Additional sources of bias in half-life estimation


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Abstract

Recently, an increasing amount of attention is being paid to biases in the measurement of time series dynamics based on calculations of half-life. In particular, this issue amplifies the controversy surrounding the purchasing power parity doctrine. Cross-sectional and temporal aggregations, along with mis-specified models, were previously identified as sources of this bias. We identified several other sources of bias, namely, sampling error, incorrect approximations, and structural breaks in time series. These sources should also receive sufficient attention for a sound measurement of half-life.

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1. Introduction

The empirical evidence suggesting the high persistence of the deviation of real exchange rates from their long-run equilibrium warranted a simple measure to capture this slow transitional dynamics. Consequently, economists borrowed the concept of “half-life” from the natural sciences. In the natural sciences, half-life is defined as the time required for the amount of radioactivity to decrease by one-half. Along the same lines, in the real exchange rate literature, we define half-life as the time required for the effects of a unit innovation to dissipate to one-half. Half-life is also used in economics as a simple measure of time series dynamics such as the income and price levels. In particular, as in Cecchetti et al. (2002) and Morshed et al. (2005), half-life is used to obtain information pertaining to the nature of the observed persistence of the deviations of city consumer price indices (CPIs) from the common trend in prices, by estimating the rate at which a mean reversion occurs. In this context, information about half-life would enable monetary policy makers to design an optimum monetary policy that can deal with the impact and persistence of regional inflation divergence.

Empirical studies on half-lives are often surrounded by controversies pertaining to the accuracy of half-life estimates. This is because, compared with the commonly expected half-life, some studies over-estimate half-life and others under-estimate it (for a detailed discussion, see Murray and Papell, 2002; Taylor, 2001). Various efforts have been made to explore the sources of differences in half-life estimates: Basker and Hernandez-Murillo (2004), Choi et al. (2004),

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The rate of convergence of the real exchange rate has been estimated to be roughly 15% (Froot and Rogoff, 1995; Cheung and Lai, 2000a).

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Cham and Devereux (2003), and Imbs et al. (2005) investigated cross-sectional aggregation as a contributing factor; Chambers (2004) investigated temporal aggregation as a contributing factor of biased estimates, and Taylor (2001) investigated temporal aggregation and mis-specified linear models as contributing factors. The expositions of the latter two articles are within the context of the autoregressive process of order one, AR(1).

In an effort to add feasible explanations to purchasing power parity (PPP) puzzles, we explore other sources for the differences in half-life estimates. These sources are the sensitivity of the half-life formula, an inappropriate formula commonly used for half-life estimations, and mis-specified models that are attributable to structural breaks. Our simulations revealed that the commonly used half-life formula is very sensitive to sampling errors even if the autoregressive process is AR(1). The half-life formula can be quite inaccurate when the time series is a higher order (for example, AR(2)) or a mixed process (for example, ARMA(1,1)). Moreover, when a structural break exists in time series, and this is not taken into consideration, half-lives are over-estimated.

This paper comprises five sections. In Section 2, we discuss the sensitivity of the commonly used half-life formula obtained from an AR(1) model. Biases resulting from using the half-life formula for higher order autoregressive processes and mixed processes are discussed in Section 3. Section 4 discusses the effects of structural breaks on half-life calculations. Concluding remarks are in Section 5.

2. Sensitivity of the half-life formula

Based on the cumulative impulse response analysis of Campbell and Mankiw (1987), researchers define the moving average (MA) coefficients of the MA representation of the process as impulse responses. More specifically, for a linear process \( y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \), where \( \psi_0 = 1 \) and \( \varepsilon_t \) are i.i.d. random variables, the half-life denoted by \( h \) is such that \( \psi_h = \frac{1}{2} \); this is the lag where the impulse response \( \psi_j \) becomes half the initial impulse response. However, unlike radioactive material, the impulse response does not always decay monotonically. If \( \psi_j \) is not a monotonically decreasing function of lag \( j \), then the half-life is not well-defined (Cheung and Lai, 2000b; Choi et al., 2004).

In econometric literature, the commonly used formula for the half-life of a (stationary) time series \( y_t \) is \( h = -\log 2 / \log \rho_1 \), where \( \rho_1 \) is the autocorrelation of \( y_t \) at lag one, i.e., \( \rho_1 = \text{corr}(y_t, y_{t-1}) \). This formula is valid only when \( \rho_1 > 0 \), and is correct if \( y_t \) is an AR(1) satisfying

\[
y_t = \rho_1 y_{t-1} + \varepsilon_t.
\]  

This is because for AR(1), \( \psi_j = \rho_1^j \). If \( \rho_1 < 0 \) for an AR(1) process, then the impulse response \( \psi_j \) oscillates between positive and negative values; as a result, the half-life is not well-defined.

Given a sample of size \( n \), the half-life of an AR(1) process is usually estimated by

\[
\hat{h} = -\frac{\log 2}{\log \hat{\rho}_1},
\]  

where \( \hat{\rho}_1 \) is the least squares estimator (LSE) of \( \rho_1 \). From the first order Taylor series expansion, we obtain

\[
\hat{h} - h \approx \frac{(\log 2) \hat{\delta}}{\rho_1 (\log \rho_1)^2},
\]  

where \( \hat{\delta} = \hat{\rho}_1 - \rho_1 \). It is well known that \( \text{var}(\hat{\rho}_1) \approx (1 - \rho_1^2)/n \). Therefore,

\[
\text{var}(\hat{h} - h) \approx \frac{\log 2}{\rho_1 (\log \rho_1)^2} \frac{1 - \rho_1^2}{n}.
\]  

Using the above equation, we tabulate the (approximate) coefficient of variation (CV) of \( \hat{h} \) in Table 1 for 0.8 \( \leq \rho_1 \leq 0.95 \) and for sample sizes \( n = 100 \) and 200. The CV varies from 33% to 64% within the range of \( \rho_1 \) for the sample of size 100; this amounts to 100 years of annual data. More specifically, for an AR(1) process with \( \rho_1 = 0.9 \), the half-life is 6.58. With \( n = 100 \), the CV of \( \hat{h} \) is 46% and the standard error of \( \hat{h} \) is 3.02. Therefore, for annual data, a half-life estimate of

\[\footnote{Chen and Engel (2005), however, showed that the cross-sectional aggregation bias might not be sufficiently large to explain the PPP puzzle.}
Table 1
Approximate standard errors and coefficient of variations of the half-life estimates for the selected AR(1) process with sample sizes of 100 and 200

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<th>$\rho_i$</th>
<th>$h$</th>
<th>$\sqrt{\text{var}(\hat{h} - h)}$</th>
<th>CV (%)</th>
<th>$\sqrt{\text{var}(\hat{h} - h)}$</th>
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</table>

3.6 years or less is as likely as a half-life estimate of 9.6 years or more. This illustrates that half-life estimates are very sensitive to sampling errors.

3. Precision of the approximate formula

More often than not, the process under consideration is not just an AR(1) process. Rather, it is a higher order AR process or a mixed process such as an autoregressive moving-average (ARMA) process. For such models, the aforementioned half-life formula serves as an approximation, and the quality of this approximation requires further investigation.

For $y_t$ following an autoregressive process of order $p$, AR$(p)$, satisfying

$$y_t = \sum_{j=1}^{p} \phi_j y_{t-j} + \epsilon_t,$$  
(5)

the impulse response $\psi_j$ satisfies the linear difference equation

$$\psi_j - \phi_1 \psi_{j-1} - \cdots - \phi_{p-1} \psi_{j-(p-1)} - \phi_p = 0,$$  
(6)
and the half-life $h$ is obtained by solving $\psi_h = \frac{1}{2}$. It is well known that the impulse response $\psi_j$ is obtained from the roots of the auxiliary equation

$$m^p - \phi_1 m^{p-1} - \cdots - \phi_{p-1} m - \phi_p = 0.$$  

(7)

Since $\psi_j$ does not necessarily decay monotonically, the half-life is not always well-defined. The commonly employed practice in economics literature is to approximate the half-life based on the formula

$$h = \frac{\log 2}{\log(1 + \beta)}.$$  

(8)

regardless of the existence of the well-defined half-life, by obtaining the “convergence speed” $\beta$ from the following error correction representation of the AR($p$) model:

$$\Delta y_t = \beta y_{t-1} + \sum_{j=1}^{p-1} \phi_j^* \Delta y_{t-j} + \varepsilon_t,$$  

(9)

where $\beta = \sum_{j=1}^{p} \phi_j - 1$ and $\phi_j^* = -\sum_{k=j+1}^{p} \phi_k$. We note that for an AR(1) process, $\beta = \rho_1 - 1$ and the formula in (8) is equivalent to that in (2).

For ease of exposition, we assess the quality of this approximation using the following AR(2) process:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t.$$  

(10)

It is well known that the impulse response of this process is

$$\psi_j = \begin{cases} (1 + j)(\phi_1/2)^j & \text{if } \phi_1^2 + 4\phi_2 = 0, \\ c_1 \lambda_1^j + c_2 \lambda_2^j & \text{if } \phi_1^2 + 4\phi_2 \neq 0, \end{cases}$$  

(11)

where $\lambda_1 = \left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)/2$, $\lambda_2 = \left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right)/2$, $c_1 = \lambda_1/(\lambda_1 - \lambda_2)$, and $c_2 = \lambda_2/(\lambda_2 - \lambda_1)$.

For the AR(2) process to be stationary, it is well known (see Box et al., 1994, p. 60) that the AR coefficients $\phi_1$ and $\phi_2$ lie in the triangular region

$$\phi_2 + \phi_1 < 1, \quad \phi_2 - \phi_1 < 1, \quad -1 < \phi_2 < 1.$$  

(12)

Within this triangular region, impulse response $\psi_j$ decreases monotonically only in the region for $\phi_1 > 0$ and $\phi_1^2 + 4\phi_2 > 0$. Therefore, the half-life is not well-defined in the other region. However, so long as $\phi_2 + \phi_1 > 0$, the approximate formula will yield a half-life. Even in the region where the half-life is well defined, the approximate formula can be quite inaccurate. The shaded region of Fig. 1 represents the region where the difference between the half-life by (11) and the approximate half-life of (8) is greater than $3$. From the shaded region, we see that the difference is more pronounced as the process approaches nonstationarity, i.e., $\phi_1 + \phi_2 = 1$, where $\phi_2 > 0$. This is because formula (8) diverges to infinity as the value of $\phi_1 + \phi_2$ approaches one.

When an AR(1) process at a higher frequency is aggregated and observed at a lower frequency, this observed process becomes an ARMA (1,1) process,

$$y_t = \phi y_{t-1} + \varepsilon_t - \theta \varepsilon_{t-1},$$  

(13)

see Wei (1996) and Chambers (2004). The impulse response $\psi_j$ is obtained by

$$\psi_j = (\phi - \theta) \phi^{j-1},$$  

(14)

and the exact half-life $h$ is

$$h = \frac{\log 2}{\log \phi} - \frac{\log(\phi - \theta)}{\log \phi} + 1.$$  

(15)
which is obtained by solving \((\phi - \theta)\phi^{h-1} = 1/2\), provided \(\phi > \theta\) and \(\phi > 0\). Since the lag one autocorrelation of the ARMA(1,1) process is

\[
\rho_1 = \frac{(1 - \theta \phi)(\phi - \theta)}{(1 - 2\theta \phi + \theta^2)},
\]  

(16)
the approximate formula (based on an AR(1) model) yields a half-life of

$$- \log 2 / \log \left\{ \frac{(1 - \theta \phi)(\phi - \theta)}{(1 - 2\theta \phi + \theta^2)} \right\},$$

(17)

In addition, an approximated model of AR(2) can be taken into consideration instead of an AR(1) model. In such a case, the approximate formula based on an AR(2) model yields a half-life of

$$- \log 2 / \log \left\{ \frac{(1 - \theta \phi)(\phi - \theta)}{1 - (\phi + 1)\theta + \theta^2} \right\},$$

(18)

the proof is provided in the Appendix.

In order to illustrate the inaccuracies of the approximate formulae, the shaded region in Fig. 2 represents the region where the difference between the half-life by (15) and the approximate half-lives of (17) or (18) is greater than 3, even when the parameters are known. Similar to Fig. 1, this difference becomes larger as the process approaches nonstationarity, i.e., $\phi$ approaches one. This large difference is attributable to the fact that the half-lives in (17) and (18) as well as in (15) diverge to infinity as the value of $\phi$ increases to one. There is a higher degree of inaccuracy when models are estimated; however, this has not been discussed in the present study.

Therefore, when we estimate the half-life for a higher order AR or a mixed process, it is recommended that researchers adopt the exact half-life formula from the impulse response function. The half-life formulae in (6), (11), and (15) are based on the impulse response functions of their respective models.

4. The effect of structural breaks

It is well known that the Dickey–Fuller unit root test lacks the power, when a true process is trend stationary with structural breaks; see Perron (1989). This implies that the LSE (of the Dickey–Fuller type) of $\hat{\beta}$ in (1) or $\hat{\beta}$ in (9) is

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<td>0.4</td>
<td>0.1</td>
<td>0.95</td>
<td>0.83</td>
<td>16.03</td>
<td>4.59</td>
<td>175.23</td>
<td>13.43</td>
</tr>
<tr>
<td>(175)</td>
<td></td>
<td>(0.03)</td>
<td>(0.07)</td>
<td>(9.27)</td>
<td>(3.08)</td>
<td>(377.13)</td>
<td>(64.79)</td>
</tr>
</tbody>
</table>

Note: 1. $\hat{\beta}_1$ and $\hat{\beta}_2$ are the estimators of $\beta$ in model (19) by the estimated models (20) and (21), respectively.
2. $\hat{h}_j = - \log 2 / \log \hat{\beta}_j$ for $j = 1, 2$ denotes the estimator of half-life (not adjusted to integers) and $\text{MSE}_j = (\hat{h}_j - h_0)^2$ for $j = 1, 2$, where $h_0$ is the true half-life, 1.36, 3.11, and 6.58 corresponding to $\alpha = 0.6, 0.8$, and 0.9, respectively.
3. The parentheses in the second column denote the number of cases where $\hat{\beta}_1 > 1$ or $\hat{\beta}_2 > 1$. We do not consider these cases in the results because the corresponding half-lives cannot be calculated. The parentheses in the other columns denote the corresponding standard deviations.
over-estimated. Macro-economic data, such as price indices and exchange rates, often have structural breaks in the trend (or level). Therefore, an analysis that does not incorporate such breaks yields over-estimated half-lives.

In order to assess the effect of a structural break in the trend (at a single point in time) on the estimation of half-lives, we conduct a small Monte Carlo experiment. We generated 10,000 replications of a series \( \{y_t\} \) of length \( T = 100 \), defined by

\[
y_t = \gamma D_t + \alpha y_{t-1} + \epsilon_t,
\]

(19)

where \( D_t = t - T_0 \) if \( t > T_0 \), and 0 otherwise, representing a structural break in the trend at \( T_0 \). For simplicity, we assume that \( T_0 = 50 \), and the innovations \( \epsilon_t \) are i.i.d. \( N(0, 1) \). For various values of \( \alpha \) and \( \gamma \), we consider \( \alpha = 0.6, 0.8, 0.9 \) and \( \gamma = 0.1, 0.2, 0.4 \). For \( \alpha = 0.6, 0.8, 0.9 \), the corresponding half-lives are 1.36, 3.11, and 6.58.

In order to estimate half-life when a structural break is not considered, we computed half-life \( \hat{h}_1 = - \log 2 / \log \hat{y}_1 \) based on the following model:

\[
y_t = \mu_1 + \gamma y_{t-1} + \epsilon_t.
\]

(20)

In order to calculate half-life when a structural break is considered, we computed half-life \( \hat{h}_2 = - \log 2 / \log \hat{y}_2 \) based on the following model:

\[
y_t = \alpha y_{t-1} + \epsilon_t.
\]

(21)

We assume that \( T_0 \) is known so that the comparison is unaffected by the estimation of the break point, \( T_0 \).

In Table 2, we compare the results of the estimation from both models (20) and (21). In the fourth and sixth columns, it is observed, similar to Andrews (1993) and Murray and Papell (2005), that all the estimators \( \hat{y}_2 \) are biased downward. Therefore the half-life estimators \( \hat{h}_2 \) are all under-estimated even though the structural breaks are considered.
In the third and fifth columns, it is observed that the estimators of $\hat{z}_1$ are biased upward with the exceptions of $(x, \gamma) = (0.8, 0.1), (0.9, 0.1),$ and $(0.9, 0.2),$ and all the estimators of half-life $\hat{h}_1$ are over-estimated with the exceptions of $(x, \gamma) = (0.9, 0.1).$ Further, from the last two columns, it is observed that all the mean squared errors (MSEs) of $\hat{h}_1$ are larger than those of $\hat{h}_2.$ In the cases of $(x, \gamma) = (0.8, 0.1), (0.9, 0.1),$ and $(0.9, 0.2),$ $\hat{h}_1$ has a larger MSE than $\hat{h}_2,$ although the corresponding $\hat{z}_1$ is less biased than $\hat{z}_2.$ This can be explained from Fig. 3, which shows the distributions of $\hat{z}_1, \hat{z}_2, \hat{h}_1,$ and $\hat{h}_2$ when $(x, \gamma) = (0.9, 0.2).$ The distribution of $\hat{z}_1$ has a higher concentration near one than that of $\hat{z}_2,$ which makes the right tail of $\hat{h}_1$ longer than that of $\hat{h}_2.$

This over-estimation phenomenon is not surprising because $\hat{z}_1$’s are ready to converge to one as sample size becomes larger regardless of the value of $x;$ see Perron (1989). Therefore, when there is a doubt regarding the existence of structural breaks, it is desirable to consider a model that incorporates structural breaks.

As an empirical example, we consider the monthly CPI of the U.S. during the period January, 1981–December, 2004 ($T = 288$), obtained from the Bureau of Labor Statistics, U.S. Department of Labor. The time series plot of the logarithm of the CPI is shown in Fig. 4. From a visual inspection, it is determined that a decline in the slope occurred in January 1991. This decline is attributed to low energy and food prices,3 and the sustained low energy prices during the entire 1990s are manifested in the CPI estimates. Therefore, assuming $T_0 = 120$ (December, 1990), we estimate the half-lives from models (20) and (21). From model (20), we obtain $\hat{z}_1 = 0.988$ (t-ratio = 209.97) and $\hat{h}_1 = 57.415,$ that is approximately 57 months or 4.8 years, while from model (21), we obtain $\hat{z}_2 = 0.962$ (t-ratio = 98.07) and $\hat{h}_2 = 17.892,$ that is approximately 18 months or 1.5 years. The difference in both the half-lives is approximately 3.3 years, and this illustrates the over-estimation of half-life from a model without any structural break.

5. Conclusions

Previous researchers have identified a number of sources of bias in half-life estimation, namely, cross-sectional aggregation, temporal aggregation, and mis-specified models. However, in this paper, we have identified several additional sources of instability in the conventional half-life estimation. We found that even for an AR(1) process, the sampling bias cannot be ignored. For higher order or mixed time series process, the biases resulting from the use of conventional

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3 Stewart (1992) provides a detailed discussion of the reasons for this drop in the slope of the CPI.
formula are quite large. The presence of structural breaks in time series creates additional noise in half-life calculation. Thus, to ensure a more accurate calculation of half-life, more attention must be paid to these issues.

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6. Appendix. Proof of (18)

Assume AR(2) to be an approximate model for ARMA(1,1). By a property of the partial autocorrelation function, we can find the coefficients \( \phi_{12}, \phi_{22} \) of AR(2) using

\[
\begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}.
\]

Then we obtain

\[
\phi_{12} + \phi_{22} = \frac{1}{1 - \rho_1^2} (\rho_1 - \rho_1 \rho_2 + \rho_2 - \rho_1^2)
= (1 + \phi - \frac{\rho_1}{1 + \rho_1})
= \frac{(1 - \phi)(\phi - \theta)}{1 - (\phi + 1)\theta + \theta^2}
\]

since \( \rho_2 = \rho \rho_1 \) and \( \rho_1 = (1 - \phi)(\phi - \theta)/1 - 2\phi + \theta^2 \). Therefore, we can deduce that

\[
-\log 2/
\log(\phi_{12} + \phi_{22}) = -\log 2/ \log \left\{ \frac{(1 - \phi)(\phi - \theta)}{1 - (\phi + 1)\theta + \theta^2} \right\}.
\]

References


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= (1 + \phi - \frac{\rho_1}{1 + \rho_1})
= \frac{(1 - \phi)(\phi - \theta)}{1 - (\phi + 1)\theta + \theta^2}
\]

since \( \rho_2 = \rho \rho_1 \) and \( \rho_1 = (1 - \phi)(\phi - \theta)/1 - 2\phi + \theta^2 \). Therefore, we can deduce that

\[
-\log 2/
\log(\phi_{12} + \phi_{22}) = -\log 2/ \log \left\{ \frac{(1 - \phi)(\phi - \theta)}{1 - (\phi + 1)\theta + \theta^2} \right\}.
\]

References