On the Convergence of Approximate Message Passing with Arbitrary Matrices

Sundee Rangan, Philip Schniter, and Alyson Fletcher

Abstract—Approximate message passing (AMP) methods and their variants have attracted considerable recent attention for the problem of estimating a random vector $x$ observed through a linear transform $A$. In the case of large i.i.d. $A$, the methods exhibit fast convergence with precise analytic characterizations on the algorithm behavior. However, the convergence of AMP under general transforms is not fully understood. In this paper, we provide sufficient conditions for the convergence of a damped version of the generalized AMP (GAMP) algorithm in the case of Gaussian distributions. It is shown, that with sufficient damping the algorithm can be guaranteed to converge, but the amount of damping grows with peak-to-average ratio of the squared singular values of $A$. This condition explains the good performance of AMP methods on i.i.d. matrices, but also their difficulties with other classes of transforms. A related sufficient condition is then derived for the local stability of the damped GAMP method under more general (possibly non-Gaussian) distributions, assuming certain strict convexity conditions.

Index Terms—Belief propagation, message passing, primal-dual methods.

I. INTRODUCTION

Consider estimating a random vector $x \in \mathbb{R}^n$ with independent components $x_j \sim P(x_j)$ from observations $y \in \mathbb{R}^m$ that are conditionally independent given the transform outputs $z = Ax$, i.e., $P(y|z) = \prod_i P(y_i|z_i)$, under knowledge of the matrix $A \in \mathbb{R}^{m \times n}$ and the densities $P(x_j)$ and $P(y_i|z_i)$. Often, the goal is to compute the minimum mean-square error (MMSE) estimate $\hat{x}_{\text{MMSE}} = \int \int x P(x|y)dx = \mathbb{E}(x|y)$, where $P(x|y)$ denotes the posterior distribution, or the maximum a posteriori (MAP) estimate $\hat{x}_{\text{MAP}} = \arg\max_{x \in \mathbb{R}^n} P(x|y)$. Moreover, using $F(z) := -\ln P(y|z)$ and $G(x) := -\ln P(x)$ and Bayes rule $P(x|y) = \frac{P(y|x)P(x)}{P(y)}$, it becomes evident that MAP estimation is equivalent to the optimization problem

$$\hat{x}_{\text{MAP}} = \arg\min_{x \in \mathbb{R}^n} F(Ax) + G(x)$$

for separable $F(z) = \sum_i F_i(z_i)$ and $G(x) = \sum_j G_j(x_j)$. Such problems arise in a range of applications including statistical regression, filtering, inverse problems, and compressed sensing.

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Most current numerical methods for solving the constrained optimization problem (2) attempt to exploit the separable structure of the objective function (2) using approaches like iterative shrinkage and thresholding (ISTA) [11]–[16] or the alternating direction method of multipliers (ADMM) [17]–[20].

However, in recent years, there has also been considerable interest in approximate message passing (AMP) methods that apply Gaussian and quadratic approximations to loopy belief propagation (BP) in graphical models [11]–[16]. AMP applied to max-sum loopy BP produces a sequence of estimates that approximate $\hat{x}_{\text{MAP}}$, while AMP applied to sum-product loopy BP produces a sequence of estimates that approximate $\hat{x}_{\text{MMSE}}$.

In the large-system limit (i.e., $m, n \to \infty$ for constant $m/n$) under i.i.d sub-Gaussian $A$, AMP methods are characterized by a state evolution whose fixed points, when unique, coincide with $\hat{x}_{\text{MAP}}$ or $\hat{x}_{\text{MMSE}}$ [17], [18]. In addition, for large but finite-sized i.i.d. matrices, simulations suggest that AMP methods converge very quickly to near-optimal solutions.

Unfortunately, a rigorous characterization of AMP for generic finite-dimensional $A$ remains lacking. The recent paper [19] studied the fixed-points of the generalized AMP (GAMP) algorithm from [15], [16]. There, it was established that the fixed points of max-sum GAMP coincide with the critical points of the optimization objective in (2), and the fixed points of sum-product GAMP are critical points of a certain free-energy optimization analogous to the results in [20]. However, the paper did not discuss the convergence of the algorithm to those fixed points. Indeed, similar to other loopy BP algorithms, GAMP may diverge. In fact, a recent work [21] has shown that GAMP can diverge with apparently benign matrices and the divergence can, in fact, be predicted via a state evolution analysis.

For general loopy BP, a variety of methods have been proposed to improve convergence, including coordinate descent, tree re-weighting, and double loop methods [22]–[24]. In this paper, we propose a “damped” modification of GAMP that is similar to the technique used in Gaussian belief propagation [25], [26]—a closely related algorithm. We also point out connections between damped GAMP and the primal-dual hybrid-gradient (PDHG) algorithm [8]–[10].

Our first main result establishes a necessary and sufficient condition on the global convergence of GAMP for arbitrary $A$ in the special case of Gaussian $P(x_j)$ and $P(y_i|z_i)$ (i.e., quadratic $F$ and $G$) and fixed scalar stepsizes. This condition (see Theorem 2 below) shows that, with sufficient damping, the Gaussian GAMP algorithm can be guaranteed to converge. However, the amount of damping grows with a peak-to-average ratio of the squared singular values of $A$. This
II. DAMPED GAMP

A. Review of GAMP

The GAMP algorithm was introduced in [15], [16] and rigorously analyzed in [18]. The procedure (see Algorithm 1) produces a sequence of estimates \( \hat{x}^t, t = 1, 2, \ldots, \) that, in max-sum mode, approximate \( x_{\text{MAP}} \) and, in sum-product mode, approximate \( x_{\text{MMSE}} \). The two modes differ only in the definition of the scalar estimation functions \( g_x \) and \( g_{\tau} \) used lines 8, 9, 12, and 13 of Algorithm 1:

- In max-sum mode, \[ g_x(r, \tau_r) = \text{prox}_{\tau_r g_j}(r_j) \quad (3) \] 
- In sum-product mode, \[ [g_x(p, \tau_p)]_i = \frac{1}{\tau_p} \int p_i \text{prox}_{f_j/\tau_p}(p_j) \, dp_j \quad (4) \]

using \( \tau_r = [\tau_{r_1}, \ldots, \tau_{r_m}]^T, \tau_p = [\tau_{p_1}, \ldots, \tau_{p_m}]^T, \) and \( \text{prox}_f(r) := \arg \min_x f(x) + \frac{1}{2} \|x - r\|^2. \quad (5) \)

Note (3) implements scalar MAP denoising under prior \( P(x_j) \propto \exp(-G(x_j)) \) and variance-\( \tau_r \) Gaussian noise.

We note that the vector multiplications and divisions in lines 6, 7, 8, 10, 11, and 12 of Algorithm 1 are taken elementwise, and \( g_x \) (or \( g_{\tau} \)) is the gradient of \( g_x \) (or \( g_{\tau} \)) w.r.t its first argument.

Algorithm 1 reveals the computational efficiency of GAMP: the vector-valued MAP and MMSE estimation problems are reduced to a sequence of scalar estimation problems in Gaussian noise. Specifically, each iteration involves multiplications by \( S, S^T, A, \) and \( A^H \) along with simple scalar estimations on the components \( x_j \) and \( z_i \); there are no vector-valued estimations or matrix inversions.

We note that Algorithm 1 writes GAMP in a "symmetrized" form, where the steps in lines 6, 9 mirror those in lines 10, 13. This differs from earlier GAMP publications, which were written using the variables \( p = \frac{1}{\tau_p} p, \tau_p = 1/\tau_p, \) and \( s = -S \). We use overlines on certain variables to distinguish the notational differences between this paper and earlier works.

Algorithm 1 GAMP with vector stepizes and damping

Require: Matrix \( A \), scalar estimation functions \( g_x \) and \( g_{\tau} \), and damping constants \( \theta_s, \theta_x \in (0, 1) \).

1: \( S \leftarrow |A|^2 \) (componentwise magnitude squared)
2: \( t \leftarrow 0 \)
3: Initialize \( \tau_x^t > 0, x^t \)
4: \( \tau_{s}^{t-1} \leftarrow 0 \)
5: repeat
6: \( 1/\tau_p^t \leftarrow S \tau_p^t \)
7: \( \mathbf{p}^t \leftarrow \mathbf{s}^{t-1} + \tau_p^t A \mathbf{x}^t \)
8: \( \tau_s^t \leftarrow \tau_p^t g_s(\mathbf{p}^t, \tau_p^t) \)
9: \( \mathbf{s}^t \leftarrow (1 - \theta_s) \mathbf{s}^{t-1} + \theta_s \mathbf{g_s}(\mathbf{p}^t, \tau_p^t) \)
10: \( 1/\tau_x^t \leftarrow S \tau_x^t \)
11: \( \mathbf{r}^t \leftarrow \mathbf{x}^t - \tau_x^t A^H \mathbf{s}^t \)
12: \( \tau_{x}^{t+1} \leftarrow \tau_x^t g_x(\mathbf{r}^t, \tau_x^t) \)
13: \( x^{t+1} \leftarrow (1 - \theta_x) x^t + \theta_x g_x(\mathbf{r}^t, \tau_x^t) \)
14: until Terminated

B. Damped GAMP

Algorithm 1 includes a small but important modification to the original GAMP from [15], [16]: lines 9 and 12 perform damping using constants \( \theta_s, \theta_x \in (0, 1) \) that slow the updates of \( s^t, x^t \) when \( \theta_s, \theta_x < 1 \), respectively, while the original GAMP implicitly uses \( \theta_s = 1 = \theta_x \). In the sequel, we establish—analytically—that damping facilitates the convergence of GAMP for general \( A \), a fact that has been empirically observed in past works (e.g., [21], [27]).

C. GAMP with Scalar Stepizes

The computational complexity of Algorithm 1 is dominated by the matrix-vector multiplications involving \( A, A^H, S, \) and \( S^T \). In [15], a scalar-stepizes simplification of GAMP was proposed to avoid the multiplications by \( S \) and \( S^T \), roughly halving the per-iteration complexity. The meaning of "stepizes" will become clear in the sequel. Algorithm 2 shows the scalar-stepizes version of Algorithm 1.

For use in the sequel, we now show that scalar-stepizes GAMP is equivalent to vector-stepizes GAMP under a different choice of \( S \): while Algorithm 1 uses \( S = |A|^2 \), Algorithm 2 effectively uses \[ S = \frac{|A|^2}{mn} \mathbf{1} \mathbf{1}^T, \quad (7) \]

i.e., a constant matrix having the same average value as \( |A|^2 \).

Thus, the two algorithms coincide when \( |A_{ij}| \) is invariant to \( i \) and \( j \). To see the equivalence, we first note that, under \( S \) from (7), line 6 in Algorithm 1 would produce a version of \( 1/\tau_p^t \) containing identical elements \( 1/\tau_p^t \), where

\[ \frac{1}{\tau_p^t} = \frac{|A|^2}{mn} \mathbf{1} \mathbf{1}^T = \frac{|A|^2}{m} \tau_x^t \]

for \( \tau_x^t = (1/n) \mathbf{1} \mathbf{1}^T \). Similarly, line 10 would produce a vector \( 1/\tau_x^t \) with identical elements \( 1/\tau_x^t \), where

\[ \frac{1}{\tau_x^t} = \frac{|A|^2}{n} \mathbf{1} \mathbf{1}^T = \frac{|A|^2}{n} \tau_s^t, \]
recognized as proximal gradient ascent in the dual variable recently studied in [8]–[10] is defined by the iteration

\[
\begin{align*}
\tau^t_p & = (1/m)\mathbf{1}^\top \tau_s, \\
\mathbf{p}^- & = \mathbf{s}^- + \mathbf{g}^\top \mathbf{A} x', \\
\tau^t_s & = \tau^t_s, \\
\mathbf{s}^- & = (1 - \theta)\mathbf{s}^- + \theta \mathbf{g}^\top (\mathbf{p}^- + \tau^t_p), \\
\tau^t_r & = (\tau^t_r/m)1\mathbf{T} \tau_s (\mathbf{p}^-, \tau^t_p), \\
\mathbf{s}^- & = (1 - \theta)\mathbf{s}^- + \theta \mathbf{g}^\top (\mathbf{p}^- + \tau^t_p), \\
\mathbf{x}' & = x' - \tau^t_r x'.
\end{align*}
\]

Algorithm 2 GAMP with scalar stepizes and damping

**Require:** Matrix \( \mathbf{A} \), scalar estimation functions \( g_x \) and \( \tau^t_s \), and damping constants \( \theta_s, \theta_x \in (0, 1) \).

1. \( t \leftarrow 0 \)
2. Initialize \( \tau^t_s > 0, x' \)
3. \( \mathbf{s}^- = 0 \)
4. repeat
5. \( \begin{align*}
\tau^t_p & = (1/m)\mathbf{1}^\top \tau_s, \\
\mathbf{p}^- & = \mathbf{s}^- + \mathbf{g}^\top \mathbf{A} x'.
\end{align*} \)
6. \( \tau^t_s = \tau^t_s \)
7. \( \mathbf{s}^- = (1 - \theta)\mathbf{s}^- + \theta \mathbf{g}^\top (\mathbf{p}^- + \tau^t_p), \)
8. \( \mathbf{s}^- = (1 - \theta)\mathbf{s}^- + \theta \mathbf{g}^\top (\mathbf{p}^- + \tau^t_p), \)
9. \( \begin{align*}
\tau^t_r & = (1/n)\mathbf{1}^\top \nu (\mathbf{p}^- + \tau^t_p), \\
\mathbf{x}' & = x' - \tau^t_r x'.
\end{align*} \)
10. until Terminated

Thus, under \( \theta_x = 1 \), scalar GAMP’s update of \( \mathbf{s} \) in line 8 of Algorithm 2 matches PDHG’s in (11). Similarly, noting the connection between (5) and (12), it follows that, under \( \theta_x = 1 \), scalar GAMP’s update of \( x \) in line 12 of Algorithm 2 matches the PDHG update (13) under \( \theta = 0 \).

In summary, PDHG under \( \theta = 0 \) (the Arrow-Hurwicz [29] case) would be equivalent to non-damped scalar GAMP if the update (11) is fixed over the iterations. GAMP, however, adapts these stepizes. In fact, under the existence of the second derivative \( f'' \), it can be shown that

\[
\text{prox}_{f'}(r) = \left[ 1 + f''(\text{prox}_{f'}(r)) \right]^{-1},
\]

implying that, for smooth \( F = F' \), GAMP updates \( \tau^t_s \)

\[
\tau^t_s = \frac{1}{1 + f''(\text{prox}_{f'}(r))},
\]

and, combining these with lines 6 and 10, we get

\[
\begin{align*}
\tau^t_x & = \frac{1}{1 + f''(\text{prox}_{f'}(r))}, \\
\mathbf{x}' & = x' - \tau^t_r x'.
\end{align*}
\]

A. Relational to Primal-Dual Hybrid Gradient Algorithms

An important case of (2) is when \( F = F' \), the convex conjugate of \( F \), as given by the Legendre-Fenchel transform

\[
F^*(s) := \sup_{z \in \mathbb{R}^m} s^\top z - F(z).
\]

For closed proper convex \( F \), we have \( F^* = F \), and so

\[
F(Ax) = \sup_{s \in \mathbb{R}^m} s^\top Ax - F^*(s),
\]

giving the equivalent saddle-point formulation of (2),

\[
\min_{x \in \mathbb{R}^n} \sup_{s \in \mathbb{R}^m} s^\top Ax - F^*(s) + G(x).
\]

The so-called primal-dual hybrid-gradient (PDHG) algorithm recently studied in [8]–[10] is defined by the iteration

\[
\begin{align*}
\mathbf{s}^- & = \text{prox}_{\tau^t_s F^*} (\mathbf{s}^- + \mathbf{g}^\top \mathbf{A} x'), \\
\mathbf{x}' & = x' - \tau^t_r x', \\
\mathbf{s}^- & = (1 - \theta)\mathbf{s}^- + \theta \mathbf{g}^\top (\mathbf{p}^- + \tau^t_p).
\end{align*}
\]

B. Convergence of GGAMP Stepizes

We first establish the convergence of the GGAMP stepizes in the case of an arbitrary matrix \( A \). For the vector-stepizes case in Algorithm 1 lines 8 and 12 become

\[
\begin{align*}
\tau^t_s & = \psi_p \psi_s (\mathbf{p}, \tau^t_p), \\
\mathbf{p} & = \text{prox}_{\tau^t_s F^*} (\mathbf{p}) + \mathbf{g}^\top \mathbf{A} x', \\
\mathbf{s}^- & = \text{prox}_{\tau^t_s F^*} (\mathbf{p}^- + \tau^t_p).
\end{align*}
\]

and, combining these with lines 6 and 10 we get

\[
\begin{align*}
\tau^t_s & = \frac{1}{\tau^t_r} \mathbf{x}' \tau^t_x + \frac{1}{\tau^t_r} x', \\
\tau^t_x & = \frac{1}{\tau^t_r} \mathbf{x}' \tau^t_x + \frac{1}{\tau^t_r} x'.
\end{align*}
\]

III. DAMPED GAUSSIAN GAMP

A. Gaussian GAMP

Although Algorithms 1 and 2 apply to generic distributions \( P(x_j) \) and \( P(y_i | z_i) \), we find it useful to at first consider the simple case of Gaussian distributions, and in particular

\[
P(x_j) = \mathcal{N}(x_j | \mu_j, \sigma_j^2), \quad P(y_i | z_i) = \mathcal{N}(z_i | \mu_i, \sigma_i^2),
\]

where \( \sigma_j^2 \) are variances and \( \mu_i, \sigma_i^2 \) are precisions. In this case, the scalar estimation functions used in max-sum mode are identical to those in sum-product mode, and are linear [15]:

\[
\begin{align*}
\mathbf{g}_x (\mathbf{p}, \tau^t_p) & = \frac{\tau^t_p}{\tau^t_p + \mathbf{p}} (\mathbf{p} + \mathbf{y}) - \mathbf{y}, \\
g_x (\mathbf{r}, \tau^t_r) & = \frac{\tau^t_r}{\tau^t_r + \mathbf{r}} (\mathbf{r} - \mathbf{x}_0) + \mathbf{x}_0.
\end{align*}
\]

Henceforth, we use “Gaussian GAMP” (GGAMP) when referring to GAMP under the estimation functions (17).
Theorem 1: Consider Algorithms 1 or 2 with Gaussian estimation functions (17) defined for any vectors \( \tau_w \) and \( \tau_0 > 0 \). Then, as \( t \to \infty \), the stepsizes \( \tau_p^t, \tau_x^t \) (or their scalar versions) converge to unique fixed points that are invariant to \( \theta_x \) and \( \theta_x \).

**Proof:** See Appendix A

An important special case that we now consider is scalar-stepsize GGAMP from Algorithm 2 under identical variances, i.e.,

\[
\tau_w = \tau_0 \mathbf{1}, \quad \tau_0 = \tau_0 \mathbf{1},
\]

for some \( \tau_w \) and \( \tau_0 > 0 \). In this case, lines 7 and 11 give

\[
\tau_s^t = \frac{1}{m} \mathbf{1}^T (\tau_p^t g^T (\tau_p^t, \tau_w^t)) = \frac{\tau_p^t \tau_w^t}{\tau_p^t + \tau_w^t} \tag{21a}
\]

\[
\tau_x^{t+1} = \frac{1}{n} \mathbf{1}^T (\tau_x^t g^T (\tau_x^t, \tau_x^t)) = \frac{\tau_x^t \tau_0}{\tau_x^t + \tau_0}. \tag{21b}
\]

and, combining these with lines 6 and 9 we get

\[
\frac{1}{\tau_s^t} = \frac{1}{\tau_p^t} + \frac{1}{\tau_w^t} = \frac{1}{m} \| \mathbf{A} \|^2_p \tau_x^t + \frac{1}{\tau_w^t} \tag{22a}
\]

\[
\frac{1}{\tau_x^{t+1}} = \frac{1}{\tau_x^t} + \frac{1}{\tau_0} = \frac{1}{n} \| \mathbf{A} \|^2_x \tau_x^t + \frac{1}{\tau_0}. \tag{22b}
\]

IV. SCALAR-STEP SIZE GGAMP CONVERGENCE

A. Main Result

We now investigate the convergence of the primal and dual variables \( x^t \) and \( \mathbf{z}^t \) for scalar GGAMP. Since, for this algorithm, the previous section established that, as \( t \to \infty \), the stepsizes \( \tau_p^t \) and \( \tau_x^t \) converge independently of \( \theta_x, \theta_x, \mathbf{z}, \mathbf{x} \), we henceforth consider GGAMP with fixed stepsizes \( \tau_p^t = \tau_p \) and \( \tau_x^t = \tau_x \), where \( \tau_p \) and \( \tau_x \) are the fixed points of (22) for Algorithm 2 (A generalization to arbitrary fixed stepsizes will be given in Section V).

**Theorem 2:** Define

\[
\Gamma(\theta_x, \theta_x) := \begin{cases} \frac{2 [(2 - \theta_x) m + \theta_x n]}{\theta_x \theta_x mn} & \text{if } m \geq n \\ \frac{2 [(2 - \theta_x) n + \theta_x m]}{\theta_x \theta_x mn} & \text{if } m \leq n. \end{cases} \tag{23}
\]

Under Gaussian priors (i.e., (17)) with identical variances (20), scalar-stepsize GAMP from Algorithm 2 converges for any \( \tau_w \) and \( \tau_0 > 0 \) when

\[
\Gamma(\theta_x, \theta_x) > \| \mathbf{A} \|^2_p / \| \mathbf{A} \|^2_p. \tag{24}
\]

Conversely, it diverges for large enough \( \tau_0 \tau_w \) when

\[
\Gamma(\theta_x, \theta_x) < \| \mathbf{A} \|^2_p / \| \mathbf{A} \|^2_p. \tag{25}
\]

**Proof:** See Appendix C

Theorem 2 provides a simple necessary and sufficient condition on the convergence of scalar GGAMP. To better interpret this condition, recall that \( \| \mathbf{A} \|^2_p \) is the maximum squared singular value of \( \mathbf{A} \), and \( \| \mathbf{A} \|^2_p \) is the sum of the squared singular values of \( \mathbf{A} \) (i.e., \( \| \mathbf{A} \|^2_p = \sum_{i=1}^{\min(m,n)} \sigma_i^2(\mathbf{A}) \)), so

\[
\kappa(\mathbf{A}) := \frac{\| \mathbf{A} \|^2_p}{\| \mathbf{A} \|^2_p / \min(m,n)} \tag{26}
\]

is the peak-to-average ratio of the squared singular values of \( \mathbf{A} \). Convergence condition (24) can then be rewritten as

\[
\kappa(\mathbf{A}) < \kappa_{\max}(\theta_x, \theta_x) := \min\{m,n\} \Gamma(\theta_x, \theta_x), \tag{27}
\]

meaning that, for GGAMP convergence, it is necessary and sufficient to choose \( \kappa_{\max}(\theta_x, \theta_x) \) above the peak-to-average ratio of the squared singular values.

When there is no damping (i.e., \( \theta_x = 1 = \theta_x \)), the definitions in (23) and (27) can be combined to yield

\[
\kappa_{\max}(1, 1) = \frac{2 \min\{m,n\} (m+n)}{mn} \in [2, 4]. \tag{28}
\]

More generally, for \( \theta_x, \theta_x \in (0, 1] \), it can be shown that

\[
\frac{2}{\theta_x \theta_x} \leq \kappa_{\max}(\theta_x, \theta_x) \leq \frac{4}{\theta_x \theta_x}, \tag{29}
\]

so that the necessary and sufficient GGAMP convergence condition (27) can be rewritten as

\[
\theta_x \theta_x < \frac{C}{\kappa(\mathbf{A})} \quad \text{for some } C \in [2, 4], \tag{30}
\]

which implies that, by choosing sufficiently small damping constants \( \theta_x \) and \( \theta_x \), scalar-stepsize GGAMP can always be made to converge.

Condition (30) also helps to understand the effect of \( \kappa(\mathbf{A}) \) on the GGAMP convergence rate. For example, if we equate \( \theta_x = \theta_x = \theta \) for simplicity, then (30) implies that

\[
\theta < \sqrt{C / \kappa(\mathbf{A})}. \tag{31}
\]

Thus, if GGAMP converges at rate \( \theta \), then after \( \theta \) is adjusted to ensure convergence, GGAMP will converge at a rate below \( \sqrt{C / \kappa(\mathbf{A})} \). So larger peak-to-average ratios \( \kappa(\mathbf{A}) \) will result in slower convergence.

B. Examples of Matrices

To illustrate how the level of damping is affected by the nature of the matrix \( \mathbf{A} \), we consider several examples.

a) Large i.i.d. matrices: Suppose that \( \mathbf{A} \in \mathbb{R}^{m \times n} \) has i.i.d. components with zero mean and unit variance. For these matrices, we know from the rigorous state evolution analysis [18] that, in the large-system limit (i.e., \( m, n \to \infty \) with fixed \( m/n \)), scalar-stepsize GGAMP will converge without any damping. We can reproduce this result using our analysis as follows: By the Marcenko-Pastur Theorem [31], it can be easily shown that

\[
\kappa(\mathbf{A}) \approx \frac{\min\{m,n\}}{m} \left[ 1 + \sqrt{\frac{m}{n}} \right] \leq \frac{2 \min\{m,n\}(m+n)}{mn}, \tag{32}
\]

with equality when \( m = n \), and where the approximation becomes exact in the large-system limit. Because this Marcenko-Pastur bound coincides with the \( \theta_x = 1 = \theta_x \) case (28) of the convergence condition (27), our analysis implies that, for large
i.i.d. matrices, scalar stepsize GGAMP will converge without damping, thereby confirming the state evolution analysis. Note that we require that the asymptotic value of $m/n \neq 1$ so that the inequality in (32) is strict; when $m = n$, (32) becomes an equality and we obtain a condition $T(\theta_x, \theta_x) = \|A\|^2_F/\|F\|^2_F$ right on the boundary between convergence and divergence, where Theorem 2 does not make any statements.

b) Subsampled unitary matrices: Suppose that $A$ is constructed by removing either rows or columns, but not both, from a unitary matrix. Then, $\kappa(A) = 1$, so, from (29), $\kappa(A) < \kappa_{\max}(\theta_x, \theta_x)$ for any $\theta_x, \theta_x \in (0, 1]$. Hence, scalar GGAMP will converge with or without damping.

c) Linear filtering: Suppose that $A \in \mathbb{R}^{n \times n}$ is circulant with first column $h$, so that $(Ax)_i = (h * x)_i$, where $*$ denotes circular convolution. (Linear convolution could be implemented via zero padding.) Then, it can be shown that

\[
\kappa(A) = \frac{\max_k=0, \ldots, n-1 |H(e^{j2\pi k/n})|}{n \sum_{k=0}^{n-1} |H(e^{j2\pi k/n})|^2},
\]

(33)

where $H(e^{j\omega})$ is the DTFT of $h$. Equation (33) implies that more damping is needed as the filter becomes more narrow-band. For example, if $H(e^{j\omega})$ has a normalized bandwidth of $B \in (0, 1]$, then $\kappa(A) \approx 1/B$ and, relative to an allpass filter, GGAMP will need to slow by a factor of $O(1/\sqrt{B})$.

d) Low-rank matrices: Suppose that $A \in \mathbb{R}^{m \times n}$ has only $r$ non-zero singular values, all of equal size. Then

\[
\kappa(A) = \frac{r}{\min\{m, n\}},
\]

which, from (31), implies the need to choose a damping constant $\theta < \sqrt{\min\{m, n\}/r}$, slowing the algorithm by a factor of $\sqrt{\min\{m, n\}/r}$ relative to a full-rank matrix. Hence, more damping is needed as the relative rank decreases.

e) Walk-summable matrices: Closely related to Gaussian GAMP is Gaussian belief propagation [23], which performs a similar iterative algorithm to minimize a general quadratic function of the form $f(x) = x^H J x + \text{Real}(e^{H^H x})$ for some positive definite matrix $J$. A classic result of [32] is that Gaussian belief propagation will converge when

\[
\lambda_{\max}(|I - J|) < 1,
\]

(34)

where $|I - J|$ is the component-wise magnitude. The condition (34) is called walk summability, with the constraints $J_{ii} = 1$ being for normalization.

To compare this condition with GGAMP, first observe that, in the identical-variance case (20), GGAMP performs the same quadratic minimization with a particular $c$ and with

\[
J = \tau_0 A^H A + \tau_w^{-1} I.
\]

Now, consider the high-SNR case, where $\tau_0 = 1$ and $\tau_w^{-1} \approx 0$, so that $J \approx A^H A$. Then the walk-summability condition (34) reduces to

\[
\lambda_{\max}(|I - A^H A|) < 1,
\]

(35)

where the normalizations $J_{ii} = 1$ imply that the columns of $A$ have unit norm, i.e., that $\|A\|^2_F = n$. Note that, if (35) is satisfied, then

\[
\|A\|^2_F = \lambda_{\max}(A^H A) \leq 1 + |1 - \lambda_{\max}(A^H A)| = 1 + |\lambda_{\max}(A^H A - I)| \leq 1 + \lambda_{\max}(|I - A^H A|) = 1 + \lambda_{\max}(|I - A^H A|) < 2.
\]

Applying these results to the $\kappa(A)$ definition (26), we find

\[
\kappa(A) = \frac{\|A\|^2_F}{\|A\|^2_F / \min\{m, n\}} < \frac{2 \min\{m, n\}}{n} < \kappa_{\max}(1, 1),
\]

(36)

where the latter inequality follows from inspection of (32).

We conclude that, in the high-SNR regime, walk summability is sufficient for GGAMP to converge with or without damping.

V. LOCAL STABILITY FOR STRICTLY CONVEX FUNCTIONS

We next consider the convergence with a more general class of scalar estimation functions $\tilde{F}_* \text{ and } g_x$: those that are twice continuously differentiable with first derivatives bounded as

\[
\tilde{F}_*(\mathbf{p}, \mathbf{r}_*), g_x \in (0, 1), \quad |g_x'(r, \tau_x)| < 1,
\]

(37)

for all $\mathbf{p}, \mathbf{r}, \mathbf{p}_*$ and $\tau_*$. This condition arises in the important case of minimizing strictly convex functions. Specifically, if GAMP is used in max-sum mode so that the scalar estimation functions are given by (3) and (4) with strictly convex, twice differentiable functions $G_1$ and $F_1$, then (3), (4), and (10) show that the conditions in (36) will be satisfied.

Outside of the Gaussian scenario, we have not yet established conditions on the global convergence of GAMP for general scalar estimation functions. Instead, we now establish conditions on local stability, as defined in (34). To simplify the analysis, we will assume that the GAMP algorithm uses arbitrary but fixed stepsize vectors $\mathbf{p}_*$ and $\tau_*$. Under these assumptions, consider any fixed point $(\mathbf{p}, \mathbf{r})$ of the GAMP method, and define the matrices

\[
Q_* := \text{diag}(q_*), \quad q_* := \tilde{F}_*(\mathbf{p}, \mathbf{p}_*),
\]

(37a)

\[
Q_x := \text{diag}(q_x), \quad q_x := g_x'(r, \tau_x),
\]

(37b)

evaluated at that fixed point. Note that, under assumption (36), the components of $q_*$ and $q_x$ lie in $(0, 1)$. Define the matrix

\[
\tilde{A} := \text{diag}^{1/2}(\mathbf{p}_*, q_* \mathbf{p}_*) \text{diag}^{1/2}(\mathbf{r}, q_x).
\]

(38)

Then (37)-(38), together with lines 8 and 10 of Algorithm 1 imply

\[
\sum_{i=1}^{m} |\tilde{A}_{ij}|^2 = q_x \tau_x \sum_{i=1}^{m} S_{pi} q_* |A_{ij}|^2
\]

(39)

\[
= q_x \tau_x \sum_{i=1}^{m} S_{ij} \mathbf{p}_i q_* q_x = q_x < 1.
\]

(40)

Hence, the column norms of $\tilde{A}$ in (38) are less than one. Similar arguments can be used to establish that, for any $i$,

\[
\sum_{j=1}^{n} |\tilde{A}_{ij}|^2 = q_s < 1,
\]

(41)

so that $\tilde{A}$ also has row norms less than one. We will thus call $\tilde{A}$ the row-column normalized matrix.
Theorem 3: Consider any fixed point \((\mathbf{s}, \mathbf{x})\) of GAMP Algorithm\(^1\) or Algorithm\(^2\) with fixed vector or scalar step sizes \(\tau_p\) and \(\tau_r\), respectively, and scalar estimation functions \(\mathbf{q}_s\) and \(g_x\) satisfying the above conditions, including (36). Then, the fixed point is locally stable if
\[
\theta_x \theta_s \|\mathbf{A}^2\|_F^2 < 1,
\] (42)
for \(\mathbf{A}\) defined in (38). For the Gaussian GAMP algorithm, the same condition implies the algorithm is globally stable.

Proof: See Appendix A

To relate this condition to Theorem 2, consider the case when \(\tau_p\) and \(\tau_r\) are fixed points of (19) with \(\mathbf{S} = |\mathbf{A}|^2\). From (40) and (41), we have that
\[
\|\mathbf{A}\|_F^2 = m\mathbf{q}_s = n\mathbf{q}_x \leq \min\{m, n\} \max\{\mathbf{q}_s, \mathbf{q}_x\},
\]
where
\[
\mathbf{q}_s = \frac{1}{m} \sum_{i=1}^{m} q_x, \quad \mathbf{q}_x = \frac{1}{n} \sum_{j=1}^{n} q_x.
\]

Thus, the peak-to-average ratio of \(\mathbf{A}\) as defined in (26) is bounded below as
\[
\kappa(\mathbf{A}) \geq \frac{\|\mathbf{A}\|_F^2}{\max\{\mathbf{q}_s, \mathbf{q}_x\}}.
\]
Hence, a sufficient condition to satisfy (42) is given by
\[
\kappa(\mathbf{A}) < \frac{1}{\theta_x \theta_s \max\{\mathbf{q}_s, \mathbf{q}_x\}}.
\] (43)

In comparison, (27) and (29) show that a Gaussian GAMP with scalar step sizes converges is \(\kappa(\mathbf{A}) < C/(\theta_x \theta_s)\). We conclude that the sufficient condition for the vector-stepsize GAMP algorithm to converge is similar to the the scalar-stepsize GAMP algorithm, but where the peak-to-average ratio is measured on a certain normalized matrix.

Conclusions

A key outstanding issue for the adoption of AMP-related methods is their convergence for generic finite-dimensional linear transforms. Similar to other loopy BP-based methods, standard forms of AMP may diverge. In this paper, we presented a damped version of the generalized AMP algorithm that, when used with fixed step sizes, can guarantee global convergence for Gaussian distributions and local convergence for the minimization of strictly convex functions. The required amount of damping is related to the peak-to-average ratio of the squared singular values of the transform matrix. However, much remains unanswered: Most importantly, we have yet to derive a condition for global convergence even in the case of strictly convex functions. Secondly, our analysis assumes the use of fixed step sizes. Third, short of computing the peak-to-average singular-value ratio, we proposed no method to compute the damping constants. Hence, an adaptive method may be useful in practice.

Appendix A

Proof of Theorem 1

The variance of updates of both Algorithms\(^1\) and\(^2\) are both of the form (19) with different choices of \(\mathbf{S}\). So, the theorem will be proven by showing that the updates (19) converge for any non-negative matrix \(\mathbf{S} \geq 0\). To this end, we use the results in [34]. Specifically, for any \(\mathbf{w}\) and \(\mathbf{r}_0 \geq 0\), define the functions
\[
\Phi_s(\tau_s) := \left[\mathbf{S} \mathbf{w}_s + \frac{1}{\mathbf{w}_s}\right]^{-1} - \frac{1}{\mathbf{w}_s},
\]
\[
\Phi_x(\tau_r) := \left[\mathbf{S}^T \mathbf{r}_s + \frac{1}{\mathbf{r}_s}\right]^{-1} - \frac{1}{\mathbf{r}_s}
\]
so that the updates (19) can be written as
\[
\tau_s^t = \Phi_s(\tau_s^t), \quad \tau_r^t = \Phi_x(\tau_r^t).
\]

It is easy to check that, for any \(\mathbf{S} \geq 0\),

(i) \(\Phi_s(\tau_s) > 0\),

(ii) \(\tau_x > \tau_x' \Rightarrow \Phi_s(\tau_x) \leq \Phi_s(\tau_x')\), and

(iii) For all \(\alpha > 1\), \(\Phi(\alpha \tau_x) > (1/\alpha)\Phi(\tau_x)\).

with the analogous properties being satisfied by \(\Phi_x(\tau_r)\). Now let \(\Phi := \Phi_x \circ \Phi_s\) be the composition of the two functions so that \(\tau_s^{t+1} = \Phi(\tau_x^t)\). Then, \(\Phi\) satisfies the three properties:

(i) \(\Phi(\tau_x) > 0\),

(ii) \(\tau_x \geq \tau_x' \Rightarrow \Phi(\tau_x) \geq \Phi(\tau_x')\), and

(iii) For all \(\alpha > 1\), \(\Phi(\alpha \tau_x) < \alpha \Phi(\tau_x)\).

Also, for any \(\tau_x \geq 0\), we have \(\Phi_x(\tau_x) \leq \tau_0\) and therefore, \(\Phi(\tau_x) \leq \tau_0\) for all \(\tau_x \geq 0\). Hence, taking any \(\tau_x \geq \tau_0\), we obtain:
\[
\tau_x \geq \Phi(\tau_x).
\]

The results in [34] then show that the updates \(\tau_s^{t+1} = \Phi(\tau_x^t)\) converge to unique fixed points. A similar argument shows that \(\tau_x^t\) also converges to unique fixed points.

Appendix B

Linear System Stability Condition

The proofs of both Theorems\(^2\) and\(^3\) are based on analyzing the GAMP algorithm via an equivalent linear system and then applying results from linear stability theory. For both results we will show that the condition of the theorem is equivalent to an eigenvalue test on a certain matrix.

First consider the Gaussian GAMP algorithm with fixed vector stepsizes. With fixed stepsizes and Gaussian estimation functions (17), Algorithm\(^1\) reduces to a linear system:
\[
\mathbf{s}' = (1 - \theta_s) \mathbf{s} - \theta_s \mathbf{Q}_s(\mathbf{s} + \mathbf{T}_p \mathbf{A} \mathbf{x}) - \theta_w \mathbf{T}_w \mathbf{y}
\]
\[
\mathbf{x}^{t+1} = (1 - \theta_t) \mathbf{x}^t + \theta_t \mathbf{Q}_s(\mathbf{x} - \mathbf{T}_r \mathbf{A}^H \mathbf{y}^t - \mathbf{x}_0) + \theta_w \mathbf{x}_0
\] (44a)

where
\[
\mathbf{Q}_s = \text{diag}(\mathbf{q}_s), \quad q_s = \mathbf{T}_w / (\mathbf{T}_w + \mathbf{T}_p),
\]
\[
\mathbf{Q}_x = \text{diag}(\mathbf{q}_x), \quad q_x = \mathbf{T}_0 / (\mathbf{T}_0 + \mathbf{T}_r).
\] (45a)

\[
\mathbf{Q}_s = \text{diag}(\mathbf{q}_s), \quad q_s = \mathbf{T}_w / (\mathbf{T}_w + \mathbf{T}_p),
\]
\[
\mathbf{Q}_x = \text{diag}(\mathbf{q}_x), \quad q_x = \mathbf{T}_0 / (\mathbf{T}_0 + \mathbf{T}_r).
\] (45b)
Note that the components of \( q_s \) and \( q_x \) are in \((0,1)\). We can write the system \((44)\) in matrix form as
\[
\begin{bmatrix}
\bar{x}^t \\
x^{t+1}
\end{bmatrix} = G \begin{bmatrix}
\bar{x}^{t-1} \\
x^t
\end{bmatrix} + b,
\] (46)
for an appropriate matrix \( G \) and vector \( b \). The matrix \( G \) is given by
\[
G := \begin{bmatrix}
I & 0 \\
-\theta_x \text{diag}(\tau_x) A^H D_x & D_x \text{diag}(\tau_s) A
\end{bmatrix},
\] (47)
where
\[
D_s = (1-\theta_s)I + \theta_s Q_s \tag{48a}
\]
\[
D_x = (1-\theta_x)I + \theta_x Q_x. \tag{48b}
\]
Here we have used that
\[
q_s \tau_p = \tau_s, \quad q_x \tau_r = \tau_x. \tag{49}
\]
Note that both \( D_x \) and \( D_s \) are diagonal matrices with entries in the interval \((0,1)\).

Now, consider the case of the more general scalar estimation functions satisfying \((36)\) and other assumptions in Section \(V\). Due to the differentiability assumptions, to prove the local stability, we only have to look at the linearization of the system around the fixed points \((43)\). With fixed stepsizes, the linearization of the updates in Algorithm \(I\) around any fixed point is given by
\[
\bar{x}^t = (1-\theta_s)\bar{x}^{t-1} + \theta_s Q_s (\bar{x}^{t-1} + \tau_p A x^t) \tag{50a}
\]
\[
x^{t+1} = (1-\theta_x)x^t + \theta_x Q_x (x^t - \tau_r A^H s^t) \tag{50b}
\]
where the matrices \( Q_s \) and \( Q_x \) in \((45)\) are replaced by the derivatives \((37)\). This linear system is also of the form \((46)\) with the same matrix \((47)\). Also, under the assumptions of the theorem, \( q_s \) and \( q_x \) are vectors with components in \((0,1)\).

Hence, we conclude that to prove the global stability of Gaussian GAMP, or the local stability of GAMP under the assumptions of Theorem \(I\) it suffices to show that the linear system \((46)\) with a matrix \( G \) of the form \((47)\) is stable. The matrices \( D_s \) and \( D_x \) are given in \((48)\) where \( Q_s \) and \( Q_x \) are diagonal matrices with elements in \((0,1)\).

To evaluate this condition, first recall that the linear system \((46)\) is stable when the eigenvalues of \( G \) are in the unit circle. However, if we define
\[
T := \begin{bmatrix}
\text{diag}^{-1/2}(\theta_s \tau_s) & 0 \\
0 & \text{diag}^{-1/2}(\theta_x \tau_x)
\end{bmatrix},
\]
the eigenvalues of \( G \) are identical to those of \( H \) given by
\[
H := TG T^{-1} = \begin{bmatrix}
I & 0 \\
-\lambda F^H D_x & D_x F
\end{bmatrix},
\] (51)
where
\[
F = \sqrt{\theta_s \theta_x \tau_s \tau_x} A, \tag{57}
\]
and
\[
d_s = (1-\theta_s) + \theta_s q_s, \quad q_s = \frac{\tau_w}{\tau_p + \tau_w}, \tag{58a}
\]
\[
d_x = (1-\theta_x) + \theta_x q_x, \quad q_x = \frac{\tau_0}{\tau_0 + \tau_r}. \tag{58b}
\]

**APPENDIX C**

**PROOF OF THEOREM \(I\)**

Our first step in the proof is to simplify the condition in Lemma \(2\).

**Lemma 2:** Consider the GAMP algorithm with scalar stepsizes, Gaussian scalar estimation functions \((17)\) and identical variances \((20)\) we obtain the following:
\[
J_\lambda = (\lambda - d_x)I + \lambda \frac{\lambda}{d_x} F^H F, \tag{56}
\]
where
\[
F = \sqrt{\theta_s \theta_x \tau_s \tau_x} A, \tag{57}
\]
and
\[
d_s = (1-\theta_s) + \theta_s q_s, \quad q_s = \frac{\tau_w}{\tau_p + \tau_w}, \tag{58a}
\]
\[
d_x = (1-\theta_x) + \theta_x q_x, \quad q_x = \frac{\tau_0}{\tau_0 + \tau_r}. \tag{58b}
\]

Now, for any \( \lambda \in \mathbb{C} \), define the matrix
\[
H_\lambda := \lambda I - H = \begin{bmatrix}
\lambda I - D_x & -F \\
F^H D_s & \lambda I - D_x + F^H F
\end{bmatrix}. \tag{54}
\]
For stability, we need to show that for any \(|\lambda| \geq 1\), \(H_\lambda\) is invertible. We simplify this condition as follows: Consider any \( \lambda \) with \(|\lambda| \geq 1\). Now, \( D_s \) in \((43)\) is a diagonal matrix with entries in \([0,1]\). Hence \( \lambda I - D_s \) is invertible since \(|\lambda| \geq 1\). Therefore, taking a Schur complement, we see that \(H_\lambda\) is invertible if and only if the matrix
\[
J_\lambda := \lambda I - D_x + F^H F + F^H D_s (\lambda I - D_s)^{-1} F
\]
is invertible. We can summarize the result as follows.

**Lemma 1:** Consider the GAMP Algorithm \(I\) for any scalar estimation functions satisfying the conditions in Section \(V\) including \((36)\). The GAMP algorithm is locally stable around a fixed point if and only if \(J_\lambda\) is invertible for all \(|\lambda| \geq 1\), where
\[
J_\lambda = \lambda I - D_x + \lambda F^H (\lambda I - D_s)^{-1} F, \tag{55}
\]
and \(F\) is given in \((52)\). In the special case of Gaussian estimation functions \((17)\), the above condition implies the GAMP Algorithm \(I\) will be globally stable.

A similar calculation can be performed for the GAMP algorithm with scalar stepsizes. In this case, the vector stepsizes such as \(\tau_x\) and \(\tau_p\) are replaced with the scalar quantities \(\tau_x\) and \(\tau_p\). For the case of Gaussian estimation functions \((17)\) and identical variances \((20)\) we obtain the following:

**Lemma 2:** Consider the GAMP Algorithm \(2\) with scalar stepsizes, Gaussian scalar estimation functions \((17)\) and identical variances \((20)\). Then, the algorithm is globally stable if and only if \(J_\lambda\) is invertible for all \(|\lambda| \geq 1\), where
\[
J_\lambda = (\lambda - d_x)I + \lambda \frac{\lambda}{d_x} F^H F, \tag{56}
\]
where
\[
F = \sqrt{\theta_s \theta_x \tau_s \tau_x} A, \tag{57}
\]
and
\[
d_s = (1-\theta_s) + \theta_s q_s, \quad q_s = \frac{\tau_w}{\tau_p + \tau_w}, \tag{58a}
\]
\[
d_x = (1-\theta_x) + \theta_x q_x, \quad q_x = \frac{\tau_0}{\tau_0 + \tau_r}. \tag{58b}
\]
Proof: From Lemma 2, we know that the system is stable if and only if \( J_\lambda \) in (56) is invertible for all \( |\lambda| \geq 1 \). To evaluate this condition, suppose that \( J_\lambda \) is not invertible for some \( |\lambda| \geq 1 \). Then, there exists an \( \nu \neq 0 \) such that \( J_\lambda \nu = 0 \), which implies that

\[
F^H F \nu = \frac{(d_x - \lambda)(\lambda - d_s)}{\lambda} \nu.
\]

Using the expression for \( F \) in (57), this is equivalent to

\[
A^H \nu = \frac{(d_x - \lambda)(\lambda - d_s)}{\theta_x \theta_s \tau_x \tau_s \lambda} \nu.
\]

Thus, \( \nu \) is an eigenvector of \( A^H A \). But, \( \sigma^2 \) is an eigenvalue of \( A^H A \) if and only if \( \sigma \) is a singular value of \( A \). Hence, we conclude that \( J_\lambda \) is invertible if and only if there exists a singular value \( \sigma \) of \( A \) such that

\[
\sigma^2 \theta_x \theta_s \tau_x \tau_s \lambda = (d_x - \lambda)(\lambda - d_s).
\]

Equivalently, we have shown that the system is stable if and only if the second-order polynomial

\[
p(\lambda) := \lambda^2 + (\sigma^2 \theta_x \theta_s \tau_x \tau_s - d_x - d_s) \lambda + d_x d_s
\]

has stable roots for all singular values of \( A, \sigma \). Now recall that \( d_x \) and \( d_s \in (0, 1) \). By the Jury stability condition, the \( p(\lambda) \) has unstable roots if and only if \( p(1) > 0 \) and \( p(-1) < 0 \). Now, the first condition is always satisfied since

\[
p(1) = \sigma^2 \theta_x \theta_s \tau_x \tau_s + (1 - d_s)(1 - d_x) > 0.
\]

So, the polynomial is stable if and only if

\[
0 < p(-1) = -\sigma^2 \theta_x \theta_s \tau_x \tau_s + (1 + d_s)(1 + d_x),
\]

or equivalently,

\[
\sigma^2 \theta_x \theta_s \tau_x \tau_s < (1 + d_s)(1 + d_x).
\]

For this to be true for all singular values of \( A \), we need

\[
\sigma_{\text{max}}^2(A) \theta_x \theta_s \tau_x \tau_s < (1 + d_s)(1 + d_x).
\]

Thus, the system is stable if and only if (59) is satisfied with

\[
\gamma := \frac{(1 + d_x)(1 + d_s)}{\theta_x \theta_s \tau_x \tau_s \|A\|_{F}^2}.
\]

So, we simply need to prove that (61) matches the definition in (60). To this end, first note that

\[
\frac{1 + d_x}{\tau_x} = \frac{2 - \theta_x}{\tau_x} + \frac{\theta_x}{\tau_x} = \frac{2 - \theta_x}{\tau_0},
\]

where (a) follows from the definition \( d_x = \tau_x / \tau_s \) in (48a) and (b) follows from the fixed-point equation (22b). Similarly using (48b) and (22a), we obtain that

\[
\frac{1 + d_s}{\tau_s} = \frac{2 - \theta_s}{\tau_s} + \frac{\theta_s}{\tau_s} = \frac{2 - \theta_s}{\tau_0}.
\]

Substituting (62) and (63) into (61), we obtain (60) and the lemma is proven.

Let

\[
\Gamma := \inf_{\tau_0, \tau_w > 0} \gamma,
\]

where \( \gamma \) is defined in (60). It follows that if

\[
\sigma^2(A) < \Gamma \|A\|_{F}^2,
\]

then the system is stable for all \( \tau_0 \) and \( \tau_w \). Conversely, if

\[
\sigma^2(A) > \Gamma \|A\|_{F}^2,
\]

then there exists at least one \( \tau_0 \) and \( \tau_w \) such that the system is unstable. So, the theorem will be proven if we can show that \( \Gamma \) defined in (64) matches the expression in (23). To calculate the minima in (64), we begin with the following lemma.

Lemma 4: The minimization in (64) is given by

\[
\Gamma = \inf_{\tau_0 > 0} \lim_{\tau_w \to 0} \gamma.
\]

That is, we can take the limit as \( \tau_w \to 0 \).

Proof: Consider the derivative of \( \gamma \) with respect to \( \tau_w \) holding \( \tau_0, \ A, \theta_s, \theta_x \) constant. The stepsize terms \( \tau_x \) and \( \tau_s \) are the fixed points of the stepsize equations (22). A long derivative calculation then shows that

\[
\frac{\partial \gamma}{\partial \tau_w} \leq 0.
\]

Thus, \( \gamma \) is decreasing in \( \tau_w \). So, for any \( \tau_0 \), the minimization in (60) can be achieved taking \( \tau_w \to 0 \).

We conclude by evaluating the limit in (65). The following lemma shows that value of the minimization agrees with (23), and hence completes the proof of the theorem.

Lemma 5: For any \( A \) and damping constants \( \theta_s, \theta_x \), the limit in (65) is given by (23).

Proof: First consider the case when \( m \geq n \). In this case, \( \tau_x \to 0 \), so that \( \tau_x \ll \tau_0 \). Therefore, the fixed point equation (22b) reduces to

\[
\frac{\|A\|_{F}^2}{n} \tau_x = \frac{1}{\tau_x} - \frac{1}{\tau_0} \approx \frac{1}{\tau_x}.
\]

Substituting this into (22b),

\[
\frac{\|A\|_{F}^2}{m} \tau_x \approx m \|A\|_{F}^2 \tau_x + \tau_w,
\]

so that

\[
\tau_x \approx \frac{1}{\|A\|_{F}^2} \frac{mn}{m - n} \tau_w.
\]

Hence,

\[
\frac{2}{\tau_x} \frac{\theta_s}{\tau_0} \approx \frac{2}{\tau_x} \frac{2}{m - n} \frac{\|A\|_{F}^2}{m} \tau_w.
\]

Also,

\[
\frac{2}{\tau_s} \frac{\theta_s}{\tau_0} \approx \frac{2}{m} \|A\|_{F}^2 \tau_x + (2 - \theta_s) \tau_w
\]

\[
\tau_w \approx \frac{2n}{m - n} \tau_x + (2 - \theta_s) \tau_w
\]

\[
= \frac{(2 - \theta_s)(m - n)}{m - n} \tau_w
\]

where (a) follows from (22a) and (b) follows from (66). Substituting (67) and (68) into (60), we obtain the limit

\[
\lim_{\tau_w \to 0} \gamma = \frac{2[(2 - \theta_s)m + \theta_x n]}{\theta_s \theta_x mn}.
\]
A similar derivation can be performed for the case when $m < n$.

**APPENDIX D**

**Proof of Theorem**

We begin with a technical lemma.

**Lemma 6.** Let $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$ so that, in polar coordinates, $\lambda = re^{i\theta}$ for some $r \geq 1$. Then,

$$\text{Real}(e^{-i\theta}z) > 0,$$

for all $z \in \mathbb{C}$ of the form

$$z = \lambda - d_x + \frac{\sigma^2\lambda}{\lambda - d_s},$$

with $d_x, d_z, \sigma \in [0, 1)$.

**Proof:** Define

$$y := \text{Real}(e^{-i\theta}z) = r - d_x \cos(\theta) + \text{Real} \left[ \frac{r \sigma^2}{re^{i\theta} - d_s} \right]$$

$$= r - d_x \cos \theta + \frac{r \sigma^2 (r \cos \theta - d_s)}{r^2 - 2rd_s \cos \theta + d_s^2}.$$ 

We need to show $y > 0$. We prove this in three cases.

**Case 1:** $r \cos \theta \geq d_s$. In this case,

$$y = r - d_x \cos \theta + \frac{r \sigma^2 (r \cos \theta - d_s)}{r^2 - 2rd_s \cos \theta + d_s^2} \geq r - d_x \cos \theta > 0,$$

where the last line follows from the fact that $r \geq 1$ and $d_x \in [0, 1)$.

**Case 2:** $r \cos \theta \in [0, d_s]$. In this case, a detailed trigonometric calculation shows that $y$ is minimized when $\cos \theta = 0$. This boundary condition will be handled in case 3. Taking the derivative

$$y' = -d_x + \frac{r \sigma^2 [r (r^2 - 2rd_s + d_s^2) + 2rd_s (r \cos \theta - d_s)]}{(r^2 - 2rd_s \cos \theta + d_s^2)^2}$$

At any maximum, $y' = 0$ so

$$y = r - d_x \cos \theta + \frac{\sqrt{d_s} \sigma (r \cos \theta - d_s)}{\sqrt{d_s^2 + d_s^2}} \geq r - d_x \cos \theta > 0,$$

**Case 3:** $\cos \theta < 0$. In this case, it is useful to let $D$

$$y = r - d_x \cos \theta + \frac{r \sigma^2 (r \cos \theta - d_s)}{r^2 - 2rd_s \cos \theta + d_s^2} \geq r + \frac{r \sigma^2 (r \cos \theta - d_s)}{r^2 - 2rd_s \cos \theta + d_s^2}$$

$$= C \left[ r^2 - 2rd_s \cos \theta + d_s^2 \right] + \sigma^2 \left( r \cos \theta - d_s \right) \]$$

where $C = r/(r^2 - 2rd_s \cos \theta + d_s^2)$ and $p(r)$ is the quadratic polynomial

$$p(r) = r^2 + (\sigma^2 - 2d_s) \cos \theta r + d_s^2 - d_s.$$ 

We need to show $p(r) > 0$. If $\sigma^2 - 2d_s < 0$ then, since $\cos \theta < 0$,

$$p(r) \geq r^2 + d_s^2 - d_s > 0.$$ 

On the other hand, if $\sigma^2 - 2d_s > 0$ then, then $p(r)$ is minimized with $\cos \theta = -1$, so

$$p(r) \geq r^2 - \sigma^2 + 2d_s + d_s^2 - d_s > 0,$$

where the inequality holds since $r \geq 1$, $\sigma^2 < 1$ and $d_s \geq 0$.

We can now prove the main result. Suppose that (42) is satisfied. By the definition of $F$ in (52) and $A$ in (38), we have that

$$\sigma_{\max}(F) < 1.$$ 

Now, from Lemma 6 we need to show that the matrix $J_{\lambda}$ in (55) is invertible for all $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$. We prove this by contradiction.

Suppose that $J_{\lambda}$ in (55) is not invertible for some $\lambda$ with $|\lambda| \geq 1$. Then, there exists an $x$ with $\|x\|^2 = 1$ such that $x^H J_{\lambda} x = 0$. Therefore, if we define $y = Fx$, the definition of $J_{\lambda}$ in (55) shows that

$$x^H (\lambda - D)x + \lambda \chi^H (\lambda I - D_s)^{-1} y = 0.$$ 

Since $D_x$ and $D_s$ are diagonal, we have

$$\sum_{j=1}^{n} (\lambda - d_{x_j}) |x_j|^2 + \sum_{i=1}^{m} \frac{\lambda}{\lambda - d_{s_j}} |y_j|^2 = 0.$$ 

Since $\|x\| = 1$, $\sum_{j} |x_j|^2 = 1$. Also, since $\|F\| = \sigma_{\max}(F) < 1$,

$$\sum_{i} |y_i|^2 = \|Fx\| = 2 \sigma^2 \|x\|^2 = \sigma^2,$$

for some $\sigma^2 < 1$. Therefore, (71) shows that

$$0 \in \text{conv}(P_{\lambda}),$$

where $P$ is the set

$$P_{\lambda} := \left\{ \lambda - d_x + \frac{\sigma^2 \lambda}{\lambda - d_s} \mid d_x, d_s \in [0, 1) \right\}.$$ 

Now write $\lambda$ in polar coordinates: $\lambda = re^{i\theta}$ for some $r \geq 1$. From Lemma 6 every element in $z \in P_{\lambda}$, satisfies $\text{Real}(e^{-i\theta}z) > 0$. It follows that if $z \in \text{conv}(P_{\lambda})$, then $\text{Real}(e^{-i\theta}z) > 0$. But, this contradicts the fact that $0 \in \text{conv}(P_{\lambda})$. Hence, the assumption that $J_{\lambda}$ is not invertible must be false, and the theorem is proven.

**REFERENCES**


