Multilateral Value for TU Cooperative Games

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Abstract

A value of a TU Cooperative game represents an assessment by a player of her gains for participating in a coalition. One of the most important values in the literature of TU games is the Shapley value [18]. It is indeed an aggregation of the marginal contributions of a player based on her bilateral interactions. In this paper we introduce a new value for TU Cooperative games. The notion of multilateral interaction of a player is proposed that accounts not only for the player’s own inclusion or exclusion in a coalition as considered in computing the Shapley value but also for her influence on her peers in their decision of joining or leaving the coalition together. We characterize this value by the axioms of linearity, anonymity, efficiency and a new axiom: the axiom of parasite player. A parasite player extracts worths of other players. Our model makes her role less significant in presence of multilateral interactions.

Keywords: Cooperative game; Shapley value; Multilateral interactive value.

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1 Introduction

A transferable utility game (TU Cooperative game or simply a TU game) is a pair \((N, v)\) where \(N\) is a finite set of players and \(v : 2^N \rightarrow \mathbb{R}\) a characteristic function satisfying
$v(\emptyset) = 0$. Subsets of $N$ are called coalitions and the value $v(S)$ for each coalition $S$ is called its worth. Let $\mathcal{G}(N)$ denote the universal game space consisting of all TU games. An intuitive assumption on $\mathcal{G}(N)$ is that the grand coalition $N$ forms for each of its members. A solution concept is a suitable allocation of the profits or the costs due to the formation of the grand coalition. A one-point solution is called the value of the TU game. Thus a value determines how much to be paid to the players for participating in the game. The Shapley value [18], Banzhaf value [2], compromise value [19], the selectope value [6], weighted Shapley value [13], random order value [20], Solidarity value [15], Procedural value [14], equal surplus value [7], consensus value [12] etc., to name a few are well-known values in Cooperative game theory. None of these values however accounts for interactions of a player with her peers in presence of other coalition members. They aggregate her marginal contributions by considering only her own inclusion or exclusion in a coalition.

The purpose of this paper is to introduce a new value for TU Cooperative games that accounts for simultaneous multilateral interactions of each player with her peers. The idea is similar to the one proposed by Borkotokey et al. [3] for Network games, however their characterizations are different. The notion of multilateral interactions ensures that a value that measures the net utility of a player in a game should also give due weightage to her capacity to interact with her peers in presence of other group members. It accounts not only for the player’s own inclusion or exclusion in a coalition but also for her influence on her peers in their decision of joining or leaving the coalition. Such contributions are seen to be significant in political games where a player joins or leaves a coalition along with her followers. Further examples are that of sharing river water between upstream and downstream countries where downstream countries interact among themselves for better bargaining, cost sharing in construction of trade specific networks, viz., the trans-national gas pipelines that connect Iran, Afghanistan and China, the ongoing 32 nation Great Asian Highway project for transport connectivity to the Asian countries and road links to Europe etc., to name a few.\footnote{An interesting example of multilateral interactions pertaining to network games can be found in [3].}
Consider for example, four adjacent countries 1, 2, 3 and 4 through which a transnational highway is to be built. The benefits of building this highway is measured in terms of the amount of trades that is expected to take place between neighbouring countries. The region is geographically so located that the highway should always pass through country 1 as this is the shortest path. Country 1 is therefore benefitted without sharing any cost of construction. However to enjoy her benefits she acts as an intermediary between 3 and 4 and offers a binding agreement. Country 2 on the other hand is indifferent about the construction if we consider her trade prospects with her neighbours on a “one on one” basis. She has good connectivity with all the three countries through which she can continue the trade. However if the rest of the countries start trading through this newly built highway she will be benefitted connecting to the highway rather than the individual roads. Thus even though her bilateral interactions are not costly due to the existing roads it is beneficial for her to cooperate with the other countries multilaterally to build the Highway and share some cost. Figure 1 shows the geographical locations of the four countries.

![Figure 1: The Highway Cost Sharing Game](#)

The idea of multilateral interactions does not have a precise a priori organizational structure, on the contrary it explores the possibilities of mutual influence among the players. Therefore it is different from a Clan game [17] where some players have veto power or games with permission structure [10] where the organization has a hierarchical structure.
Under this new framework, we rename the marginal contributions that aggregate to give the Shapley value the bilateral contributions of a player. On the other hand in our proposed value her multilateral contributions aggregate in presence of her peers. We characterize the value by the axioms of linearity, anonymity, efficiency and a new axiom: the axiom of a “parasite player”. The parasite player contributes nothing in a coalition in terms of generation of the worth. On the contrary she self-styles some binding agreement among the players. The notion of a parasite player has lot of similarities with that of a middleman for non-Cooperative games due to Rubinstein and Wolinsky [16]. The original model in [16] was a simple market model of buyers and sellers that captured explicitly the role of a middleman. The middlemen reduces the waiting time (and thus the cost) of the agents (Buyers and Sellers) who would otherwise wait until a suitable partner shows up. There has been much work in studying different models of trade (e.g., various non-Cooperative bargaining models) and analysing how middlemen influence the formation of prices and the efficiency of trade, see [21]. Even though the need of a middleman in trade is a must if time consumption is costly, the extra cost for paying her will be borne by the agents that increases the liabilities of the players, an idea similar to that of the parasite player. Thus as a possible application of our model, we discuss in particular the Cost Sharing game in presence of a parasite player.\(^2\) This is not addressed so far in the literature.

A Cost Sharing game is one in which costs (and benefits) are shared among different agents in a cooperative endeavour. The group of agents wants an allocation mechanism that is efficient, equitable and has appropriate incentives to its agents for their cooperation. Among other solutions the Shapley value has a long tradition of being used for cost sharing in many organizations, see for example [23]. Here we assume that the cost sharing game involves an intermediary (synonymous with the parasite player and the middleman in [16]) who volunteers to negotiate among the players (being one among them) to build

\(^2\)Another interesting example may be that of an Academic Institution where one has three groups of people: the students, the faculty members and the non-teaching staff. If utility is measured in terms of the teaching-learning outputs it is likely that we find parasites among the members in the non-teaching staff.
the required binding agreements. For this service we show that the players have to pay him extra in addition to their shared costs under the Shapley value as a sharing alternative. Finally we show that the notion of multilateral interactions can be best used to avoid the parasite player and cost sharing can be reduced by avoiding such extra liabilities using our proposed value.

It is worth mentioning here that games involving a parasite player can never be convex and therefore they are neither superadditive nor monotonic increasing. Thus we axiom that the parasite player is denied a non-zero payoff since it reduces the coalitional worths. It follows that a parasite player is treated in the same way as that of a null player however it is worth noting that a null player is insignificant in the coalition while the parasite player increases the liabilities of the other players by extracting what the rest of the players generate. More importantly the null player contributes nothing to a coalition when her bilateral interactions are only considered. There are situations where the null player does not influence the outcome of the game on a “one on one” interaction but significant effects are seen when it interacts with more than one player simultaneously i.e., interacting multilaterally. This idea cannot be modelled with the standard Shapley like solution concepts.

The rest of the paper is organized as follows. In Section 2 we state some preliminary concepts of TU-games. In Section 3 we give a characterization of the Multilateral value along with some properties. A Cost Sharing game in presence of a parasite player is described in Section 4 followed by the concluding remarks in Section 5.

2 Preliminaries

Let the set of players $N$ be fixed. Thus the TU-game $(N, v)$ is denoted by its characteristic function $v$. With some abuse of notations we denote the singleton sets without braces. Thus we write $S \cup i$ for $S \cup \{i\}$, $S \setminus i$ for $S \setminus \{i\}$ etc. The size (cardinality) of coalition $S$ is denoted by the corresponding lower case letter $s$. A game $v$ is called superadditive if for any $S, T \subseteq N$ with $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$ and if for any $S, T \subseteq N$ with $S \cap T = \emptyset$, $v(S \cup T) \leq v(S) + v(T)$ then $v$ is called subadditive. The increase in worth
when player \( i \in N \) joins coalition \( S \subseteq N \setminus i \) is called the marginal contribution of player \( i \) to coalition \( S \) which is denoted by \( b_i^S(v) \) and is given by

\[
b_i^S(v) = v(S) - v(S \setminus i).
\]

(2.1)

The unanimity games \( \hat{u}_T : 2^N \to \mathbb{R} \) and the identity games \( u_T : 2^N \to \mathbb{R} \), \( T \subseteq N \) are respectively defined as follows.

\[
\hat{u}_T(S) = \begin{cases} 
1, & \text{if } T \subseteq S \\
0, & \text{otherwise}
\end{cases}
\]

(2.2)

\[
u_T(S) = \begin{cases} 
1, & \text{if } T = S \\
0, & \text{otherwise}
\end{cases}
\]

(2.3)

The class of unanimity games and the class of identity games are bases for the linear space \( \mathcal{G}(N) \). Formally, a value on \( \mathcal{G}(N) \) is a function that assigns a single payoff vector \( \Phi(v) = (\Phi_i(v))_{i \in N} \in \mathbb{R}^n \) to every game \( v \in G \). Here we restrict ourselves only to the study of the Shapley value for reasons already mentioned in section 1. Among the various interpretations of the Shapley value, see for example [4] we take the following as it relates to the interpretation of our proposed value.

Suppose that the “grand coalition” \( N \) forms in a way such that the players enter the coalition one by one. This order of entrance can be expressed by a permutation \( \pi : N \to N \) of the players. Let the collection of all permutations on \( N \) be denoted by \( \Pi(N) \). For every \( \pi \in \Pi(N) \), let \( P(\pi, i) = \{ j \in N | \pi(j) < \pi(i) \} \) be the set of players that enter before player \( i \) in the order \( \pi \). The Shapley value [18] is the solution \( \Phi^{Sh} : \mathcal{G}(N) \to \mathbb{R}^n \) that assigns to every player her expected marginal contribution to the coalition of players that enter before him, given that every order of entrance has equal probability of \( \frac{1}{n!} \) to occur and is given by,

\[
\Phi_i^{Sh}(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} [v(P(\pi, i) \cup i) - v(P(\pi, i))]
\]

(2.4)

which after simplifications becomes,

\[
\Phi_i^{Sh}(v) = \sum_{S \subseteq N : i \in S} \frac{(s-1)! (n-s)!}{n!} [v(S) - v(S \setminus i)]
\]

(2.5)
For the game $v \in \mathcal{G}(N)$, a player $i \in N$ is called a null player if for every coalition $S \subseteq N$, we have $v(S) = v(S \setminus i)$. There has been a number of characterizations of the Shapley value hitherto found in the literature, see \cite{1, 5, 8, 9, 20, 22} for example. In this paper our characterization goes exactly in the same line of Weber \cite{20} who used the following four axioms to characterize the Shapley value.

(a) Efficiency ($\text{Eff}$) : A value $\Phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ is efficient if for the game $v \in \mathcal{G}(N)$ :

$$\sum_{i \in N} \Phi_i(v) = v(N)$$

(b) Null Player Property ($\text{NP}$) : A value $\Phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ satisfies the null player property if for every game $v \in \mathcal{G}(N)$ it holds that $\phi_i(v) = 0$ for every null player $i \in N$ in the game $v$.

(c) Anonymity ($\text{AN}$) : A value $\Phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ satisfies anonymity($\text{AN}$) if for every permutation $\pi : N \rightarrow N$ ,

$$\phi_{\pi(i)}(\pi v) = \phi_i(v)$$

(d) Linearity ($\text{Lin}$) : A value $\Phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ is linear if for all games $u, w \in \mathcal{G}(N)$ every pair of $\alpha, \beta \in \mathbb{R}$ and every player $i \in N$ :

$$\Phi_i(\alpha u + \beta w) = \alpha \Phi_i(u) + \beta \Phi_i(w)$$

### 3 Characterizations

In this section we introduce and characterize the Multilateral interactive value for TU Cooperative games. As mentioned in the Introduction, we call $b^S_i(v)$ given in (2.1) the marginal bilateral contribution of $i$ with respect to $S$ to distinguish it from the situations where player $i$ leaves the coalition $S$, along with a group $T \subset S$ of players. These contributions are therefore the result of interactions on a “one on one” basis: we call them bilateral interactions. Thus formally we have the following.
Definition 1. Given an $S \subseteq N \setminus i$ we denote by $m^S_{T i}(v)$ the marginal multilateral contribution of $i$ in presence of $T \subseteq S$ and define as follows.

$$m^S_{T i}(v) = v(S \cup i) - v(S \setminus T), \quad \forall \ S \subseteq N \setminus i, \ T \subseteq S$$

(3.1)

Note that from marginal multilateral contributions’ perspectives, the null player may have positive contributions to a coalition when interacting multilaterally with a group of players even though her contribution is zero on a “one on one” basis. This will be shown in the example given at the end of the paper.

Lemma 1. Let $\Phi_i$ be a value for $\mathcal{G}(N)$ that satisfies Lin for each $i \in N$. Then there exist real constants $\alpha^i_S$ for all $S \subseteq N$ such that for every $v \in \mathcal{G}(N)$,

$$\Phi_i(v) = \sum_{\emptyset \neq S \subseteq N} \alpha^i_S v(S)$$

(3.2)

Proof. Since the class of identity games $\{u_S : S \subseteq N\}$ forms a basis for $\mathcal{G}(N)$, every $v \in \mathcal{G}(N)$ can be uniquely determined by its values on the basis as follows.

$$v = \sum_{\emptyset \neq S \subseteq N} v(S)u_S$$

By Lin, we have,

$$\Phi_i(v) = \sum_{\emptyset \neq S \subseteq N} v(S) \Phi_i(u_S)$$

The result follows by setting $\alpha^i_S := \Phi_i(u_S)$.

In what follows we formally introduce the notion of a parasite player.

Definition 2. A player $i \in N$ is called a parasite player if for each $S \subseteq N \setminus i$, the coalition $S \cup i$ is endowed with a worth given by the following rule.

$$v(S \cup i) = \frac{1}{2^{|S|}} \sum_{T \subseteq S} v(S \setminus T), \quad \forall \ S \subseteq N \setminus i$$

(3.3)
One can observe from the definition of a parasite player that the worth of the coalition involving the parasite player is the mean worth generated by the remaining players. In particular a parasite player contributes nothing to the game rather it acquires the worths generated by the other players. Following are few interesting results pertaining to games with a parasite player.

**Proposition 1.** There cannot be a parasite player in a convex game. In particular if $(N,v)$ is superadditive or monotone increasing it cannot have a parasite player.

*Proof.* The proof is a direct consequence of the definition of a parasite player. 

**Proposition 2.** If $v$ is subadditive, the parasite player gets a negative payoff under the Shapley value.

*Proof.* Let $i \in N$ be a parasite player in $v$. It follows that for any $S \subseteq N \setminus i$,

$$v(S \cup i) = \frac{1}{2^{|S|}} \sum_{T \subseteq S} v(S \setminus T).$$

In particular when $S = \emptyset$ we have $v(i) = 0$.

Since $v$ is subadditive we have $v(S \cup i) \leq v(S) + v(i)$. This further implies that $v(S \cup i) - v(S) \leq 0$ for all $S \subseteq N \setminus i$. The result follows from (2.5).

**Proposition 3.** In any game $v$, the Shapley value for any two parasite players in $v$ gives the same payoff.

*Proof.* The proof is a direct consequence of the definition of parasite players.

In view of the above propositions, one can argue that the synergy in a coalition gets hindered by the presence of a parasite player as it reduces the coalitional worths in case of a profit sharing game while it reduces the coalitional cost in a cost sharing game.

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3As mentioned in the Introduction, this may be a binding agreement initiated by the parasite player for which the worth of the coalition involves her extracts under the rule of average. The rule may vary depending on the binding agreements. For making our model simple we have taken this particular rule. One may take a more general rule however it would result altogether in a different solution concept.
However Proposition 2 suggests that as a reward for the reduction of the coalitional cost the Shapley value provides the \textit{parasite player} some profit in cost sharing games which usually are subadditive in nature. This adds to the shares of the costs of the other players. Thus in either case (profit sharing or cost sharing) a \textit{parasite player} should be marginalized and be awarded with zero payoff. Therefore we have the following axiom (PA).

\textbf{Definition 3.} \textit{Parasite player} axiom (PA): If \(i \in N\) is a \textit{parasite player} for \(v\), then \(\Phi_i(v) = 0\).

\textbf{Lemma 2.} Let the value \(\Phi_i\) satisfy Lin and PA. Then for all \(i \in N\), there exist real constants \(\gamma^i_S\) for all \(S \subseteq N \setminus i\) such that for every \(v \in \mathcal{G}(N)\),

\[
\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{T \subseteq S} \gamma^i_S m^S_{T_i}(v)
\]

\textit{Proof.} When \(\Phi_i\) satisfies Lin, we have from (3.2)

\[
\Phi_i(v) = \sum_{\emptyset \neq S \subseteq N \setminus i} \alpha^i_S v(S)
= \sum_{S \subseteq N \setminus i} \{\alpha^i_{S \cup i} v(S \cup i) + \alpha^i_S v(S)\}
\]  

(3.4)

After rearranging the coefficients \(\alpha^i_S\) in (3.4), we obtain

\[
\Phi_i(v) = \sum_{S \subseteq N \setminus i} \left\{ \alpha^i_{S \cup i} v(S \cup i) + \sum_{T \subseteq S} \beta^i_{S \setminus T} v(S \setminus T) \right\}
\]  

(3.5)

where

\[
\beta^i_{S \setminus T} = \frac{1}{2^n - t - 1} \alpha^i_S
\]

Let \(\Phi\) satisfy PA. If \(i\) is a \textit{parasite player} for \(v\), then

\[
\Phi_i(v) = \sum_{S \subseteq N \setminus i} \left\{ \alpha^i_{S \cup i} \frac{1}{2^n} \sum_{T \subseteq S} v(S \setminus T) + \sum_{T \subseteq S} \beta^i_{S \setminus T} v(S \setminus T) \right\}
\]  

\[
= \sum_{S \subseteq N \setminus i} \left\{ \sum_{T \subseteq S} \left( \frac{1}{2^n} \alpha^i_{S \cup i} + \beta^i_{S \setminus T} \right) v(S \setminus T) \right\}
\]  

(3.6)
(3.6) holds for any TU Cooperative game $v$ such that $i$ is a parasite player for $v$, in particular for all games satisfying for any $S \subseteq N \setminus i$ and any $T \subseteq S$

$$v(S' \cup i) = \sum_{T' \subseteq S'} \frac{1}{2^{s'}} v(S' \setminus T') \quad \forall S' \subseteq N \setminus i$$

$$v(S') = u_{(S \setminus T)}(S')$$

By $PA$ we have,

$$\Phi_i(v) = 0$$

$$\Rightarrow \sum_{S' \subseteq N \setminus i} \left\{ \sum_{T' \subseteq S'} \left( \frac{1}{2^{s'}} \alpha_{S_1}^{i} + \beta_{S_2 \setminus T'}^{i} \right) u_{(S_1 \setminus T')} (S' \setminus T') \right\} = 0$$

$$\Rightarrow \frac{1}{2^{s'}} \alpha_{S_1}^{i} = -\beta_{S_2 \setminus T'}^{i}, \quad \forall T \subseteq S$$

Setting $\gamma_i^S := \frac{1}{2^{s'}} \alpha_{S_1}^{i} = -\beta_{S_2 \setminus T'}^{i}$ in (3.5), we obtain

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{T \subseteq S} \gamma_i^S \left\{ 2^{s} v(S \cup i) - \sum_{T \subseteq S} v(S \setminus T) \right\}$$

$$= \sum_{S \subseteq N \setminus i} \sum_{T \subseteq S} \gamma_i^S \{ v(S \cup i) - v(S \setminus T) \}$$

Thus the result follows by virtue of (3.1).

**Lemma 3.** Under Lin, PA, AN for all $i \in N$, there exist real constants $\gamma_i$ for all $S \subseteq N \setminus i$ such that for every $v \in G(N)$,

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{T \subseteq S} \gamma_i^S m_{T_i}^S(v)$$

**Proof.** By Lemma 2, we have for all $i \in N$, there exist real constants $\gamma_i^S$ for all $S \subseteq N \setminus i$ such that for every $v \in G(N)$,

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{T \subseteq S} \gamma_i^S m_{T_i}^S(v)$$

Given $S \subseteq N \setminus i$, let $\pi$ be a permutation on $N$ such that $\pi(S \cup i) = S \cup i$. Considering the game $u_{S \cup i}$ given by equation (2.3), it follows from AN,

$$\Phi_i(u_{S \cup i}) = \Phi_{\pi i}(u_{n(S \cup i)})$$

$$\Rightarrow \gamma_i^S = \gamma_{\pi(i)}^S$$
This implies that there exist real constants $\gamma_s$ depending on the size of $S \subseteq N \setminus i$ such that for every $v \in \mathcal{G}(N)$,

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{T \subseteq S} \gamma_s \, m^S_{T_i}(v) \quad (3.7)$$

Combining Lemma 1, 2 and 3, we have the following theorem.

**Theorem 1.** Under Lin, PA, AN, Eff for all $i \in N$, there exists a unique solution concept for $(N,v)$ given by

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{T \subseteq S} \gamma_s \, m^S_{T_i}(v) \quad (3.8)$$

where

$$\gamma_{n-1} = \frac{1}{n \cdot 2^{n-1}}$$

$$\gamma_{s-1} = \frac{(n-s)}{s \cdot 2^{s-1}} \sum_{t=1}^{n-s-1} C_{t-1} \gamma_{s+t-1} \quad (3.9)$$

Equation (3.9) gives a recursive formula for the coefficients $\gamma_s$'s.

**Proof.** Under Lin, PA, AN and by Lemma 3, we have for all $i \in N$, there exist real constants $\gamma_s$ for all $S \subseteq N : i \in S$ such that for every $v \in \mathcal{G}(N)$,

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{T \subseteq S} \gamma_s \, m^S_{T_i}(v)$$
Using $Eff$, we have

$$v(N) = \sum_{i \in N} \Phi_i(v)$$

$$= \sum_{S \subseteq N \setminus i} \sum_{T \subseteq S} \gamma_s m^S_{T} (v)$$

$$= \sum_{i \in N} \sum_{S \subseteq N \setminus i} \sum_{T \subseteq S} \gamma_s \{v(S \cup i) - v(S \setminus T)\}$$

$$= \sum_{\emptyset \neq S \subseteq N} v(S) \left\{ \sum_{i \in S} 2^{s-1} \gamma_{s-1} - \sum_{j \in N \setminus S} \sum_{\emptyset \neq T \subseteq N \setminus S : j \in T} \gamma_{s+t-1} \right\}$$

$$= \sum_{\emptyset \neq S \subseteq N} v(S) \left\{ s.2^{s-1} \gamma_{s-1} - (n-s) \sum_{t=1}^{n-s} C_{t-1} \gamma_{s+t-1} \right\}$$

$$= \sum_{\emptyset \neq S \subseteq N} v(S) \left\{ s.2^{s-1} \gamma_{s-1} - (n-s) \sum_{t=1}^{n-s} C_{t-1} \gamma_{s+t-1} \right\}$$

It follows from the non negativity of the terms inside the summation sign that the coefficient of $v(N)$ is 1 and all $v(S)$’s are 0 for all $S \subset N$, which yields the following recursive relations

$$\gamma_{n-1} = \frac{1}{n.2^{n-1}}$$

$$\gamma_{s-1} = \frac{(n-s)}{s.2^{s-1}} \sum_{t=1}^{n-s} C_{t-1} \gamma_{s+t-1}$$

We call the solution $\Phi$ given by (3.8) a multilateral interactive solution and the value $\Phi_i(v)$ for a TU game $v$ as the Multilateral interactive value or simply a Multilateral value. It is worth mentioning here that unlike the Shapley value which is the average of the marginal contributions (bilateral) of the players entering one by one into a room described in section 2, the Multilateral value is the average of the marginal contributions (multilateral) of the players each entering into the room with a coalition of any size of the players’ set $N$. 

\[\square\]
4 Decomposition of the Multilateral value

In what follows next, we express the Multilateral value of a TU game in terms of the Shapley values of a class of associate games. For any $T \subseteq N$ denote by $[T]$ a hypothetical player [11] that represents the collective action of the players in $T$. Following this assumption, we say that the players in $N \setminus T$ interact with the players in $T$ at level $t$. Fix a $T \subseteq N$ and set $\bar{N} = N \setminus T \cup [T]$. Then for any $\bar{K} \subseteq \bar{N}$, there is a $K \subseteq N \setminus T$ such that $\bar{K} = K \cup [T]$ or $\bar{K} = K$. Define the TU game $\bar{V}_{[T]} : 2^{\bar{N}} \to \mathbb{R}$ as follows. For any $\bar{K} \subseteq \bar{N}$

$$\bar{V}_{[T]}(\bar{K}) = \begin{cases} \bar{\gamma}_{k+t} v(K \cup T), & \text{if } [T] \in \bar{K} \\ \bar{\gamma}_k v(K), & \text{otherwise} \end{cases}$$

(4.1)

where $\bar{\gamma}_s = \frac{\gamma_s}{\alpha^t_s}$ with $\alpha^t_s = \frac{(s-t+1)!(n-s-1)!}{(n-t+1)!}$, and $s = k + t - 1$.

Following [4], a possible interpretation of the class of games of the form $\bar{V}_{[T]}$ is given below. The quantity $\bar{\gamma}_s$ may be taken as the conditional probability that a coalition of size $\bar{s}$ which is equal to $s-t+1$ is chosen from $\bar{N}$ given that a subset $T \subseteq S$ is always present in $S$. Here $\alpha^t_s$ represents the probability of choosing a coalition from $\bar{N}$ with size varying from 0 to $(n-t+1)$ where players in $T$ behave as a single hypothetical player. Therefore $\bar{V}_{[T]}$ in (4.1) is the game that estimates the worth of the coalitions with respect to $v$ in which players in $T$ act as a single player. A bilateral interaction of each of the players in $N \setminus T$ with $[T]$ is indeed a multilateral interaction with the players in $T$. This suggests that we level each such interaction cardinally and call the game $\bar{V}_{[T]}$ the associate game at interaction level $t$.

Following theorem states that the payoff to a player under the Multilateral value is the sum of her payoffs obtained from the Shapley values of the associate games at all interaction levels in the range $\{1, 2, ..., n\}$.

**Theorem 2.** *The Multilateral value $\Phi$ is the sum of the Shapley values $\Phi^{Sh}$ for the games $\bar{V}_{[T]}$ given by,*

$$\Phi_i(v) = \sum_{[T] \in N : i \in T} \Phi^{Sh}_{[T]}(\bar{V}_{[T]})$$
Proof. Using (4.1), it follows from (3.8) that,

\[
\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{T \subseteq S} \gamma_s m^S_{T_i}(v)
\]

\[
= \sum_{S \subseteq N \setminus i} \sum_{T \subseteq S} \gamma_s \{v(S \cup i) - v(S \setminus T)\}
\]

\[
= \sum_{T \subseteq N : i \in T} \sum_{S \subseteq N : T \subseteq S} \gamma_s \{v(S) - v(S \setminus T)\}
\]

\[
= \sum_{T \subseteq N : i \in T} \sum_{S \subseteq N : T \subseteq S} \frac{(n - t + 1)!}{(s - t + 1)!(n - s - 1)!} \frac{(s - t + 1)!(n - s - 1)!}{(n - t + 1)!} \gamma_s \{v(S) - v(S \setminus T)\}
\]

\[
= \sum_{[T] \in \bar{N} : i \in T} \sum_{\bar{S} \subseteq \bar{N} : [T] \subseteq \bar{S}} \frac{(\bar{s} - 1)!(\bar{n} - \bar{s})!}{\bar{n}!} \{\bar{V}_{[T]}(S) - \bar{V}_{[T]}(S \setminus [T])\}
\]

\[
= \sum_{[T] \in \bar{N} : i \in T} \Phi^\text{Sh}_{[T]}(\bar{V}_{[T]})
\]

where \(\bar{s} = s - t + 1\), \(\bar{n} = n - t + 1\)

\[
\square
\]

Remark 1. Proposition 2 suggests that the Multilateral value of a game aggregates the Shapley values of its associate games where players in the associate games interact among themselves bilaterally at all possible levels. Thus it is a decomposition theorem that connects the Multilateral value with the Shapley values of the associate games.

5 An Application to the Cost Sharing Game

Revisiting the example of cost sharing in the construction of a trade specific trans-national highway mentioned in the introduction, we consider here the corresponding cost sharing game in presence of a parasite player and a null player. Their roles under the Shapley value and the Multilateral value are compared. The parasite player here acts as an intermediary, similar to a middleman of [16] under a cooperative setup and her role is to build the binding agreements among the others using (3.3). Let \(N = \{1, 2, 3, 4\}\) be the players’ set that represents the four countries through which the highway is to be built. Based on the situation we define the cost function \(c : \mathcal{G}(N) \to \mathbb{R}\) as shown in the following table.
Table 1: The Highway Cost Sharing Game

<table>
<thead>
<tr>
<th>$S$</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(1, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c(S)$</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$S$</td>
<td>{1, 3}</td>
<td>{1, 4}</td>
<td>{2, 3}</td>
<td>{2, 4}</td>
<td>{3, 4}</td>
</tr>
<tr>
<td>$c(S)$</td>
<td>1.5</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$S$</td>
<td>{1, 2, 3}</td>
<td>{1, 2, 4}</td>
<td>{1, 3, 4}</td>
<td>{2, 3, 4}</td>
<td>{1, 2, 3, 4}</td>
</tr>
<tr>
<td>$c(S)$</td>
<td>1.5</td>
<td>1</td>
<td>2.25</td>
<td>4</td>
<td>2.25</td>
</tr>
</tbody>
</table>

Note that here 1 is a parasite player and 2 a null player and the game $(N, v)$ is subadditive meaning players can reduce cost by cooperating. It is further interesting to note that player 2 being a null player contributes nothing through her bilateral interactions however her marginal multilateral contributions are non-zero since we have, $m_{T_2}^S(v) = 0.25$, when $S = \{1, 2, 3, 4\}$ and $T = \{1, 2, 3\}$, $m_{T_2}^S(v) = 1.5$, when $S = \{1, 2, 3, 4\}$ and $T = \{2, 3, 4\}$ etc. Thus player 2 contributes to the marginal contributions (costs) in presence of the other players. The Shapley and the Multilateral values are respectively given below.

$$\Phi_{sh}(v) = (-1, 0, 2, 1.25) \quad \text{and} \quad \Phi(v) = (0, 0.24, 1.13, 0.83).$$

Observe that the Shapley value rewards the parasite player a payoff 1 that adds to the shares of the costs to be borne by player 3 and 4 and player 2 does not have to share any cost. However in view of the foregoing discussion, if player 3 and 4 indulge in multilateral interactions by marginalizing the intermediary (player 1), and the marginal multilateral contributions of player 2 are considered, the Multilateral value becomes a better alternative to the Shapley value as it suggests no cost sharing for player 1 (yet she enjoys the benefits of the highway) and some share of cost to player 2.

### 6 Conclusion

This paper proposes a new value for TU Cooperative games that accounts for multilateral interactions among players. The characterization includes the standard Shapley like axioms of linearity, anonymity, efficiency along with a new axiom: the axiom of a parasite player. It is shown that the notion of multilateral contributions extends the ordinary marginal
contributions whom we call here as the marginal bilateral contributions. We obtain an interesting relationship between the proposed value with the Shapley value. An example of a cost sharing problem is provided to illustrate the usefulness of the value.

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References


