implies \( (c) \). So, let \( \delta \) be a diversity on \( Y \) such that the tight-span \( T_\delta \) is a tree. Corollary \( 7.2 \) now implies that \( T_\delta \) is a tree and Theorem \( 1.5 \) gives us a compatible set \( \mathcal{P} \) of partial splits of \( \mathcal{P}^*(Y) \) and a function \( \alpha : \mathcal{P} \to \mathbb{R}_{\geq 0} \) such that \( d_\delta = d_{(\mathcal{P}, a)} \). By Corollary \( 7.7 \) there exists a compatible set \( \mathcal{S} = \{p(P) \mid P \in \mathcal{P}\} \) of partial splits of \( Y \) such that \( \mathcal{A} = \mathcal{P}^*(A) \) and \( \mathcal{B} = \mathcal{P}^*(B) \). It remains to show that all \( S \in \mathcal{S} \) are splits.

Suppose one of these partial splits, say \( \{A, B\} \), is not a split, and let \( i \in A \), \( j \in B \) and \( l \in Y \setminus (A \cup B) \). By \( (D1) \), we get

\[
\delta(|i, k|) + \delta(|k, j|) \geq \delta(|i, j|),
\]

which is equivalent to

\[
d_\delta(|i|, |k|) + d_\delta(|k|, |j|) \geq d_\delta(|i|, |j|)
\]

by the definition of \( d_\delta \) and \( (D2) \). Now any partial split \( \{\mathcal{P}^*(C), \mathcal{P}^*(D)\} \) that separates \( k \) from either \( i \) or \( j \) must also separate \( i \) and \( j \) since it is compatible to \( \{\mathcal{P}^*(A), \mathcal{P}^*(B)\} \). So each split making a contribution to the left of Equation \( 7.3 \) makes the same contribution to the right of the equation and \( \{\mathcal{P}^*(A), \mathcal{P}^*(B)\} \) only contributes to the right, a contradiction.

So \( d_\delta = d_{(\mathcal{S}, a)} \) which implies \( \delta = \delta_{(\mathcal{S}, a)} \). Hence \( \delta \) is a phylogenetic diversity.

\[\Box\]

8. Discussion

8.1. k-Dissimilarity Maps. We have seen how to define the tight-span of various maps that generalise metrics. Another kind of map that we could consider taking the tight-span of is a \( k \)-dissimilarity map on a set \( X \), that is, a function \( D : \binom{X}{k} \to \mathbb{R} \). In this case, to obtain a tight-span one could take the set of vertices of the hypersimplex \( \Delta(k, X) \subset \mathbb{R}^X \) as the corresponding point configuration, that is, the set of all functions \( \sum_{x \in A} e_x \) for all \( A \in \binom{X}{k} \). More specifically, motivated by Proposition \( 4.1 \) for the case \( k = 2 \), given a \( k \)-dissimilarity map \( D \) on \( X \), define the function \( w_D : \Delta(k, X) \subset \mathbb{R}^X \to \mathbb{R} \) that sends \( \sum_{x \in A} e_x \) to \( -D(A) \), set \( P_D := \mathcal{E}_{w_D}(\Delta(k, X)) \), and \( T_D := \mathcal{T}_{w_D}(\Delta(k, X)) \). It follows from Lemma \( 2.1 \) that \( T_D \) is the set of minimal elements of \( P_D \).
Even though one might expect that $T_D$ has similar properties to the tight-spans we have so far considered, this is not the case. Indeed, given a weighted tree $T$ with leaf set $X$, one can define a $k$-dissimilarity map $D^k_T$ by assigning to each $k$-subset $A \subset X$ the total length of the induced subtree. But, as can be easily checked, the tight-span $T_{D^k_T}$ does not in general also have to be a tree, and so there is no obvious generalisation of the Tree Metric Theorem. Even so, the tree $T$ can be reconstructed from $T_{D^k_T}$ [16, Section 8.1], and so it could still be of interest to further study these tight-spans.

8.2. Coherent decompositions. Coherent decompositions of metrics were introduced by Bandelt and Dress [1] and are intimately related to tight-spans. Thus the question arises whether a similar decomposition theory could be developed for the different generalisations of metrics that we have considered.

In [15], the concept of coherent decompositions of metrics was generalised to weight functions of polytopes (as discussed in Section 2). For directed distances, and also symmetric and non-symmetric functions, this directly leads to a theory of coherent decompositions. Moreover, a decomposition theorem for $k$-dissimilarities in terms of “split $k$-dissimilarities” was recently derived in [16], which might be extended if an appropriate theory was worked out for tight-spans of $k$-dissimilarities, as suggested above. For diversities, as we have seen, a diversity on a set $Y$ can be considered as a distance on $P^*(Y)$, but such diversities form only a subset of all such distances. So to develop a theory of coherent decomposition for diversities, one could maybe try to first answer the following question:

**Question 8.1.** Given a diversity $\delta$ how can one compute coherent decompositions of the distance $d_\delta$? Moreover, which coherent components of such a decomposition are of the form $d_\delta^*$ for some diversity $\delta^*$?

8.3. Infinite Sets and Injective Hulls. In this paper, we have only considered tight-spans arising from finite sets. However, many of the results concerning tight-spans (and not point configurations) can be translated to infinite sets. For example, much of the theory for the tight-span of a metric space was originally developed for arbitrary metric spaces [8, 20], which is important since the tight-span of a metric (diversity) — which is of course an infinite set — comes equipped with a canonical metric (diversity), such that the tight-span of this metric is nothing other than itself (see [20, 3], respectively). Stated differently, this means that tight-spans are injective objects in the appropriate category [20], a property that would be interesting to understand in the setting of point configurations. However, if the theory for point configurations is to be extended to infinite sets, a first crucial step would be to understand how to generalise splits of polytopes, which appears to have no obvious generalisation in the infinite setting.

**References**

5. ——, *Generosity helps or an 11-competitive algorithm for three servers*, J. Algorithms 16 (1994), no. 2, 234–263. MR 1258238 (94m:90046)


School of Computing Sciences, University of East Anglia, Norwich, NR4 7TJ, UK