Multi-Order Exact Solutions for a generalized shallow water wave equation and other nonlinear PDEs

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Abstract

We seek multi-order exact solutions of a generalized shallow water wave equation along with those corresponding to a class of nonlinear systems described by the KdV, modified KdV, Boussinesq, Klein-Gordon and modified Benjamin-Bona-Mahony equation. We employ a modified version of a generalized Lamé equation and subject it to a perturbative treatment identifying the solutions order by order in terms of Jacobi elliptic functions. Our solutions are new and hold the key feature that they are expressible in terms of an auxiliary function $f$ in a generic way. For appropriate choices of $f$ we recover the previous results reported in the literature.

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1 Introduction

Seeking tractable solutions of nonlinear evolution equations has been the focus of intense study for the past several decades (see, e.g., \cite{1}-\cite{9}). While many strategies have been employed, such as the Hirota bilinear method \cite{2}, homogeneous balance method \cite{3}, trigonometric method \cite{4}, the hyperbolic method \cite{5} and the Jacobi

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elliptic function method \[6\] to name a few, addressing derivations of exact solutions, perturbative techniques have also been used to extract multi-order exact periodic solutions based on Lamé equation and Jacobi elliptic functions \[7, 8, 9\]. In this note we propose a generalized class of Lamé equation comprising an auxiliary function \(f(\xi)\) to consider correlations between it and the perturbatively reduced nonlinear evolution equation for different orders of the perturbative parameter.

A generalized Lamé equation is expressed as

\[
\frac{d^2y}{d\xi^2} + \left[ \lambda - (n + 1)(n + 2)a_1a_2(f(\xi))^2 \right] y = 0, \tag{1}
\]

where \(\lambda\) stands for an eigenvalue, \(n\) is a positive integer and we restrict the auxiliary function \(f(\xi)\) to obey an elliptic equation

\[
\left( \frac{df}{d\xi} \right)^2 = \left( 1 + a_1f^2 \right) \left( a_2f^2 + a_3 \right). \tag{2}
\]

In (1) and (2), \(a_1, a_2, a_3\) are real constants. It is well known that equation (2) admits of several categories of solutions for different values of \(a_1, a_2, a_3\) all expressible in terms of modulus \(k\) of the Jacobi elliptic functions. For later use we summarize a few relevant ones in Table 1.

Table 1: \(f(\xi)\) is provided along with the values of the parameters \(a_j, j = 1, 2, 3\). The complementary modulus is denoted by \(k'^2 = 1 - k^2\) \[10\].

<table>
<thead>
<tr>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
<th>(f(\xi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>(-k^2)</td>
<td>1</td>
<td>(sn(\xi, k))</td>
</tr>
<tr>
<td>(-1)</td>
<td>(k^2)</td>
<td>(k'^2)</td>
<td>(cn(\xi, k))</td>
</tr>
<tr>
<td>(-1)</td>
<td>1</td>
<td>(-k^2)</td>
<td>(dn(\xi, k))</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-k'^2)</td>
<td>(-k^2)</td>
<td>(nc(\xi, k)) i.e. (\frac{1}{cn(\xi, k)})</td>
</tr>
<tr>
<td>(-k'^2)</td>
<td>1</td>
<td>(-1)</td>
<td>(nd(\xi, k)) i.e. (\frac{1}{dn(\xi, k)})</td>
</tr>
<tr>
<td>1</td>
<td>(k^2)</td>
<td>1</td>
<td>(sc(\xi, k)) i.e. (\frac{sn(\xi, k)}{cn(\xi, k)})</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-k^2)</td>
<td>1</td>
<td>(cd(\xi, k)) i.e. (\frac{sn(\xi, k)}{dn(\xi, k)})</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-1)</td>
<td>(k^2)</td>
<td>(dc(\xi, k)) i.e. (\frac{cn(\xi, k)}{dn(\xi, k)})</td>
</tr>
<tr>
<td>(k^2)</td>
<td>(-k^2)</td>
<td>1</td>
<td>(sd(\xi, k)) i.e. (\frac{cn(\xi, k)}{sn(\xi, k)})</td>
</tr>
</tbody>
</table>

It should be mentioned that while we would attend to the enlisted set of \(f(\xi)\) in Table 1 to widen the scope of enquiry, some particular cases of \(f(\xi)\), namely \(sn(\xi, k)\), \(dn(\xi, k)\) and \(cd(\xi, k)\), have been studied in the literature \[7, 8, 9\] to arrive at solutions of the relevant PDEs by applying the perturbation method on the latter. However, as is evident from Table 1 there do remain other variants of \(f(\xi)\) which open up different cases of new solutions not only for the already considered PDEs
but other types as well by perturbatively reducing them to an ODE form so as to match with (1).

We have selected a physically meaningful model of a generalized shallow water wave equation \cite{11, 12, 13, 14} which is amenable to a perturbative treatment by looking for a travelling wave variable \( u(x, t) = u(\xi) \) with \( \xi = \gamma(x - ct) \) where \( \gamma \) is the wave number and \( c \) is the wave speed. We discuss the procedure of obtaining the solutions in the next section. In the following section we also take up applicability of our scheme to other nonlinear evolution equations namely, the Korteweg de Vries (KdV) equation, modified KdV equation, Boussinesq equation, Klein-Gordon equation and modified Benjamin-Bona-Mahony (mBBM) equation with a view to tracking down new hitherto unexplored multi-order solutions. The main point of our analysis is to demonstrate that all such solutions can be written down in a generic form in terms of the function \( f \).

To make equation (1) accessible in terms of known Lamé functions, we recast it in terms of a new variable \( \eta \) by applying the transformation \( f(\xi) = \sqrt{\eta} \). This gives

\[
\frac{d^2 y}{d\eta^2} + \frac{1}{2} \left[ \frac{1}{\eta} + \frac{1}{\eta + \frac{1}{a_1}} + \frac{1}{\eta + \mu} \right] \frac{dy}{d\eta} - \frac{\nu + (n+1)(n+2)\eta}{4\eta(\eta + \frac{1}{a_1})(\eta + \mu)} y = 0,
\]

where the parameters \( \mu \) and \( \nu \) are defined by

\[
\mu = a_1 \frac{a_2}{a_1} \quad \text{and} \quad \nu = -\frac{\lambda}{a_1 a_2}.
\]

Equation (3) is readily solvable for the special cases of \( n = 1 \) and \( 2 \) for different choices of \( \mu \) and \( \nu \) subject to certain relation between them. The solutions expressible by the corresponding Lamé functions are \cite{7}

\begin{align*}
\lambda &= -\left( a_2 + a_1 a_3 \right) \quad \text{[provided } \nu = (\mu + \frac{1}{a_1})\text{]}.
\end{align*}

\begin{align*}
\lambda &= -(a_2 + 4a_1 a_3) \quad \text{[provided } \nu = (4\mu + \frac{1}{a_1})\text{]}.
\end{align*}

\begin{align*}
\lambda &= -(4a_2 + a_1 a_3) \quad \text{[provided } \nu = (4\mu + \frac{4}{a_1})\text{]}.
\end{align*}

\begin{align*}
\lambda &= -4(a_2 + a_1 a_3) \quad \text{[provided } \nu = 4(\mu + \frac{1}{a_1})\text{]}.
\end{align*}

Henceforth we will be guided by the solutions (4) - (7) along with an appropriate \( f \) from Table 1 for the various multi-order cases that follow from a nonlinear PDE.
2 A generalized shallow water wave equation

We turn attention to a generalized class of shallow water wave (GSWW) equation given by \[11, 12, 13, 14\]

\[u_{xxxx} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \quad (8)\]

where \(\alpha, \beta \in \mathbb{R} - \{0\}\). Under Boussinesq approximation the derivation of (8) results from the classical study of water waves. More interestingly, there also follows various classical and non-classical reductions from GSWW such as the KdV and BBM equations. Investigations of Painlevé tests reveal complete integrability for specific values of \(\alpha\) and \(\beta\) namely \(\alpha = \beta\) or \(\alpha = 2\beta\) \[12\]. Recently we extended \[14\] the Jacobian elliptic function method to classify new exact travelling wave solutions expressible in terms of quasi-periodic elliptic integral function and doubly-periodic Jacobian elliptic functions.

For the travelling wave solutions \(u = u(\xi)\), equation (8) can be reduced in terms of the variable \(v(\xi) \equiv \frac{du}{d\xi}\) to the ODE form

\[\frac{d^2 v}{d\xi^2} + P v^2 + Q v + c_1 = 0. \quad (9)\]

where \(P = \frac{\alpha + \beta}{2\gamma}, \quad Q = \frac{1 - \gamma^2}{\gamma^2}\) and \(c_1\) is an integrating constant.

To tackle (9) perturbatively we set

\[v(\xi) = v_0(\xi) + \epsilon v_1(\xi) + \epsilon^2 v_2(\xi) + \cdots \quad (10)\]

where \(\epsilon(>0)\) is a small parameter and \(v_0(\xi),\ v_1(\xi),\ v_2(\xi),\ldots\) represent various multi-order solutions like the zeroth-order, first-order, second-order solutions etc. of equation (9). Accordingly we can write \(u(\xi)\) as

\[u(\xi) = u_0(\xi) + \epsilon u_1(\xi) + \epsilon^2 u_2(\xi) + \cdots \quad (11)\]

where \(u_i(\xi) = \int_0^\xi v_i(\tau) \, d\tau\).

Substituting the series (10) in equation (9) we obtain for each power of \(\epsilon\) the corresponding equations

\[\epsilon^0 : \frac{d^2 v_0}{d\xi^2} + P v_0^2 + Q v_0 + c_1 = 0, \quad (12)\]

\[\epsilon^1 : \frac{d^2 v_1}{d\xi^2} + (2P v_0 + Q) v_1 = 0, \quad (13)\]

\[\epsilon^2 : \frac{d^2 v_2}{d\xi^2} + (2P v_0 + Q) v_2 = -P v_1^2, \quad (14)\]

and so on.
We now proceed to solve the above chain of equations by first expanding $v_0$ namely,

$$v_0 = \sum_{i=0}^{l} A_i f^i,$$

where $A_i$’s are constants and then comparing highest order linear and nonlinear terms in \(1^2\). In this way we obtain $l = 2$ by making use of \(2\). Thus \(15\) gets reduced to the quadratic form

$$v_0 = A_0 + A_1 f + A_2 f^2.$$  \hspace{1cm} (16)

2.1 Zeroth-order exact solution

With $v_0$ given by \(16\) we are led to the following system of coupled equations:

$$2a_3 A_2 + PA_0^2 + QA_0 + c_1 = 0,$$  \hspace{1cm} (17)

$$A_1[(a_2 + a_1 a_3) + 2PA_0 + Q] = 0,$$  \hspace{1cm} (18)

$$4(a_2 + a_1 a_3)A_2 + P(A_1^2 + 2A_0 A_2) + QA_2 = 0,$$  \hspace{1cm} (19)

$$A_1[a_1 a_2 + PA_2] = 0,$$  \hspace{1cm} (20)

$$A_2[6a_1 a_2 + PA_2] = 0.$$  \hspace{1cm} (21)

It is readily seen that we generate the following set of consistent solutions

$$A_0 = -\frac{1}{2P}[Q + 4(a_2 + a_1 a_3)],$$

$$A_1 = 0$$

and $A_2 = -\frac{6a_1 a_2}{P}$ along with the constraint

$$[16(a_2 + a_1 a_3)^2 - 48a_1 a_2 a_3] - Q^2 - 4Pc_1 = 0.$$  \hspace{1cm} (22)

From \(22\) we obtain wave speed as

$$c = \left[1 \pm 4\gamma^2 \left((a_2 + a_1 a_3)^2 - 3a_1 a_2 a_3 + \frac{\alpha + \beta}{8\gamma}c_1 \right)^{\frac{1}{2}} \right]^{-1}$$  \hspace{1cm} (23)

where the two signs signal the two directions. Note that in order to have $c$ real we can always adjust the integration constant $c_1$ to keep $(a_2 + a_1 a_3)^2 + \frac{\alpha + \beta}{8\gamma}c_1 > 3a_1 a_2 a_3 + \frac{1}{16\gamma}$.

Further we get from \(9\) and \(16\) the first integral

$$v_0 = \frac{du_0}{d\xi} = -\frac{1}{2P}[Q + 4(a_2 + a_1 a_3)] - \frac{6a_1 a_2}{P} \int f^2 \, d\xi.$$  \hspace{1cm} (24)

which in turn gives the zeroth-order solution

$$u_0 = -\frac{1}{2P}[Q + 4(a_2 + a_1 a_3)]\xi - \frac{6a_1 a_2}{P} \int f^2 \, d\xi.$$  \hspace{1cm} (25)

As is evident, $u_0$ depends upon the choice of the auxiliary function $f$. In Table 2 we furnish the various forms for $u_0$ corresponding to different elliptic functions of Table 1. A sample graph of the zeroth-order solution $u_0$ for $f = sn(\xi, k)$ is depicted in Figure 1 and a plot of wave-speed for variation of the constant $c_1$ is shown in Figure 2.
Figure 1: Zeroth-order exact solution corresponding to $f(\xi) = sn(\xi, k)$ for $k = 0.5, \alpha = 1, \beta = 4, \gamma = 1$

Figure 2: Wave speed $c$ vs constant of integration $c_1$ for $f(\xi) = sn(\xi, k)$ for $k = 0.5, \alpha = 1, \beta = 4, \gamma = 1$
Table 2: Wave velocity $c = \frac{1}{1 \pm \frac{k^2}{1 - k^2} + \frac{\alpha + \beta}{2 \gamma} c_1}$, $E(\phi, k)$ is the incomplete elliptic integral function of second kind where $\sin \phi = \text{sn}(\xi, k)$ and $\Lambda = -\frac{1}{2\pi}[Q - 4(1 + k^2)]$.

<table>
<thead>
<tr>
<th>$f(\xi)$</th>
<th>Zeroth-order solution $u_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{sn}(\xi, k)$ or $\text{cn}(\xi, k)$ or $\text{dn}(\xi, k)$</td>
<td>$\Lambda \xi + \frac{6}{\pi} E(\phi, k)$</td>
</tr>
<tr>
<td>$\text{nc}(\xi, k)$ or $\text{sc}(\xi, k)$ or $\text{dc}(\xi, k)$</td>
<td>$\Lambda \xi + \frac{6}{\pi} [E(\phi, k) - \text{sn}(\xi, k)\text{dc}(\xi, k)]$</td>
</tr>
<tr>
<td>$\text{nd}(\xi, k)$ or $\text{sd}(\xi, k)$ or $\text{cd}(\xi, k)$</td>
<td>$\Lambda \xi + \frac{6}{\pi} [E(\phi, k) - k^2 \text{sn}(\xi, k)\text{cd}(\xi, k)]$</td>
</tr>
</tbody>
</table>

2.2 First-order exact solution

Knowing $v_0$ from (24), equation (13) reads

$$\frac{d^2v_1}{d\xi^2} + [-4(a_2 + a_1a_3) - 12a_1a_2f^2]v_1 = 0,$$

(26)

which matches with (1) for $\lambda = -4(a_2 + a_1a_3)$. This gives $n = 2$ implying that the solution of (26) can be written as

$$v_1 = \frac{du_1}{d\xi} = c_2 L_2(\xi) = c_2 f(1 + a_1f^2)^{\frac{1}{2}}(1 + \frac{a_2}{a_3}f^2)^{\frac{1}{2}},$$

(27)

where $c_2$ is an arbitrary constant.

Integration of (27) with the use of (2) immediately provides

$$u_1 = \frac{c_2}{2\sqrt{a_3}} f^2$$

(28)

which serves as a first order approximation to equation (8). Figure 3 gives a plot of $u_1$ for $f = \text{sn}(\xi, k)$.

Figure 3: First-order exact solution corresponding to $f(\xi) = \text{sn}(\xi, k)$ for $k = 0.5$, $\alpha = 1$, $\beta = 4$, $\gamma = 1$, $c_2 = 1$
2.3 Second-order exact solution

Using (24) and (27) the second-order equation (14) takes the form

\[ \frac{d^2v_2}{d\xi^2} + [-4(a_2 + a_1a_3) - 12a_1a_2 f^2]v_2 = -Pc_2^2 f^2 (1 + a_1 f^2) (1 + \frac{a_2}{a_3} f^2). \]  

(29)

We observe that the coefficients of (29) are polynomials in \( f \). This prompts us to take a polynomial ansatz for \( v_2 \) namely

\[ v_2 = \sum_{i=0}^{l} B_i f^i. \]  

(30)

Substitution of the above into (29) we get \( l = 4 \) on equating the highest power of \( f \) from both sides. Thus (30) is reduced to the biquadratic form

\[ v_2 = B_0 + B_1 f + B_2 f^2 + B_3 f^3 + B_4 f^4. \]  

(31)

Putting (31) into (29) and equating the coefficients of \( f^i \) (\( i = 0 \) to 6) to zero, we have the following system of coupled equations,

\[ a_3B_2 - 2(a_2 + a_1a_3)B_0 = 0, \]
\[ (a_2 + a_1a_3)B_1 - 2a_3B_3 = 0, \]
\[ 12a_3B_4 - 12a_1a_2B_0 + Pc_2^2 = 0, \]
\[ (a_2 + a_1a_3)B_3 - 2a_1a_2B_4 = 0, \]
\[ -6a_1a_2B_2 + 12(a_2 + a_1a_3)B_4 + Pc_2^2 (a_1 + \frac{a_2}{a_3}) = 0, \]
\[ a_1a_2B_3 = 0, \]
\[ 8a_1a_2B_4 + Pc_2^2 \frac{a_1a_2}{a_3} = 0. \]  

(32)

Solving for \( B_0, B_1, B_2, B_3, B_4 \) we obtain

\[ B_0 = -\frac{c_2^2 P}{24a_1a_2}, \quad B_1 = 0, \quad B_2 = -\frac{(a_2 + a_1a_3)c_2^2 P}{12a_1a_2a_3}, \quad B_3 = 0, \quad B_4 = -\frac{c_2^2 P}{8a_3}. \]  

(33)

Hence the solution of equation (29) can be written as

\[ v_2 = \frac{du_2}{d\xi} = -\frac{c_2^2 P}{24a_1a_2a_3} [a_3 + 2(a_2 + a_1a_3) f^2 + 3a_1a_2 f^4]. \]  

(34)

Using (2) we can integrate (34) to get

\[ u_2 = -\frac{c_2^2 P}{24a_1a_2a_3} f (1 + a_1 f^2)^{\frac{1}{2}} (a_3 + a_2 f^2)^{\frac{1}{2}}. \]  

(35)

\( u_2 \) is plotted for \( f = sn(\xi, k) \) in Figure 4.
Figure 4: Second-order exact solution corresponding to $f(\xi) = sn(\xi, k)$ for $k = 0.5$, $\alpha = 1$, $\beta = 4$, $\gamma = 1$, $c_2 = 1$

2.4 A second-order perturbative result

Given our findings (25), (28) and (35) we can write down a second-order perturbative solution to the GSWW equation (8) as follows

$$u = -\frac{1}{2P}[Q + 4(a_2 + a_1a_3)]\xi - \frac{6a_1a_2}{P} \int f^2 \, d\xi + \frac{\epsilon c_2}{2\sqrt{a_3}} f^2$$

$$-\frac{\epsilon^2 c_2^2 P}{24a_1a_2a_3} f(1 + a_1f^2)^{\frac{1}{2}}(a_3 + a_2f^2)^{\frac{3}{2}}.$$ (36)

This result is quite general in the sense that it can admit of any form of $f$ as listed in Table 1. For instance, taking $f = sn(\xi, k)$ and $\epsilon = 0.1$ the corresponding solution can be displayed graphically as shown in Figure 5.

Figure 5: Second-order perturbative solution corresponding to $f(\xi) = sn(\xi, k)$ for $\epsilon = 0.1$, $k = 0.5$, $\alpha = 1$, $\beta = 4$, $\gamma = 1$, $c_2 = 1$
3 Application to other nonlinear evolution equations

In the previous section, we considered a nonlinear second-order ODE containing a functional parameter \( f \) and obtained specific solutions in terms of \( f \). It was shown that the use of Jacobi elliptic function method in a perturbative way yielded new multi-order exact solutions that depended upon \( f \) in a generic way. In this section, we show that our approach can be profitably applied to other nonlinear equations resulting in more general types of solutions that have not been reported to the best of our knowledge. A complete list of our results are presented in Table 3.

Table 3: New general solutions for various non-linear evolution equations

<table>
<thead>
<tr>
<th>Equation</th>
<th>Multi-order exact solutions ( u_n, n = 0, 1, 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) KdV equation ( u_t + \alpha uu_x + \beta u_{xxx} = 0 )</td>
<td>( u_0 = \pm \frac{c}{a} - \frac{3\beta c^2}{a} [(a_2 + a_1a_3 + 3a_1a_2f^2], ) [ where ( c = \pm \beta \gamma^2 {16(a_2 + a_1a_3)^2 - 48a_1a_2a_3 + 2\frac{a}{\beta \gamma^2} - c_1)^{\frac{1}{2}} \right) ] ( u_1 = c_2f (1 + a_1f^2)\frac{1}{2} (1 + \frac{a_2}{a_3}f^2)\frac{1}{2} ) ( u_2 = -\frac{c_3^2}{4a_1a_2a_3\beta \gamma^2} [a_3 + 2(a_2 + a_1a_3)f^2 + 3a_1a_2f^4] )</td>
</tr>
<tr>
<td>(ii) mKdV equation ( u_t + \alpha u^2 u_x + \beta u_{xxx} = 0 )</td>
<td>( u_0 = \pm \sqrt{-\frac{6\beta c^2a_1a_2}{a} f}, ) [ where ( c = \beta \gamma^2 (a_2 + a_1a_3) ] ( u_1 = c_1(1 + a_1f^2)\frac{1}{2} (1 + \frac{a_2}{a_3}f^2)\frac{1}{2} ) ( u_2 = \pm \frac{c_2^2}{2a_3} \sqrt{-\frac{6\beta c_1a_1a_2}{a} f} [(a_2 + a_1a_3) + 2a_1a_2f^2] )</td>
</tr>
<tr>
<td>(iii) Boussinesq equation ( u_{tt} - c_0^2 u_{xx} - \alpha u_{xxxx} - \beta (u^2)_{xxx} = 0 )</td>
<td>( u_0 = -\frac{1}{2\beta} (c_0^2 - c^2 + 4a\gamma^2 (a_2 + a_1a_3)) - \frac{6\alpha c^2}{\beta} a_1a_2f^2; ) [ where ( c = {c_0^2 \pm 4a\gamma^2 (a_2 + a_1a_3)^2 - 3a_1a_2a_3 }^{\frac{1}{2}} ] ( u_1 = c_1(1 + a_1f^2)\frac{1}{2} (1 + \frac{a_2}{a_3}f^2)\frac{1}{2} ) ( u_2 = -\frac{c_2^2}{2a_3} \sqrt{-\frac{6\beta c_1a_1a_2}{a} f} [(a_2 + a_1a_3) + 2a_1a_2f^2] )</td>
</tr>
<tr>
<td>(iv) Klein-Gordon equation ( u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0 )</td>
<td>( u_0 = \pm \sqrt{-\frac{2(a_2-1)\gamma^2 a_1a_2}{\beta} f}, ) [ where ( c = \pm {1 - \frac{\alpha}{\gamma^2 (a_2 + a_1a_3)} }^{\frac{1}{2}} ] ( u_1 = c_1(1 + a_1f^2)\frac{1}{2} (1 + \frac{a_2}{a_3}f^2)\frac{1}{2} ) ( u_2 = \pm \frac{c_2^2}{2a_3} \sqrt{-\frac{2(a_2-1)\gamma^2 a_1a_2}{\beta} f} [(a_2 + a_1a_3) + 2a_1a_2f^2] )</td>
</tr>
<tr>
<td>(v) mBBM equation ( u_t + c_0 u_x + u^2 u_x + \beta u_{xxx} = 0 )</td>
<td>( u_0 = \pm \sqrt{6\beta^2 c a_1a_2 f}, ) [ where ( c = \frac{c_0}{1 + (a_2 + a_1a_3)^{\frac{1}{2}}} ] ( u_1 = c_1(1 + a_1f^2)\frac{1}{2} (1 + \frac{a_2}{a_3}f^2)\frac{1}{2} ) ( u_2 = \pm \frac{c_2^2}{2a_3} \sqrt{6\beta^2 c a_1a_2 f} [(a_2 + a_1a_3) + 2a_1a_2f^2] )</td>
</tr>
</tbody>
</table>
(i) KdV equation

The concerned equation is of the form

\[ u_t + \alpha uu_x + \beta u_{xxx} = 0 \]  

(37)

where \( \alpha \) and \( \beta \) are real parameters. Let us choose \( f = cd(\xi, k) \) for which \( a_1 = -1, a_2 = -k^2 \) and \( a_3 = 1 \). Substituting these values into the zeroth-order, first-order and second-order exact solutions listed in Table 3, we find

\[
\begin{align*}
    u_0 &= c + \frac{4\beta \gamma^2}{\alpha} (1 + k^2) - \frac{12\beta \gamma^2}{\alpha} k^2 cd(\xi, k) \\
    u_1 &= c_2 k^2 cd(\xi, k) sd(\xi, k) nd(\xi, k) \\
    u_2 &= -\frac{c_2^2 \alpha}{48\beta \gamma^2 k^2} [1 - 2(1 + k^2) cd^2(\xi, k) + 3k^2 cd^4(\xi, k)].
\end{align*}
\]

(38)

These solutions have been obtained in [8] for \( \alpha = 1 \) where the authors used the notations \( m \) and \( k \) in places of \( k \) and \( \gamma \) respectively and their scale is \( A = k^2 c_2, c_1 = 0 \). The KdV equation has also been studied in [7] and their results can be extracted from ours for \( f = sn(\xi, k) \) corresponding to \( \alpha = 1 \).

(ii) mKdV equation

\[ u_t + \alpha u^2 u_x + \beta u_{xxx} = 0. \]  

(39)

Our general result listed in Table 3 contains, as special cases, the following results reported in earlier works.

- \( f = sn(\xi, k) \) (Ref [7])

\[
\begin{align*}
    u_0 &= \pm \gamma k \sqrt{-\frac{6\beta}{\alpha}} sn(\xi, k) \\
    u_1 &= c_1 cn(\xi, k) dn(\xi, k) \\
    u_2 &= \pm \frac{(1 + k^2)c_1^2}{12\gamma k} \sqrt{-\frac{6\alpha}{\beta}} sn(\xi, k) \left[ \frac{2k^2}{1 + k^2} sn^2(\xi, k) - 1 \right]
\end{align*}
\]

(40)

- \( f = cd(\xi, k) \) (Ref [8])

\[
\begin{align*}
    u_0 &= \pm \gamma k \sqrt{-\frac{6\beta}{\alpha}} cd(\xi, k) \\
    u_1 &= (1 - k^2)c_1 sd(\xi, k) nd(\xi, k) \\
    u_2 &= \pm \frac{(1 + k^2)c_1^2}{12\gamma k} \sqrt{-\frac{6\alpha}{\beta}} cd(\xi, k) \left[ \frac{2k^2}{1 + k^2} cd^2(\xi, k) - 1 \right]
\end{align*}
\]

(41)
(iii) **Boussinesq equation**

\[ u_{tt} - c_0^2 u_{xx} - \alpha u_{xxxx} - \beta (u^2)_{xx} = 0. \]  

From the general solution given in terms of \( f \) (see Table 3) one can deduce the ones obtained in [7] as special case for \( f = sn(\xi, k) \).

(iv) **Klein-Gordon equation**

\[ u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0. \]  

The above equation was considered in [9]. Their results correspond to the case \( f = dn(\xi, k) \) and we do indeed recover their solutions from Table 3 as given below:

\[
\begin{align*}
    u_0 &= \pm \sqrt{\frac{2\alpha}{\beta \gamma^2 (k^2 - 2)}} \, dn(\xi, k) \\
    u_1 &= \frac{k^2}{k'} \sqrt{-c_1^2} \, sn(\xi, k) cn(\xi, k) \\
    u_2 &= \mp \frac{c_1^2}{2(1 - k^2)} \sqrt{\frac{\beta}{2 \gamma^2 (c^2 - 1)}} \, dn(\xi, k) [2 - k^2 - 2dn^2(\xi, k)]
\end{align*}
\]

(v) **mBBM equation**

\[ u_t + c_0 u_x + u^2 u_x + \beta u_{xxx} = 0. \]  

For \( f = sn(\xi, k) \) the multi-order solutions to (45) read from Table 3

\[
\begin{align*}
    u_0 &= \pm \gamma \sqrt{6c\beta} \, sn(\xi, k) \\
    u_1 &= c_1 cn(\xi, k) dn(\xi, k) \\
    u_2 &= \mp \frac{c_1^2 (1 + k^2)}{12 \gamma k} \sqrt{\frac{6}{c\beta}} \, sn(\xi, k) [12 - \frac{2k^2}{1 + k^2} sn^2(\xi, k)]
\end{align*}
\]

These results were obtained in Ref [7].

### 4 Concluding remarks

We have reported exact multi-order periodic solutions for a generalized shallow water wave equation based on various types of Jacobi elliptic functions in a perturbative framework. The presence of an auxiliary function, which makes reference to any particular Jacobi function explicit, can be well adjusted to connect with other classes of PDE such as the KdV, modified KdV, Boussinesq, Klein-Gordon and modified Benjamin-Bona-Mahony equation. The general character of our results encompasses those special cases which have earlier been studied in the literature.
References


