ON TOTALLY WEIGHTED INTERCONNECTION NETWORKS

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Received 3 December 2012
Revised 21 March 2013

In a network, each arc is assigned a weight. The weight of a path or a cycle is defined as the minimum weight of its arcs. The maximum of weights of all paths between two nodes is defined as the strength of connectedness between the nodes. In network applications, the reduction in the strength of connectedness is more relevant than the total disconnection of the graph. A graph is totally weighted if both node set and arc set are weighted. A precisely weighted graph is a totally weighted complete graph with weight functions \( \sigma : V \to \mathbb{R}^+ \) and \( \mu : E \to \mathbb{R}^+ \) such that \( \mu(x, y) = \sigma(x) \land \sigma(y) \) for any pair of nodes \( x, y \) of \( G \) where \( \land \) represents the minimum. Different types of arcs and the existence of partial bridges in precisely weighted complete graphs are studied. The minimum and maximum strong degrees of such a graph are also obtained. Also the concept of node strength sequence is introduced and relations between node strength sequence and strong degrees are obtained.

Keywords: Totally weighted graph, precisely complete graph, partial cutnode, partial bridge, strong path, node strength sequence.

1. Introduction

Wireless networks are studied extensively due to their wide applications. The power consumption of a station determines its transmission range, and thus also the stations it can send messages to; the power typically increases at least quadratically in the transmission range. Assigning power levels to the stations (nodes) determines the resulting communication network. The lifetime of wireless networks also proportional to the battery capacity of the sensor in the transmitting station. A graph with one node weight function and one arc weight function will be the most suitable model of such a network.¹

Weighted graphs are as old as that of graphs. In majority of applications related with graphs, especially in networks, weighted graph models are used. Minimum and maximum spanning tree problems are also well known. In the designing of a
network, only the capacity is relevant. The reduction in the flow is more relevant and frequently occurring than the total disconnection of the graph. Presently all investigations are related with cut-sets; that is set of nodes or arcs whose removal disconnects the graph. This is the motivation for this paper and we investigate the properties of sets of nodes and arcs whose removal weakens the flow between some pair of nodes.

Several authors have made remarkable contributions to weighted graph theory. Bondy and Fan\cite{4,5} studied different types of paths and cycles in weighted graphs. Zang, Li and Broersma\cite{17} introduced heavy paths and cycles in networks. Batagelj and Mrvar\cite{2,3} studied distance and degree concepts in networks and introduced a subquadratic triad census algorithm for large sparse networks. Everett\cite{7–9} applied graph connectivity problems in Social network theory. Kleinberg \textit{et al.}\cite{10,11} applied graph connectivity parameters to the problems involving network failure. Some contributions in graph networks related to the strength of connectedness and flow were made by Mathew and Sunitha.\cite{13,15,16} They have generalized the celebrated Menger’s theorem in Graph Theory.\cite{14}

Section 2 contains the preliminaries required for the development of this paper. The concepts like strength of connectedness, partial cutnode, partial bridge, etc., are explained. Section 2 also contains some characterizations of partial bridges (Theorems 1 and 2), a characterization of partial cutnodes (Theorem 3) and a sufficient condition for an edge to be a partial cutnode (Proposition 4). The concept of precisely weighted graphs are introduced in Section 3. A precisely weighted graph is a totally weighted complete graph with some flow restrictions. It is proved that a precisely weighted graph has no partial cutnodes, but can have partial bridges (Corollary to Proposition 5 and Theorem 4). Partial bridges in precisely weighted graphs are characterized in Theorem 5. Different types of arcs in a precisely weighted graphs are studied in Section 4, and is shown that they have no $\delta$-arcs (Lemma 1). Some results related to strong paths in precisely weighted graphs are also obtained (Theorem 8 and 9). Section 5 deals with the strong degree in precisely weighted graphs. The minimum and maximum strong degrees of a precisely weighted graph are also obtained (Lemma 3). In Section 6, the concept of node strength sequences are introduced and number of nodes with minimum and maximum strong degrees are evaluated (Proposition 9).

2. Strength of Connectedness

We consider only undirected graphs without loops or multiple edges. Let $V(G)$ and $E(G)$ denote the set of vertices and edges of a graph $G$, respectively. $G$ is called a weighted graph if each edge $e$ is assigned a nonnegative real number $w(e)$, called the weight of $e$. For a subgraph $H$ of $G$, the weight of $H$ is defined by $w(H) = \sum_{e \in E(H)} w(e)$. An unweighted graph can be regarded as a weighted graph in which each edge $e$ is assigned weight $w(e) = 1$. 

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In a weighted graph $G$, we can associate, to each pair of nodes in $G$, a real number called strength of connectedness. It is evaluated using strengths of different paths joining the given pair of nodes. Consider the definition of the strength of a path which is given below. The following basic definitions are from Ref. [13].

**Definition 1.** Let $G$ be a weighted graph. The strength of a path $P = v_0e_1v_1......e_nv_n$, denoted by $s(P)$ is defined as $s(P) = w(e_1) \land w(e_2) \land w(e_3) \land ...... \land w(e_n)$, where $w(e_i)$ is the weight of the arc $e_i$ and $\land$ denotes the minimum.

Consequently the strength of a cycle $C$ in a weighted graph $G$ is defined as the minimum of the weights of arcs in $C$.

**Definition 2.** Let $G$ be a weighted graph. The strength of connectedness of a pair of nodes $u, v \in V(G)$, denoted by $\text{CONN}_G(u,v)$ is defined as $\text{CONN}_G(u,v) = \lor \{s(P) : P$ is a $u-v$ path in $G$, where $\lor$ denotes the maximum.$\}$

**Example 1.** Let $G(V,E)$ be a weighted graph with $V = \{a,b,c,d\}$ and $E = \{e_1 = (a,b),e_2 = (b,c),e_3 = (c,d),e_4 = (d,a)\}$ with $w(e_1) = 1, w(e_2) = 3, w(e_3) = 8, w(e_4) = 5$. Here $\text{CONN}_G(a,b) = 3, \text{CONN}_G(a,c) = 5, \text{CONN}_G(a,d) = 5, \text{CONN}_G(b,c) = 3, \text{CONN}_G(b,d) = 3, \text{CONN}_G(c,d) = 8$.

Next we have an obvious result.

**Proposition 1.** Let $G$ be a weighted graph and $H$, a weighted subgraph of $G$. Then for any pair of nodes $u, v \in G$, we have $\text{CONN}_H(u,v) \leq \text{CONN}_G(u,v)$.

**Definition 3.** A $u-v$ path in a weighted graph $G$ is called a strongest $u-v$ path if $s(P) = \text{CONN}_G(u,v)$.

**Definition 4.** Let $G$ be a weighted graph. A node $w$ is called a partial cutnode (p-cutnode) of $G$ if there exists a pair of nodes $u, v \in G$ such that $u \neq v \neq w$ and $\text{CONN}_{G-w}(u,v) < \text{CONN}_G(u,v)$.

**Definition 5.** Let $G$ be a weighted graph. An arc $e = (u,v)$ is called a partial bridge (p-bridge) if $\text{CONN}_{G-e}(u,v) < \text{CONN}_G(u,v)$.

As in graphs without weights, we can give simple characterizations for p-cutnodes and p-bridges in a weighted graph using paths as given below.

**Proposition 2.** Let $G$ be a weighted graph and let $w$ be a node in $G$. Then $w$ is a p-cutnode if and only if $w$ is an internal node in every strongest $x-y$ path for some pair of nodes $x, y \in V(G)$ other than $w$.

**Proposition 3.** Let $G$ be a weighted graph and let $e$ be an arc in $G$. Then $e$ is a p-bridge if and only if $e$ is an arc in every strongest $x-y$ path for some pair of nodes $x, y \in V(G)$.

**Remark 1.** Note that every cutnode of $G$ is a p-cutnode and every bridge of $G$, a p-bridge. The converse is not true as seen from the following example.

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Example 2. Let \( G(V, E) \) be a weighted graph with \( V = \{ a, b, c, d \} \) and \( E = \{ e_1 = (a, b), e_2 = (b, c), e_3 = (c, d), e_4 = (d, a), e_5 = (d, b) \} \) with \( w(e_1) = 1, w(e_2) = 3, w(e_3) = 7, w(e_4) = 4, w(e_5) = 3 \). Here \( (d, c) \) is a p-bridge since its deletion from \( G \) reduces the strength of connectedness between \( d \) and \( c \) from 7 to 3. Similarly \( (a, d) \) also is a p-bridge. Also \( c \) and \( d \) are p-cutnodes of \( G \) since \( \text{CONN}_{G-\{c,d\}}(b,d) = 1 < 3 = \text{CONN}_G(b,d) \) and \( \text{CONN}_{G-\{a,c\}}(a,c) = 1 < 4 = \text{CONN}_G(a,c) \). Note that \( G \) is 2-connected and hence have no cutnodes or bridges.

Now we give some equivalent conditions for an arc to be a p-bridge.

**Theorem 1.** Let \( G \) be a weighted graph and \( e = (x, y) \in E(G) \). Then the following statements are equivalent.

(i) \( e \) is a p-bridge.

(ii) \( \text{CONN}_{G-e}(x, y) < w(e) \).

(iii) \( e \) is not a weakest arc of any cycle in \( G \).

**Proof.** (ii) \( \Rightarrow \) (i) Assume (2) and suppose that \( e \) is not a p-bridge. Then \( \text{CONN}_{G-e}(x, y) = \text{CONN}_G(x, y) \geq w(e) \), which contradicts (ii) and hence (i) holds.

(i) \( \Rightarrow \) (iii) Assume (i) and suppose that \( e \) is a weakest arc of a cycle say \( C \). Then any path \( P \) involving the arc \( (x, y) \) can be converted into a path not containing the arc \( (x, y) \), whose strength is at least equal to that of the path \( P \) (using the path \( C - (x, y) \)). Thus arc \( (x, y) \) cannot be a p-bridge.

(iii) \( \Rightarrow \) (ii) If \( \text{CONN}_{G-e}(x, y) \geq w(e) \), then there is a path say \( P \) from \( x \) to \( y \) not involving arc \( e \) that has strength at least \( w(e) \). Then \( P \cup \{ e \} \) is a cycle of which \( e \) is a weakest arc.

If \( e = (x, y) \) is a p-bridge, it is not necessary that \( x \) or \( y \) is a p-cutnode. But common node of two p-bridges is always a p-cutnode as seen from the following proposition.

**Proposition 4.** If \( w \) is a common node of at least two p-bridges, then \( w \) is a p-cutnode.

**Proof.** Let \( (u_1, w) \) and \( (w, u_2) \) be two partial bridges. Then there exist some \( u, v \) such that \( (u_1, w) \) is on every strongest \( u - v \) path. If \( w \) is distinct from \( u \) and \( v \), it follows that \( w \) is a p-cutnode. Now suppose that one of \( u, v \) is \( w \) so that \( (u_1, w) \) is on every strongest \( u - w \) path or \( (w, u_2) \) is on every strongest \( w - v \) path. if possible, let \( w \) be not a p-cutnode. Then between every two nodes of \( G \), there exist at least one strongest path not containing \( w \). In particular, there exist at least one strongest path \( P \) joining \( u_1 \) and \( u_2 \), not containing \( w \). This path \( P \) together with \( (u_1, w) \) and \( (w, u_2) \) forms a cycle \( C \)(say).

Case I: If \( u_1, w, u_2 \) is not a strongest path, then clearly one of \( (u_1, w), (w, u_2) \) or both becomes the weakest arcs of the cycle \( C \). It contradicts the fact that \( (u_1, w) \) and \( (w, u_2) \) are p-bridges of \( G \).
Case II: If $u_1, w, u_2$ is also a strongest path joining $u_1$ to $u_2$, then $\text{CONN}_G(u_1, u_2) = \mu(u_1, w) \land \mu(w, u_2)$, the strength of $P$. Thus arcs of $P$ are at least as strong as $\mu(u_1, w)$ and $\mu(w, u_2)$ which implies that $(u_1, w)$ or $(w, u_2)$ or both are the weakest arcs of the cycle $C$, which is again a contradiction.

Now we characterize $p$-bridges using the concept of maximum spanning tree (MST) of a weighted graph.

**Theorem 2.** Let $G$ be a weighted graph. Then an arc $e \in E(G)$ is a $p$-bridge if and only if $e$ is in every maximum spanning tree of $G$.

**Proof.** Let $e = (x, y)$ be a $p$-bridge in a weighted graph $G$. Then, arc $(x, y)$ is the unique strongest $x - y$ path in $G$ and hence in every MST of $G$.

Conversely suppose that $e = (x, y)$ is an arc of $G$ which is in every MST of $G$. If $e$ is not a $p$-bridge, by Theorem 1, $e$ must be a weakest arc of some cycle say $C$ in $G$ and hence $\text{CONN}_G(x, y) \geq w(e)$. Thus there exist at least one MST of $G$ not containing the arc $e$. □

**Remark 2.** If $G$ is a weighted graph with $n$ nodes, then $G$ can have at most $n - 1$ $p$-bridges. The equality occurs when $G$ is a weighted tree with $n$ nodes.

Next we characterize $p$-cutnodes of a weighted graph $G$ using the concept of maximum spanning trees.

**Theorem 3.** A node $w$ in a weighted graph $G$ is a $p$-cutnode if and only if $w$ is an internal node of every MST.

**Proof.** Let $w$ be a $p$-cutnode in a weighted graph $G$. Then for some nodes $x, y \in V(G)$, $w$ is in every strongest $x - y$ path. Now every MST contains a unique strongest $x - y$ path and hence $w$ is an internal node of each MST of $G$.

Conversely suppose that $w$ is an internal node in every MST. Let $T$ be any MST and let $(u, w)$ and $(w, v)$ are arcs incident on $w$ in $T$. Clearly $uwv$ is a strongest $u - v$ path in $T$. If possible suppose that $w$ is not a $p$-cutnode. Then between every pair of nodes $u, v \in V(G)$, there exists a strongest path which avoids $w$. Consider one such $u - v$ path $P$. Clearly $P$ contains arcs which are not in $T$. $\text{CONN}_G(u, v)$ is either weight of arc $(u, w)$ or weight of $(w, v)$. If $\text{CONN}_G(u, v) = w((u, w))$, then arcs in $P$ have weight at least $w((u, w))$. Removal of arc $(u, w)$ and addition of $P$ in $T$ give another MST in which $w$ is a pendent node, a contradiction. □

**Remark 3.** From the above theorem it follows that in any weighted graph, there always exist at least two nodes which are non $p$-cutnodes.

In the following section we shall introduce a particular type of weighted graphs which are complete. The definition of a totally weighted graph is as follows.

**Definition 6.** A graph is said to be totally weighted if both its node set and arc set are weighted.
3. Precisely Weighted Graphs

Precisely weighted graphs are introduced in this section. Strength of connectedness between any two vertices in a precisely weighted graph is obtained and its partial bridges are characterized.

In any network, we note that flow through an arc cannot exceed the minimum of capacities of its end nodes. The maximum flow can be achieved by giving the minimum of weights of the end nodes to each arc. Motivated by this concept, we give the following definition.

**Definition 7.** A precisely weighted graph (PWG) is a complete graph $G(V, E)$ with weight functions $\sigma : V \rightarrow \mathbb{R}^+$ and $\mu : E \rightarrow \mathbb{R}^+$ such that $\mu(x, y) = \sigma(x) \land \sigma(y)$ for any pair of nodes $x, y$ of $G$, where $\land$ denote the minimum. It will be denoted by $G(\sigma, \mu)$.

Note that if $G$ is an PWG, then it is a weighted complete graph.

**Example 3.** Let $G(\sigma, \mu)$ be a totally weighted graph with $V = \{u, v, w, x\}$, $\sigma(u) = 1, \sigma(v) = 0.5, \sigma(w) = 0.8, \sigma(x) = 0.1, \mu(u, v) = 0.5 = \mu(v, w), \mu(w, x) = 0.1 = \mu(x, u), \mu(u, w) = 0.8, \text{ and } \mu(v, x) = 0.1$. Then $G$ is an PWG.

The strength of connectedness between two nodes of a precisely weighted graph can be easily evaluated from the following proposition.

**Proposition 5.** If $G(\sigma, \mu)$ is a precisely weighted graph with node set $V$, then $\text{CONN}_G(u, v) = \mu(u, v)$ for any two nodes in $V$.

**Proof.** Let $G(\sigma, \mu)$ be a precisely weighted graph with node set $V$ and arc set $E$. Let $u, v \in V$. Consider all paths of length two from $u$ to $v$. We denote the maximum of strengths of all such paths from $u$ to $v$ by $\mu^2(u, v)$. Then
\[
\mu^2(u, v) = \lor_{w \in V} \{\mu(u, w) \land \mu(w, v)\} = \lor_{w \in V} \{\sigma(u) \land \sigma(w) \land \sigma(v)\} \leq \sigma(u) \land \sigma(v) = \mu(u, v)
\]
Similarly the maximum of strengths of all $u - v$ paths of length 3, denoted by $\mu^3(u, v) \leq \mu(u, v)$ and in the same way we can show that $\mu^k(u, v) \leq \mu(u, v)$ for all positive integer $k$. By definition $\text{CONN}_G(u, v) = \text{Sup}\{s(P) : P \text{ is a } u - v \text{ path in } G\} = \mu(x, y)$, which completes the proof.

Thus in a precisely weighted graph an arc joining any two nodes $u$ and $v$ is a shortest strongest $u - v$ path.

The concept of a partial cutnode is defined in Ref. [13]. A node $x$ is a partial cutnode of $G$ if deleting $x$ reduces the strength of connectedness between some other pair of nodes. Hence we have the following corollary.

**Corollary.** A precisely weighted graph has no partial cutnodes.
The concept of a partial bridge is given in Definition 5. A complete graph will not have any bridges. But a precisely weighted graph can have a partial bridge as seen from the following theorem.

**Theorem 4.** A precisely weighted graph can have at most one partial bridge.

**Proof.** (By induction on \(|V|\)). Let \(G(\sigma, \mu)\) be a precisely weighted graph with node set \(V\) and arc set \(E\) where \(|V| = 3\). Then \(G\) can have at most one p-bridge by Proposition 4 and Corollary above. Assume that any precisely weighted graph with less than \(k\) nodes possesses at most one p-bridge. Let \(G_k\) denote a PWG with \(k\) nodes \(v_1, v_2, ..., v_k\). Remove the node \(v_k\) from \(G_k\) and the resulting weighted graph is \(G_{k-1}\). By assumption \(G_{k-1}\) can have at most one p-bridge. If \(G_{k-1}\) has no p-bridges then the proof is over because all the \(k - 1\) arcs \((v_1, v_k), (v_2, v_k), ..., (v_{k-1}, v_k)\) are adjacent and any two adjacent p-bridges will contribute a p-cutnode (namely the common node), which is not possible, by corollary above. If \(G_{k-1}\) has a p-bridge say \((v_r, v_l) : 1 \leq r, l \leq k - 1\), then none of the arcs \((v_1, v_k), (v_2, v_k), ..., (v_{k-1}, v_k)\) can be a p-bridge, for otherwise let \((v_p, v_k) : 1 \leq p \leq k - 1\) be a p-bridge. Now consider the four nodes \(v_r, v_l, v_p\) and \(v_k\). Arrange them so that their \(\sigma\) values are in ascending order and rename them as \(u_1, u_2, v_3\) and \(u_4\). Then \((u_1, u_2), (u_1, u_3)\) and \((u_1, u_4)\) are not p-bridges, they being the weakest arcs of some cycle in the subgraph induced by \(u_1, u_2, u_3\) and \(u_4\). A p-bridge cannot be the weakest arc of any cycle, by Theorem 1.)

Now let the arcs \((u_2, u_3), (u_2, u_4)\) and \((u_3, u_4)\) form a triangle say \(C\). Since we have two p-bridges from among the combinations of \(u_1, u_2, u_3\) and \(u_4\), they must be adjacent arcs in \(G\), which is a contradiction since any two adjacent p-bridges contribute a p-cutnode and a precisely weighted graph has no p-cutnodes. Thus it follows that \(G_k\) has at most one p-bridge and the theorem is proved by induction.

More over we can locate the p-bridge as given in the following theorem.

**Theorem 5.** Let \(G(\sigma, \mu)\) be a precisely weighted graph with node set \(V\) and arc set \(E\) where \(|V| = n\). Then \(G\) has a partial bridge if and only if there exists an increasing sequence \(\{t_1, t_2, ..., t_n\}\) of positive real numbers such that \(t_{n-2} < t_{n-1} \leq t_n\) where \(t_i = \sigma(u_i)\) for \(i = 1, 2, ..., n\). Also the arc \((u_{n-1}, u_n)\) is the p-bridge of \(G\).

**Proof.** Assume that \(G : (\sigma, \mu)\) is a precisely weighted graph and that \(G\) has a p-bridge \((u, v)\). Now \(\mu(u, v) = \sigma(u) \wedge \sigma(v)\). Without loss of generality let \(\sigma(u) \leq \sigma(v)\), so that \(\mu(u, v) = \sigma(u)\). Also note that \((u, v)\) is not a weakest arc of any cycle in \(G\). Now it is required to prove that \(\sigma(u) \geq \sigma(w)\) for every \(w \neq v\). On the contrary assume that there is at least one node \(w \neq v\) such that \(\sigma(u) \leq \sigma(w)\). Now consider the cycle \(C : u, v, w, u\).

Then \(\mu(u, v) = \mu(u, w) = \sigma(u)\) and
\[\mu(v, w) = \sigma(v)\] if \(\sigma(u) = \sigma(v)\) or \(\sigma(u) < \sigma(v) \leq \sigma(w)\) and
\[\mu(v, w) = \sigma(w)\] if \(\sigma(u) < \sigma(w) < \sigma(v)\).

In either case \((u, v)\) becomes a weakest arc of a cycle which contradicts the fact that \((u, v)\) is a p-bridge.
Conversely, let $t_1 \leq t_2 \leq \ldots \leq t_{n-2} \leq t_{n-1} \leq t_n$ and $t_i = \sigma(u_i)$ for all $i$. We claim that arc $(u_{n-1}, u_n)$ is the p-bridge of $G$. We have $\mu(u_{n-1}, u_n) = \sigma(u_{n-1}) \land \sigma(u_n) = \sigma(u_{n-1})$ and by hypothesis, all other arcs of $G$ will have strength strictly less than $\sigma(u_{n-1})$. Thus the arc $(u_{n-1}, u_n)$ is not a weakest arc of any cycle in $G$ and hence is a p-bridge.

4. Arcs in a Precisely Weighted Graph

In\textsuperscript{13} Mathew and Sunitha have introduced the concepts of strong arcs in weighted graphs. The authors have classified the strong arcs of a weighted graph into three and analyzed the types of arcs in different weighted graph structures. Different types of arcs in a precisely weighted graph are discussed in this section. It is observed that it has no $\delta$-arcs and can have at most one $\alpha$-strong arc. Characterization of partial bridges using $\alpha$-strong arcs and the existence of a $\beta$-strong path between any two nodes of a precisely weighted graph are also presented. Now some of the definitions in Ref. [13] may be recalled.

**Definition 8.** An arc of a weighted graph $G(V, E)$ with weight function $\mu : E \to \mathbb{R}^+$ is called strong if its weight is at least as great as the connectedness of its end nodes when it is deleted, that is $\mu(x, y) \geq \text{CONN}_{G−e}(x, y)$ where $G − e$ is the weighted subgraph of $G$ obtained by deleting the weighted arc $(x, y)$. An $x−y$ path $P$ is called a strong path if $P$ contains only strong arcs. If $(x, y)$ is a strong arc, we say that $x$ and $y$ are strong neighbors. Also a cycle is called a strong cycle, if all its arcs are strong.

**Example 4.** Let $G(V, E)$ be a weighted graph with $V = \{u, v, w\}$, $\mu(u, v) = 0.6 = \mu(v, w)$ and $\mu(w, u) = 0.1$. Then arcs $(u, v)$ and $(v, w)$ are strong arcs, while arc $(w, u)$ is not strong as $\mu(w, u) = 0.1 < 0.6 = \text{CONN}_{G−e}(w, u)$.

**Proposition 4.** An arc $(x, y)$ in a weighted graph $G$ is strong if and only if $w(e) = \text{CONN}_G(x, y)$.

**Proposition 5.** Let $G$ be connected weighted graph and let $x, y$ be any two nodes in $G$. Then there exists a strong path from $x$ to $y$.

In a graph without weights, all pairs of nodes have strength of connectedness one. But in weighted graphs, the strength of connectedness is different for different pairs of nodes. We classify strong arcs as $\alpha$-strong and $\beta$-strong as in the following definition.

**Definition 9.** Let $G$ be a weighted graph. Then an arc $e = (x, y) \in E$ is called $\alpha$-strong if $\text{CONN}_{G−e}(x, y) < w(e)$, $\beta$-strong if $\text{CONN}_{G−e}(x, y) = w(e)$ and a $\delta$-arc if $\text{CONN}_{G−e} > w(e)$. A $\delta$-arc $e$ is called a $\delta^*$-arc if $e$ is not a weakest arc of $G$. 
Clearly an arc \( e \) is strong if it is either \( \alpha \)-strong or \( \beta \)-strong. That is arc \((x, y)\) is strong if its weight is at least equal to the strength of connectedness between \( x \) and \( y \) in \( G \).

**Definition 10.** A \( u-v \) path \( P \) in \( G \) is called a strong \( u-v \) path if all arcs in \( P \) are strong. In particular if all arcs of \( P \) are \( \alpha \)-strong, then \( P \) is called an \( \alpha \)-strong path and if all arcs of \( P \) are \( \beta \)-strong, then \( P \) is called a \( \beta \)-strong path.

**Example 5.** Let \( G(V, E) \) be a weighted graph with \( V = \{a, b, c, d\} \) and \( E = \{e_1 = (a, b), e_2 = (b, c), e_3 = (c, d), e_4 = (d, a), e_5 = (a, c)\} \) with \( w(e_1) = 7, w(e_2) = 8, w(e_3) = 2, w(e_4) = 2, w(e_5) = 4 \). \( (a, b) \) and \( (b, c) \) are \( \alpha \)-strong, \( (c, d) \) and \( (d, a) \) are \( \beta \)-strong and arc \( (a, c) \) is a \( \delta^* \)-arc. Clearly arc \( (a, c) \) is \( \delta^* \) since it is not a weakest arc in \( G \). Also \( P_1 = abc \) is an \( \alpha \)-strong path, \( P_2 = cda \) is a \( \beta \)-strong path. In \( G \), \( C_1 = abcd \) is a strong cycle but \( C_2 = abca \) is not a strong cycle.

**Theorem 6.** An arc \( e \) in a weighted graph \( G \) is a partial bridge if and only if \( e \) is \( \alpha \)-strong.

Next the types of arcs in an PWG are discussed. The number of \( \beta \)-strong arcs in an PWG is obtained and the existence of a \( \beta \)-strong path between any two nodes of an PWG is shown. Also in a PWG without \( \alpha \)-strong arcs, it is established that the concepts of strong path and strongest path coincide. It is observed that all arcs in an PWG are strong. In a complete graph, there are no bridges, but an PWG may contain \( p \)-bridges. Hence we have the following two Lemmas.

**Lemma 1.** A precisely weighted graph has no \( \delta \)-arcs.

**Proof.** Let \( G(\sigma, \mu) \) be a precisely weighted graph. If possible assume that \( G \) contains a \( \delta \)-arc \((u, v)\)(say). Then,

\[
\mu(u, v) < \text{CONN}_{G-\{u, v\}}(u, v).
\]

That is, there exists a stronger path \( P \) other than the arc \((u, v)\) from \( u \) to \( v \) in \( G \). Let \( \mu(u, v) = p \) and the strength of the path \( P \) be \( q \). Then \( p < q \). Let \( w \) be the first node in \( P \) after \( u \). Then,

\[
\mu(u, w) > p \ldots \ldots (1)
\]

Similarly let \( x \) be the last node in \( P \) before \( v \). Then,

\[
\mu(x, v) > p \ldots \ldots (2)
\]

Since \( \mu(u, v) = p \), at least one of \( \sigma(u) \) or \( \sigma(v) \) should be \( p \). Now \( G \) being an PWG, (1) gives a contradiction if \( \sigma(u) = p \) and (2) gives a contradiction if \( \sigma(v) = p \); which completes the proof.

By Theorem 4, an PWG has at most one \( p \)-bridge and by Theorem 6, an arc \( e \) in a weighted graph \( G \) is a partial bridge if and only if \( e \) is \( \alpha \)-strong. So we have the lemma given below.
Lemma 2. There exists at most one $\alpha$-strong arc in an PWG.

Using Lemmas 1 and 2 we have the following two theorems.

Theorem 7. Let $G(\sigma, \mu)$ be a PWG with $|V| = n$. Then the number of $\beta$-strong arcs in $G$ is $^nC_2$ or $^nC_2 - 1$ where $^nC_2$ denotes the number of combinations of $n$ things taken two at a time given by the formula $^nC_2 = \frac{n!}{2!(n-2)!}$.

Theorem 8. Let $G(\sigma, \mu)$ be an PWG. Then there exist $\beta$-strong paths between any two nodes of $G$.

Theorem 9. Let $G$ be a PWG without $\alpha$-strong arcs and let $P$ be any $x-y$ path in $G$ with maximum strength. Then the following two conditions are equivalent.

(i) $P$ is a strong $x-y$ path.

(ii) $P$ is a strongest $x-y$ path.

Proof. $(i) \implies (ii)$

Let $G$ be a PWG without $\alpha$-strong arcs and let $P$ be any $x-y$ path in $G$. Assume that $P$ is a strong $x-y$ path. By Lemma 1, all arcs of $G$ are $\beta$-strong. Then by definition of a $\beta$-strong arc,

$$\text{CONN}_{G-(x,y)}(x,y) = \mu(x,y) = \text{strength of } P.$$

Now, since $G$ is an PWG, by Proposition 4,

$$\text{CONN}_G(x,y) = \mu(x,y).$$

From (1) and (2),

$$\text{CONN}_G(x,y) = \text{strength of } P,$$

which implies that $P$ is a strongest $x-y$ path.

$(ii) \implies (i)$

Let $P$ be a strongest $x-y$ path in $G$. By Lemma 1, $P$ contains only $\beta$-strong arcs and hence is a strong $x-y$ path, which completes the proof.

Remark 6. Converse of Theorem 9 does not hold generally as illustrated in the following example.

Example 6. Let $G(\sigma, \mu)$ be a weighted graph with $V = \{u, v, w, x\}$ and $\sigma(u) = 0.5$, $\sigma(v) = 0.4, \sigma(w) = 0.7, \sigma(x) = 0.5$ and $\mu(u, v) = \mu(v, w) = \mu(w, x) = \mu(x, u) = \mu(v, x) = \mu(u, w) = 0.1$. In $G$, all arcs are $\beta$-strong and all strong paths are strongest paths but $G$ is not an PWG.

5. Strong Degree of a Node

In an unweighted graph, every arc is strong, since every edge has a uniform weight one. But in weighted graphs, both strong arcs and non strong arcs exist and hence the concepts of strong degree and usual degree are different. First we shall rewrite
the usual definition of degree of a node in a weighted graph in terms of the weight function $\mu$ as follows.

The degree of a node $v$ in a weighted graph $G(V,E)$ with arc weight function $\mu : E \rightarrow \mathbb{R}^+$ is defined as $d(v) = \sum \mu(u,v)$. The minimum degree of $G$ is $\delta(G) = \land \{d(v)|v \in \sigma^*\}$ and the maximum degree of $G$ is $\triangle(G) = \lor \{d(v)|v \in \sigma^*\}$.

Sameena and Sunitha$^{12}$ have defined the strong degree $d_s(v)$ of a node in a particular type of weighted graphs (weights between 0 and 1) as follows.

**Definition 11.$^{12}$** Let $G : (V,E)$ be a weighted graph. The strong degree of a node $v \in V$ is defined as the sum of weights of all strong arcs incident at $v$. It is denoted by $d_s(v)$.

Also if $N_s(v)$ denote the set of all strong neighbors of $v$, then $d_s(v) = \sum_{u \in N_s(v)} \mu(u,v)$.

**Example 7.** Let $G(V,E)$ be a weighted graph with $V = \{u,v,w\}$, $\mu(u,v) = 1.5, \mu(v,w) = 2.5$ and $\mu(w,u) = 3$. Here arc $(u,v)$ is not strong and hence $d_s(u) = 3, d_s(v) = 2.5$ and $d_s(w) = 5.5$.

Now the minimum strong degree of $G$ is $\delta_s(G) = \land \{d_s(v)|v \in \sigma^*\}$ and maximum strong degree of $G$ is $\triangle_s(G) = \lor \{d_s(v), v \in \sigma^*\}$.

**Example 8.** Let $G(V,E)$ be with $V = \{u,v,w,x\}$ and $\mu(u,v) = 0.1 = \mu(v,x), \mu(v,w) = 1 = \mu(u,w), \mu(w,x) = 0.3$. In $G$, all arcs except $(u,v)$ and $(v,x)$ are strong. Thus $d_s(u) = 1 = d_s(v), d_s(w) = 2.3$ and $d_s(x) = 0.3$. Hence $\delta_s(G) = 0.3$ and $\triangle_s(G) = 2.3$.

Now we have the following two easy propositions,

**Proposition 6.$^{12}$** The sum of strong degrees of all nodes in a weighted graph $G$ is equal to twice the sum of membership values of all strong arcs in $G$.

**Proposition 7.** In a precisely weighted graph there always exists at least one pair of nodes $u$ and $v$ such that $d_s(u) = d_s(v)$.

The concept of strong degree is relevant in weighted graph applications especially problems related with flows as the flow through arcs, which are not strong, can be redirected through a different strongest path.

There exists at least one strongest path at each node of a nontrivial connected weighted graph and hence we have the following proposition.

**Proposition 8.** In a non trivial connected weighted graph $G : (\sigma,\mu), 0 < d_s(v) \leq d(v)$ for all nodes $v \in V$.

Clearly $d(v) = d_s(v)$ for every node $v$ in an unweighted graph. Also by Definition, all arcs of an PWG are strong and hence $d_s(v) = d(v)$ for all $v \in V$. 

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Also strong degree of a node $v$ in a PWG is given by $d_s(v) = \sum_{u \neq v} \{\sigma(u), \sigma(v)\}$

where $u \in V$.

Next lemma is related to the minimum and maximum degrees of an PWG. Since $\mu(u, v) = \sigma(u) \land \sigma(v)$ for all arcs $(u, v)$ of an PWG $G$, the minimum and maximum degrees of nodes in $G$ can be evaluated in terms of the weights of its nodes.

**Lemma 3.** Let $G(\sigma, \mu)$ be an PWG with $V = \{u_1, u_2, ..., u_n\}$ such that $\sigma(u_1) \leq \sigma(u_2) \leq ... \leq \sigma(u_n)$. Then $(u_1, u_i)$ is an arc of minimum weight at $u_j$ for $2 \leq j \leq n$ and $(u_i, u_n)$ is an arc of maximum weight at $u_i$ for $1 \leq i \leq n - 1$. Also, 

$$d(u_1) = \delta_s(G) = (n-1)\sigma(u_1) \text{ and}$$

$$d(u_n) = \Delta_s(G) = \sum_{i=1}^{n-1} \sigma(u_i).$$

**Proof.** Throughout the proof, we suppose that $\sigma(u_1) < \sigma(u_2) \leq \sigma(u_3) \leq ... \leq \sigma(u_{n-1}) < \sigma(u_n)$. If there are more than one node with minimum node strength or maximum node strength, the proof will be similar. First we prove that for $2 \leq j \leq n$, $(u_1, u_j)$ is an arc of minimum weight at $u_j$. If possible, suppose that $(u_1, u_j)$, $2 \leq j \leq n$ is not an arc of minimum weight at $u_j$. Also let $(u_k, u_j), 2 \leq k \leq n, k \neq j$ be an arc of minimum weight at $u_i$. Being an PWG,

$$\mu(u_1, u_j) = \sigma(u_1) \land \sigma(u_j) \text{ and, } \mu(u_k, u_i) = \sigma(u_k) \land \sigma(u_i).$$

Since $\mu(u_k, u_i) < \mu(u_1, u_i)$, we have, $\sigma(u_k) \land \sigma(u_i) < \sigma(u_1) \land \sigma(u_i) = \sigma(u_1).$

That is either $\sigma(u_k) < \sigma(u_1)$ or $\sigma(u_i) < \sigma(u_1)$. Since $l, k \neq 1$, this is a contradiction to our assumption that $\sigma(u_1)$ is the unique minimum node degree. Thus for $2 \leq j \leq n, (u_1, u_j)$ is an arc of minimum weight at $u_j$.

Next, we prove that $(u_i, u_n)$ is an arc of maximum weight at $u_i$ for $1 \leq i \leq n - 1$. On the contrary suppose that $(u_k, u_n), 1 \leq k \leq n - 1$ is not an arc of maximum weight at $u_k$ and let $(u_k, u_r), 1 \leq r \leq n - 1, k \neq r$ be an arc of maximum weight at $u_k$.

Then, $\mu(u_k, u_r) > \mu(u_k, u_n)$ and hence, $\sigma(u_k) \land \sigma(u_r) > \sigma(u_k) \land \sigma(u_n) = \sigma(u_k)$, which implies that $\sigma(u_r) > \sigma(u_k)$. Therefore, $\mu(u_k, u_r) = \sigma(u_k) = \mu(u_k, u_n)$, which is a contradiction to our assumption. Thus $(u_k, u_n)$ is an arc of maximum weight at $u_k$.

Now we have,

$$d_s(u_1) = \sum_{i=2}^{n} \mu(u_1, u_i) = \sum_{i=2}^{n} (\sigma(u_1) \land \sigma(u_i)) = \sum_{i=2}^{n} \sigma(u_i) = (n-1)\sigma(u_1).$$

If possible suppose that $d_s(u_1) \neq \delta_s(G)$ and let $u_k, k \neq 1$ be a node in $G$ with minimum strong degree.

Now $d_s(u_1) > d_s(u_k)$, implies $\sum_{i=2}^{n} \mu(u_1, u_i) > \sum_{k \neq 1, j \neq k} \mu(u_k, u_j)$.

That is, $\sum_{i=2}^{n} (\sigma(u_1) \land \sigma(u_i)) > \sum_{k \neq 1, j \neq k} (\sigma(u_k) \land \sigma(u_j))$. 

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Since \( \sigma(u_1) \land \sigma(u_i) = \sigma(u_1) \) for \( i = 2, 3, ..., n \), \( \sigma(u_k) \land \sigma(u_1) = \sigma(u_1) \) and for all other indices \( j, \sigma(u_k) \land \sigma(u_j) > \sigma(u_1) \), hence it follows that

\[
(n-1)\sigma(u_1) > \sum_{k \neq 1, j \neq k} (\sigma(u_k) \land \sigma(u_j)) > (n-1)\sigma(u_1).
\]

That is, \( d_s(u_1) > d_s(u_1) \), a contradiction. Thus \( d_s(u_1) = \delta_s(G) = (n-1)\sigma(u_1) \).

Finally we show that \( d_s(u_n) = \Delta_s(G) = \sum_{i=1}^{n-1} \sigma(u_i) \).

Since \( \sigma(u_n) > \sigma(u_i) \) for \( i = 1, 2, ..., n - 1 \) and \( G \) is a PWG,

\[
\mu(u_n, u_i) = \sigma(u_n) \land \sigma(u_i) = \sigma(u_i).
\]

Therefore, \( d_s(u_n) = \sum_{i=1}^{n-1} \mu(u_n, u_i) = \sum_{i=1}^{n-1} \sigma(u_i) \).

Now if possible suppose that \( d_s(u_n) \neq \Delta_s(G) \). Let \( u_l, 1 \leq l \leq n - 1 \) be a node in \( G \) such that \( d_s(u_l) = \Delta_s(G) \) and \( d_s(u_n) < d_s(u_l) \).

Now, \( d_s(u_l) = \sum_{i=1}^{l-1} \mu(u_l, u_i) + \sum_{i=l+1}^{n-1} \mu(u_l, u_i) + \mu(u_n, u_l) \).

\[
\leq \sum_{i=1}^{l-1} \sigma(u_i) + (n-l)\sigma(u_l) + \sigma(u_l) \leq \sum_{i=1}^{n-1} \sigma(u_i) = d_s(u_n).
\]

That is, \( d_s(u_l) \leq d_s(u_n) \), a contradiction to our assumption. Thus the Lemma is proved.

6. Node Strength Sequence

To each weighted graph, we can associate a sequence of real numbers namely the node strength sequence \((n-s \text{ sequence})\) as given below.

**Definition 12.** Let \( G(\sigma, \mu) \) be a precisely weighted graph with \( |V| = n \). Then the node-strength sequence \((n-s \text{ sequence})\) of \( G \) is defined to be \((p_1, p_2, ..., p_n)\) with \( p_1 \leq p_2 \leq ... \leq p_n \) where \( p_i > 0 \) is the weight of node \( i \) when nodes are arranged so that their weights are non decreasing. In particular \( p_1 \) is the smallest node weight and \( p_n \), the largest node weight.

The following example illustrates this concept.

**Example 9.** Let \( G(\sigma, \mu) \) be with \( V = \{a, b, c, d\} \) and \( \sigma(a) = \sigma(c) = \sigma(d) = 0.1, \sigma(b) = 0.2 \). Then the node-strength sequence of \( G \) is \((0.1, 0.1, 0.1, 0.2)\) or \((0.1^3, 0.2)\).
Lemma 3, we have, The proofs of (i) and (ii) are obvious. We present the proofs for (iii) and (iv).

(iii) Let \( v_i^{(j)}; j = 1, 2, ..., r_1 \) be the set of nodes in \( G \) with \( d_s(v_i^{(j)}) = p_1, 1 \leq i \leq k \). By Lemma 3, we have,

\[
d_s(v_i^{(j)}) = \delta_s(G) = (n - 1)p_1 \quad \text{for} \quad j = 1, 2, ..., r_1
\]

No node with weight more than \( p_1 \) can have degree \( \delta_s(G) \) since,

\[
\mu(v_i^{(j)}, v_i^{(l)}) = \sigma(v_i^{(j)}) > p_1 \quad \text{for} \quad 2 \leq i \leq k, j = 1, 2, .., r_1, l = 1, 2, ..., r_i+1.
\]

Thus there exists exactly \( r_1 \) nodes with strong degree \( \delta_s(G) \). Next we prove that \( d_s(v_k^t) = \Delta_s(G), t = 1, 2, ..., r_k \).

Since \( \sigma(v_k^t) \) is the maximum node weight, we have \( \mu(v_k^t, v_k^s) = p_k, t \neq j; t, j = 1, 2, ..., r_k \) and \( \mu(v_k^t, v_i^j) = \sigma(v_k^t) \wedge \sigma(v_i^j) = \sigma(v_i^j) \) for \( t = 1, 2, ..., r_k; j = 1, 2, ..., r_i; i = 1, 2, ..., k - 1 \).

Thus for \( t = 1, 2, ..., r_k \),

\[
d_s(v_k^t) = \sum_{i=1}^{k-2} \sum_{j=1}^{r_i-1} \sigma(v_i^j) + (r_k - 1)p_k.
\]

\[
= \sum_{i=1}^{k-2} \sigma(u_i) = \Delta_s(G) \quad \text{(By Lemma 3)}.
\]

Now if \( u \) is a node such that \( \sigma(u) = p_{k-1} \), we have,

\[
d_s(u) = \sum_{i=1}^{k-2} \sum_{j=1}^{r_i-1} \mu(u, v_i^j) + (r_k - 1 + r_k)p_{k-1}.
\]

\[
= \sum_{i=1}^{k-2} \sum_{j=1}^{r_i-1} \sigma(v_i^j) + \sum_{j=1}^{r_k-1} \sigma(v_k^j) + (r_k - 1)p_{k-1}.
\]

\[
< \sum_{i=1}^{k-2} \sum_{j=1}^{r_i-1} \sigma(v_i^j) + \sum_{j=1}^{r_k-1} \sigma(v_k^j) + (r_k - 1)p_k = \Delta_s(G).
\]
Thus there exists exactly $r_k$ nodes with degree $\Delta_s(G)$.

(iv) Let $v^{(1)}_k = v_k$ be the node in $G$ such that $d_s(v_k) = p_k$.

Then by Lemma 3, $d_s(v_k) = \Delta_s(G) = \sum_{i=1}^{n-1} \sigma(u_i)$.

Now let $v^{(t)}_{k-1}, t = 1, 2, ..., r_k - 1$ be the nodes in $G$ with $d_s(v^{(t)}_{k-1}) = p_{k-1}$.

Then for $t = 1, 2, ..., r_k - 1$,

$$d_s(v^{(t)}_{k-1}) = \sum_{i=1}^{k-2} \sum_{j=1}^{r_i} \mu(v^{(t)}_i, v^{(t)}_{k-1}) + \sum_{l \neq m} \mu(v^{(t)}_{k-1}, v^{(t)}_{k-1}) + \mu(v^{(t)}_{k-1}, v_k).$$

But, $\mu(v^{(t)}_i, v^{(t)}_{k-1}) = \sigma(v^{(t)}_i)$ for $i = 1, 2, ..., k-2$ and $j = 1, 2, ..., r_i$,

$\mu(v^{(t)}_{k-1}, v^{(t)}_{k-1}) = p_{k-1}$ and $\mu(v^{(t)}_{k-1}, v_k) = p_{k-1}$.

Thus, $d_s(v^{(t)}_{k-1}) = \sum_{i=1}^{k-2} \sum_{j=1}^{r_i} \sigma(v^{(t)}_i) + (r_{k-1} - 1)p_{k-1} + p_{k-1}$.

$$= \sum_{i=1}^{k-2} \sum_{j=1}^{r_i} \sigma(v^{(t)}_i) + r_{k-1}p_{k-1}.$$  

$$= \sum_{i=1}^{n-1} \sigma(u_i) = \Delta_s(G).$$

Thus there exist $r_{k-1} + 1$ nodes with strong degree $\Delta_s(G)$.

Now if $u$ is a node such that $\sigma(u) < p_{k-1}$, as in the proof of (iii), we can show that $d_s(u) < \Delta_s(G)$. Thus there exist exactly $r_{k-1} + 1$ nodes with strong degree $\Delta_s(G)$ and the proof is complete. 

7. Conclusion

In this article, an attempt is made to analyze the basic connectivity concepts in weighted graphs in terms of the reduction of strength of connectedness between pairs of nodes. The concepts of partial cutnodes and partial bridges are characterized. Also in order to model some type of wireless networks, we have introduced a new type of weighted graph called precisely weighted graphs. Different types of arcs in precisely weighted graphs and relation between degrees of nodes and node weights are studied. Also the concept of node strength sequence is introduced and some of its properties are analyzed. Even though it seems preliminary at a glance, it is promising. The authors will concentrate more on the connectivity of node weighted graphs and their applications in wireless networks in the forthcoming papers.

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