Comparison of two nonparametric estimators for reliability of discrete-time semi-Markov systems based on multiple independent observations

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***Abstract*—We consider a discrete-time semi-Markov system, with a finite state space. The empirical and the exact maximum likelihood estimator for the semi-Markov kernel are given in the case of multiple parallel observations of the same process. Afterwards, we describe a reliability model described by a discrete-time semi-Markov process and we derive basic reliability measures, such as reliability, availability, failure rates and mean hitting times. Finally, we present a comparison between empirical and exact maximum likelihood estimators for these measures through a numerical application.**

***Index Terms*—Discrete-time semi-Markov system, nonparametric estimation, exact maximum likelihood estimation, reliability, mean hitting times.**

I. Discrete-time semi-Markov system

In recent literature, discrete-time semi-Markov models have achieved significant importance in probabilistic and statistical modeling especially the ones with a finite space state. System reliability and relative dependability measures consist, amongst others, an important application field. The term chain will be used for a discrete-time semi-Markov process. A general study on the semi-Markov chains is given by Barbu and Limnios [1] toward applications. Some statistical inference problems, such as the proposition of a computation procedure for solving the corresponding Markov renewal equation and the study of an empirical estimator of the semi-Markov kernel and other measurements in the case of one observed trajectory, are presented.

We consider a semi-Markov chain with finite state space and the sequence of the backward recurrence times, which form a coupled Markov chain. The basic properties of this Markov chain have been studied in Chryssaphinou et al. [2]. Trevezas and Limnios (2011) [3] present the exact maximum likelihood (EML) estimation of the semi-Markov kernel for a single trajectory of a semi-Markov system up to an arbitrary fixed time, when the length of the observation tends to infinity, and, next, when multiple independent observed trajectories generated by the same semi-Markov kernel, censored at a fixed time, when the number of trajectories tends to infinity, and study its asymptotic properties. In the present work, we focus on the latter case for a nonparametric semi-Markov model, which, from a practical point of view, corresponds to the evolution of multiple identical components of a repairable system or systems. Based on the maximum likelihood estimation of the coupled Markov chain, we examine the estimation of several reliability measures of a discrete-time semi-Markov system.

We give now all the necessary preliminaries concerned a semi-Markov chain. From now on we will use the following notation for the non-zero natural numbers $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and take by convention that $0/0 := 0$.

Consider the finite set $E = \{1, \dots, s\}$, $s \in \mathbb{N}^*$, and an E-valued stochastic chain $\boldsymbol{Z} := (Z_k)_{k\in\mathbb{N}}$. Let $\boldsymbol{J} := (J_n)_{n\in\mathbb{N}}$ be the successive visited states of $\boldsymbol{Z}$ with state space E and $\boldsymbol{S} := (S_n)_{n\in\mathbb{N}}$ are the jump times of $\boldsymbol{Z}$ with values in $\mathbb{N}$ with $0 = S_0 \le S_1 \le \cdots \le S_n \le S_{n+1} \le \cdots$. Also, let us denote $X_n := S_n - S_{n-1}$, $n \in \mathbb{N}^*$, as the sojourn times in these states with values in $\mathbb{N}$.

Definition 1. *The stochastic process $(\boldsymbol{J}, \boldsymbol{S}) := (J_n, S_n)_{n\in\mathbb{N}}$, with state space E, is said to be a Markov renewal chain (MRC), if, for all $j \in E$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$, it satisfies a.s. the following equality*

$$\mathbb{P}(J_{n+1} = j, X_{n+1} = k | J_0, \dots, J_n; S_1, \dots, S_n) = \mathbb{P}(J_{n+1} = j, X_{n+1} = k | J_n).$$

In this case, $\boldsymbol{Z}$ is called a semi-Markov chain (SMC).

Actually, $\boldsymbol{Z}$ gives the state of the process at time k. We assume that the MRC $(\boldsymbol{J}, \boldsymbol{S})$ is time homogeneous, that is, the above probability is independent of n and S_n. The process $\boldsymbol{J}$ is a Markov chain (MC) with state space E and transition kernel $\boldsymbol{p} := (p_{ij};\ i, j \in E)$, where

$$p_{ij} := \mathbb{P}(J_{n+1} = j | J_n = i),$$

called the embedded Markov chain (EMC) of $\boldsymbol{Z}$. We denote by $N(k)$, $k \in \mathbb{N}$, the process which counts the number of jumps of $\boldsymbol{Z}$ in the interval $(0, k]$, defined by $N(k) := \max\{n \ge 0 :$

$S_n \leq k\}$. The SMC $\boldsymbol{Z}$ is associated with the MRC $(\boldsymbol{J}, \boldsymbol{S})$ by

$$Z_k := J_{N(k)}, \quad k \in \mathbb{N}.$$

Let $N_i(k)$ be the number of visits of $\boldsymbol{Z}$ to state $i \in E$ up to time k, and $N_{ij}(k)$ the number of direct jumps of $\boldsymbol{Z}$ from state i to state j up to time k. To be specific,

$$N_i(k) := \sum_{m=1}^{N(k)} \mathbf{1}_{\{J_{m-1}=i\}}$$
$$N_{ij}(k) := \sum_{m=1}^{N(k)} \mathbf{1}_{\{J_{m-1}=i, J_m=j\}},$$

where $\mathbf{1}_A$ is the indicator function of the set A.

Definition 2. *The transition kernel $\boldsymbol{q}(k) := (q_{ij}(k);\ i,j \in E)$, $k \in \mathbb{N}$, is called the discrete-time semi-Markov kernel (DTSMK) of the SMC $\boldsymbol{Z}$ and it is defined by*

$$q_{ij}(k) := \mathbb{P}(J_{n+1} = j, X_{n+1} = k | J_n = i). \qquad (1)$$

For all $i, j \in E$, let $\boldsymbol{f}(k) := (f_{ij}(k); i, j \in E)$ be the conditional distribution function of the sojourn time in any state i, given that the next visited state is j, $j \neq i$, defined as follows

$$\begin{aligned} f_{ij}(k) &:= \mathbb{P}(X_{n+1} = k | J_n = i, J_{n+1} = j) \\ &= \begin{cases} \frac{q_{ij}(k)}{p_{ij}}, & \text{if } p_{ij} \neq 0, \\ \mathbf{1}_{\{k=\infty\}}, & \text{if } p_{ij} = 0. \end{cases} \end{aligned}$$

Definition 3. *For all $i, j \in E$, let us denote by $\boldsymbol{H}(k) := diag(H_i(k);\ i \in E)^\top$, $k \in \mathbb{N}$, the sojourn time cumulative distribution function in any state i*

$$H_i(k) := \mathbb{P}(X_{n+1} \leq k | J_n = i) = \sum_{j \in E} \sum_{l=0}^{k} q_{ij}(l).$$

and by $\bar{\boldsymbol{H}}(k) := (\bar{H}_i(k);\ i \in E)^\top$, $k \in \mathbb{N}$, the survival function in any state i.

Let us denote by μ_{ii} the mean recurrence time of state i for the SMC $\boldsymbol{Z}$, by $\boldsymbol{\pi} = (\pi_i; i \in E)$ and $\boldsymbol{\nu} = (\nu_i; i \in E)$, the stationary distribution of the SMC $\boldsymbol{Z}$ and the EMC $\boldsymbol{J}$, respectively. Let $\boldsymbol{m} := (m_i; i \in E)^\top$ be the vector with m_i to be the mean sojourn time of $\boldsymbol{Z}$ in state $i \in E$, i.e. $m_i := \sum_{n \in \mathbb{N}} [1 - H_i(n)]$, and $\bar{m}$ the mean sojourn time of $\boldsymbol{Z}$ defined as $\bar{m} := \sum_{k \in E} \nu_k m_k$.

Definition 4. *The matrix function $\boldsymbol{\psi}(k) := (\psi_{ij}(k); i, j \in E)$, $k \in \mathbb{N}$, is called Markov renewal function and it is defined by*

$$\begin{aligned} \psi_{ij}(k) &:= \mathbb{P}(\bigcup_{n=0}^{k} \{J_n = j, S_n = k\} | J_0 = i) \\ &:= \sum_{n=0}^{k} q_{ij}^{(n)}(k), \end{aligned}$$

where $\boldsymbol{q}^{(n)}(k) := (q_{ij}^{(n)}(k);\ i,j \in E)$, $n, k \in \mathbb{N}$, is the n-fold discrete-time convolution (see [1]), given as

$$q_{ij}^{(n)}(k) := \mathbb{P}(J_n = j, S_n = k | J_0 = i).$$

Let $\boldsymbol{I} := (I(k);\ k \in \mathbb{N})$, where $I(k) := (\mathbf{1}_{\{i=j\}}(k);\ i, j \in E)$ and

$$\mathbf{1}_{\{i=j\}}(k) := \begin{cases} 1, & \text{if } i = j,\ k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $*$, the convolution between two (matrix-valued) functions.

Definition 5. *The transition function $\boldsymbol{P}(k) := (P_{ij}(k);\ i, j \in E)$, $k \in \mathbb{N}$, of the SMC $\boldsymbol{Z}$ is defined by $P_{ij}(k) := \mathbb{P}(Z_k = j | Z_0 = i)$ and, in matrix form, is written as*

$$\boldsymbol{P}(k) = \boldsymbol{\psi} * (\boldsymbol{I} - \boldsymbol{H})(k).$$

The definition of the sequence of the backward recurrence times is now given.

Definition 6. *For all $k \in \mathbb{N}$, we define $\boldsymbol{U} := (U_k)_{k \in \mathbb{N}}$ as the sequence of the backward recurrence times for the SMC $\boldsymbol{Z}$ given by*

$$U_k := \begin{cases} k, & \text{if } k < S_1, \\ k - S_{N(k)}, & \text{if } k \geq S_1. \end{cases}$$

We note that, for all $k \in \mathbb{N}$, $U_k \leq k$. The stochastic process $(\boldsymbol{Z}, \boldsymbol{U}) := (Z_k, U_k)_{k \in \mathbb{N}}$ is a MC with values in $E \times \mathbb{N}$. In our case, where $S_0 = 0$, we get that $U_0 = 0$.

Definition 7. *The transition matrix $\boldsymbol{P}^B := (p_{i,u;j};\ i, j \in E, u \in \mathbb{N})$ of the MC $(\boldsymbol{Z}, \boldsymbol{U})$ is defined as*

$$\begin{aligned} & p_{i,u;j} \\ & := \begin{cases} \mathbb{P}(Z_{k+1} = j, U_{k+1} = 0 | Z_k = i, U_k = u), & j \neq i, \\ \mathbb{P}(Z_{k+1} = i, U_{k+1} = u + 1 | Z_k = i, U_k = u), & j = i. \end{cases} \end{aligned}$$

The value of U_{k+1} is fully determined by the value of Z_{k+1}. So, for all $(i, u) \in E \times \mathbb{N}$ and all $k \in \mathbb{N}$ such that $\mathbb{P}(Z_k = i, U_k = u) > 0$, the transition probabilities of the MC $(\boldsymbol{Z}, \boldsymbol{U})$ are written as

$$p_{i,u;j} = \begin{cases} \frac{q_{ij}(u+1)}{\bar{H}_i(u)}, & j \neq i, \\ \frac{\bar{H}_i(u+1)}{\bar{H}_i(u)}, & j = i. \end{cases}$$

We assume that the MRC $(\boldsymbol{J}, \boldsymbol{S})$ is irreducible and aperiodic, with finite mean sojourn time. The MC $(\boldsymbol{Z}, \boldsymbol{U})$ is therefore irreducible.

II. Empirical and Exact Maximum Likelihood Estimation

In this section, we present the nonparametric estimation of semi-Markov chains by two different aspects; the empirical and EML estimation. We observe a SMC in the interval $[0, M]$, where $M \in \mathbb{N}^*$ a fixed censoring time.

Definition 8. *Let us define the observation of the SMC $\boldsymbol{Z}$ censored at time $M \in \mathbb{N}^*$*

$$\begin{aligned} \mathcal{H}_M &:= \{Z_u;\ 0 \leq u \leq M\} \\ &:= \{J_0, X_1, J_1, \ldots, X_{N(M)}, J_{N(M)}, U_M\}, \end{aligned}$$

where $U_M := M - S_{N(M)}$.

Now, we suppose the realization of L, $L \geq 2$ independent observed trajectories observations censored at a time $M \in \mathbb{N}^*$, fixed for all, when the number of the observations tends to infinity. We collect the total information in the interval $[0, M]$ and we exclude the results without separating the different trajectories.

Let $N^l(M)$, $N_i^l(M)$ and $N_{ij}^l(M)$ be the l-th realizations of the counting processes $N(M)$, $N_i(M)$ and $N_{ij}(M)$, respectively, as defined in the previous section.

For all $i, j \in E$, $k \in \{0, 1, \ldots, M\}$, $M \in \mathbb{N}^*$, and $l = \{1, \ldots, L\}$, we define the following discrete-time counting process $N_{ij}^l(k, M)$ that gives the number of visits from state i to state j, up to time M, with sojourn time in state i equal to k, for the l-th trajectory, defined as

$$N_{ij}^l(k, M) := \sum_{n=1}^{N^l(M)} \mathbf{1}_{\{J_{n-1}^l = i, J_n^l = j, X_{n+1}^l = k\}}.$$

Definition 9. *Let L be the number of independent observed trajectories up to fixed time $M \in \mathbb{N}^*$. For any $i, j \in E$ and any $k \in \{0, 1, \ldots, M\}$, we define the following counting processes*

$$N_i(M, L) := \sum_{l=1}^{L} N_i^l(M),$$
$$N_{ij}(M, L) := \sum_{l=1}^{L} N_{ij}^l(M),$$
$$N_{ij}(k, M, L) := \sum_{l=1}^{L} N_{ij}^l(k, M).$$

For both empirical and EML estimation, the estimated initial law $\hat{\boldsymbol{\alpha}}(M, L) := (\hat{\alpha}_i(M, L);\ i \in E)$ and the estimated transition matrix $\hat{\boldsymbol{p}}(M, L) := (\hat{p}_{ij}(M, L);\ i, j \in E)$ of L trajectories, for any $M \in \mathbb{N}^*$, are given by

$$\hat{\alpha}_i(M, L) := \frac{N_i^{\alpha}(L)}{L} := \frac{1}{L} \sum_{l=1}^{L} \mathbf{1}_{\{Z_0^l = i\}},$$
$$\hat{p}_{ij}(M, L) = \frac{N_{ij}(M, L)}{N_i(M, L)}.$$

Definition 10. *Let L independent observations of a SMC $\boldsymbol{Z}$ up to a fixed censoring time $M \in \mathbb{N}^*$. For any $i, j \in E$ and $k \in \{0, 1, \ldots, M\}$, the empirical estimator $\tilde{\boldsymbol{q}}(k, M) := (\tilde{q}_{ij}(k, M);\ i, j \in E)$, for the DTSMK (2) is given as follows*

$$\hat{q}_{ij}(k, M, L) = \frac{N_{ij}(k, M, L)}{N_i(M, L)}. \tag{2}$$

The EML estimator is based on the time from the last jump of an observation up the time k.

Definition 11. *For all $i, j \in \mathbb{N}$, $k \in \{0, 1, \ldots, M\}$, $M \in \mathbb{N}^*$, and $l = \{1, \ldots, L\}$, we define the following discrete-time counting processes*

1) $N_{i,u}^{B,l}(M) := \sum_{n=1}^{M} \mathbf{1}_{\{Z_{n-1}^l = i, U_{n-1}^l = u\}}$ *: the number of visits in the state $(i, u) \in E \times \{0, 1, \ldots, M-1\}$, up to time $M \in \mathbb{N}^*$, neglecting the last visited state (J_M, U_M).*
2) $N_{i,u}^{B,l}(j, M) := \sum_{n=1}^{M} \mathbf{1}_{\{Z_{n-1}^l = i, Z_n^l = j, U_{n-1}^l = u\}}$ *: the number of visits of $\boldsymbol{Z}$ from state i to state j, with backward recurrence time u, up to time $M \in \mathbb{N}^*$.*

Definition 12. *For all $i, j \in E$ and $u \in \{0, 1, \ldots, M-1\}$, $M \in \mathbb{N}^*$, we define the counting processes*

$$N_{i,u}^B(M, L) := \sum_{l=1}^{L} N_{i,u}^{B,l}(M),$$
$$N_{i,u}^B(j, M, L) := \sum_{l=1}^{L} N_{i,u}^{B,l}(j, M).$$

Proposition 1 ([3])**.** *For any fixed time $M \in \mathbb{N}^*$, the EML estimator $\tilde{\boldsymbol{q}}(k, M, L) := (\tilde{q}_{ij}(k, M, L);\ i, j \in E)$, $i, j \in E$, $i \neq j$, $k \in \{1, \ldots, M\}$, for the DTSMK (2) in case of L trajectories is given as follows*

$$\tilde{q}_{ij}(k, M, L) = \begin{cases} \tilde{p}_{i,0}(j, M, L), & k = 1, \\ \tilde{p}_{i,k-1}(j, M, L) \prod_{u=0}^{k-2} \tilde{p}_{i,u}(i, M, L), & 2 \leq k \leq M, \end{cases} \tag{3}$$

where

$$\tilde{p}_{i,u}(j, M, L) = \frac{N_{i,u}^B(j, M, L)}{N_{i,u}^B(M, L)}, \quad u \in \{0, 1, \ldots, M-1\}.$$

III. Reliability model

A scientific field that semi-Markov models have been applied is, among others, reliability theory. We present the main measures of reliability, availability, failure rates and mean hitting times and how the theory of semi-Markov chains contribute to their study.

For a stochastic system with state space E, described by a SMC, we distinguish the up and down states of the system, denoted by U and D accordingly, i.e. $E = U \cup D$, with $U \cap D = \emptyset$ and $U, D \neq \emptyset$. For a finite state space $E = \{1, \ldots s\}$, we enumerate first the up states, $U = \{1, \ldots, r\}$, and next the down states, $D = \{r+1, \ldots, s\}$. For $m, n \in \mathbb{N}^*$, with $m > n$, let $\mathbf{1}_{m,n}$ denote the m-column vector whose the n first elements are 1 and the last $m-n$ ones are 0. For $m \in \mathbb{N}^*$, let $\mathbf{1}_m$ denote the m-column vector with all elements equal to one.

Now, let us denote by $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$, the vectors of the initial law on U and D respectively (in the same manner, we consider the partitions of the sojourn time cumulative distribution function $\boldsymbol{H}(k)$ and the mean sojourn times $\boldsymbol{m}$). Considering the transition kernel $\boldsymbol{p}$, the submatrices $\boldsymbol{p_{11}}$, $\boldsymbol{p_{12}}$, $\boldsymbol{p_{21}}$ and $\boldsymbol{p_{22}}$ are the restrictions of $\boldsymbol{p}$ on $E \times E$, $E \times U$, $U \times E$ and $U \times U$ respectively (similarly, we act for the DTSMK $\boldsymbol{q}(k)$, the Markov renewal function $\boldsymbol{\psi}(k)$ and the transition function $\boldsymbol{P}(k)$).

Also, let us denote by T_D the first passage time in subset D, called *the lifetime of the system*, and by T_U the first hitting time of subset U given that $\boldsymbol{\alpha_1} = \mathbf{0}$, called *the repair time*. That is,

$$T_D := \min\{n \in \mathbb{N} :\ Z_n \in D\},$$
$$T_U := \min\{n \in \mathbb{N} :\ Z_n \in U\},$$

with $\min \emptyset := \infty$.

A. Reliability

Definition 13. *The reliability R of a system at time $k \in \mathbb{N}$, starting to function at time $k = 0$, is defined as the probability that the system has functioned without failure in the interval $[0, k]$, i.e.*

$$R(k) := \mathbb{P}(Z_n \in U;\ \forall n \in [0, k]).$$

In the framework of a semi-Markov model, for all $k \in \mathbb{N}$, the reliability is defined by the following equation

$$R(k) = \boldsymbol{\alpha_1} \boldsymbol{P_{11}}(k) \mathbf{1}_r.$$

B. Availability

Definition 14. *The pointwise availability A of a system at time $k \in \mathbb{N}$ is the probability that the system is operational at time k (independently of the fact that the system has failed or not in $[0, k)$), i.e.*

$$A(k) := \mathbb{P}(Z_k \in U).$$

That means that the system functions at the time k, ignoring its history. For all $k \in \mathbb{N}$, the pointwise availability is given by

$$A(k) = \boldsymbol{\alpha} \boldsymbol{P}(k) \mathbf{1}_{s,r}.$$

Definition 15. *The steady-state availability A_∞ of a system is defined as the limit of the pointwise availability (when the limit exists), as the time tends to infinity, i.e.*

$$A_\infty := \lim_{k \to \infty} A(k).$$

For a semi-Markov system, the steady-state availability is given by

$$A_\infty = \frac{1}{\boldsymbol{\nu m}} \boldsymbol{m}^\top diag(\boldsymbol{\nu}) \mathbf{1}_{s,r}.$$

C. Failure rate functions

1) BMP-failure rate function:

Definition 16. *The BMP-failure rate function λ of a system at time $k \in \mathbb{N}$, starting working at time $k = 0$, is the conditional probability that the failure of the system occurs at time k, given that the system has worked until time $k-1$, i.e.*

$$\lambda(k) := \mathbb{P}(T_D = k | T_D \geq k).$$

The BMP-failure rate at time $k \geq 1$ is given by

$$\lambda(k) = \begin{cases} 1 - \frac{\boldsymbol{\alpha_1} \boldsymbol{P_{11}}(k) \mathbf{1}_r}{\boldsymbol{\alpha_1} \boldsymbol{P_{11}}(k-1) \mathbf{1}_r}, & R(k-1) \neq 0, \\ 0, & otherwise, \end{cases}$$
$$= \begin{cases} 1 - \frac{R(k)}{R(k-1)}, & R(k-1) \neq 0, \\ 0, & otherwise, \end{cases}$$

with $\lambda(0) = 1 - R(0)$.

2) RG-failure rate function: Due to some difficulties in applying the BMP-failure rate function on some discrete-time systems, an alternative failure rate function r has been proposed

$$r(k) = \begin{cases} -\ln \frac{R(k)}{R(k-1)}, & k \geq 1, \\ -\ln R(0), & otherwise, \end{cases}$$

called *the RG-failure rate function* at time $k \in \mathbb{N}$.

D. Mean Hitting Times

1) Mean time to failure:

Definition 17. *The mean time to failure (MTTF) is defined as the mean lifetime, i.e. the expectation of the hitting time to the down set D, $MTTF := \mathbb{E}[T_D]$.*

The mean time to failure in a semi-Markov model follows

$$MTTF = \boldsymbol{\alpha_1} (I - \boldsymbol{p_{11}})^{-1} \boldsymbol{m_1}.$$

2) Mean time to repair:

Definition 18. *The mean time to repair (MTTR) is defined as the mean of the repair duration, i.e. the expectation of the hitting time to the up set U, $MTTR := \mathbb{E}[T_U]$.*

The mean time to repair is given as

$$MTTR = \boldsymbol{\alpha_2} (I - \boldsymbol{p_{22}})^{-1} \boldsymbol{m_2}.$$

IV. Numerical Application

In this section, we apply the previous results to a three-state semi-Markov system described as follows. The state space of the system $E = \{1, 2, 3\}$ is partitioned into the up-state set $U = \{1, 2\}$ and the down-state set $D = \{3\}$. To define it completely, we need the initial law $\boldsymbol{\alpha} = (0.9 \quad 0.1 \quad 0)$ and the transition kernel $\boldsymbol{p}$ of the EMC $\boldsymbol{J}$, given by

$$\boldsymbol{p} = \begin{pmatrix} 0 & 1 & 0 \\ 0.6 & 0 & 0.4 \\ 1 & 0 & 0 \end{pmatrix}.$$

The conditional distributions of the sojourn times are

$$\boldsymbol{f}(k) = \begin{pmatrix} 0 & f_{12}(k) & 0 \\ f_{21}(k) & 0 & f_{23}(k) \\ f_{31}(k) & 0 & 0 \end{pmatrix},$$

where the conditional distributions for the sojourn times $f_{12}(k)$ and $f_{31}(k)$ are geometric distributions with parameters $p = 0.15$ and $p = 0.20$ respectively, and the distributions $f_{21}(k)$ and $f_{23}(k)$ follow the discrete Weibull distribution with parameters $(q, b) = (0.9, 1.2)$ for the transition $2 \to 1$ and $(q, b) = (0.8, 1.2)$ for the transition $2 \to 3$. The realization of a trajectory of fixed length $M \in \mathbb{N}^*$ of the SMC $\boldsymbol{Z}$ with state space E, transition matrix $\boldsymbol{p}$ and initial law $\boldsymbol{\alpha}$ is simulated through a Monte Carlo method.

We observe 20000 independent trajectories of the SMC $\boldsymbol{Z}$ up to the censoring time $M = 100$. The empirical and EML estimations for all the measures are based on the estimators (2) and (3) of the DTSMK. We present now the plots for the

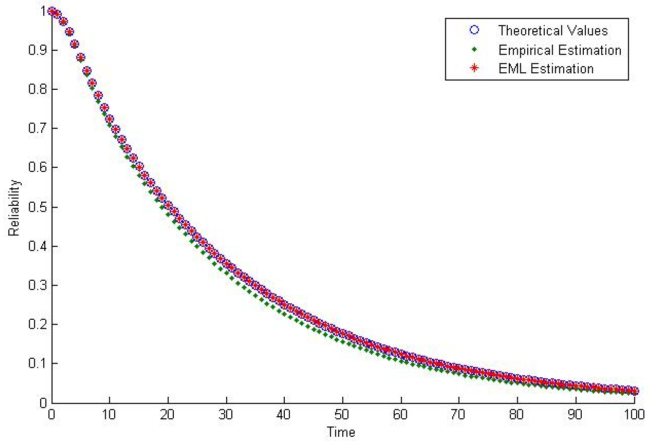

Fig. 1. Reliability plot.

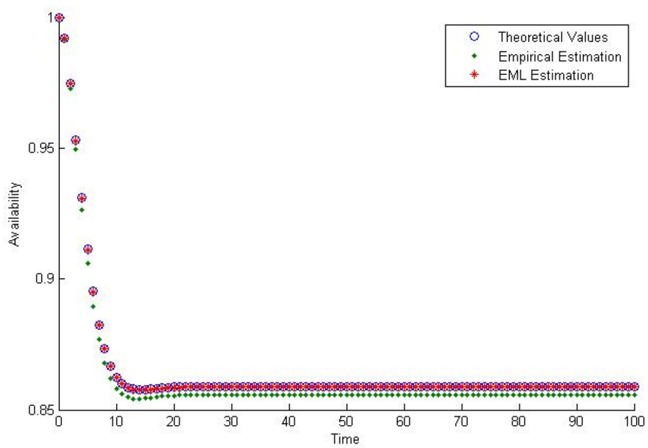

Fig. 2. Availability plot.

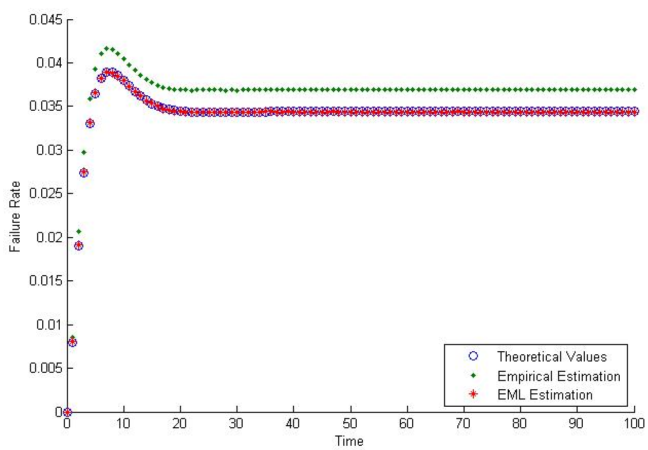

Fig. 3. BMP-failure rate plot.

reliability, availability and failure rates, presented in Section III and their estimations.

In Table I, the empirical and EML estimation are given for

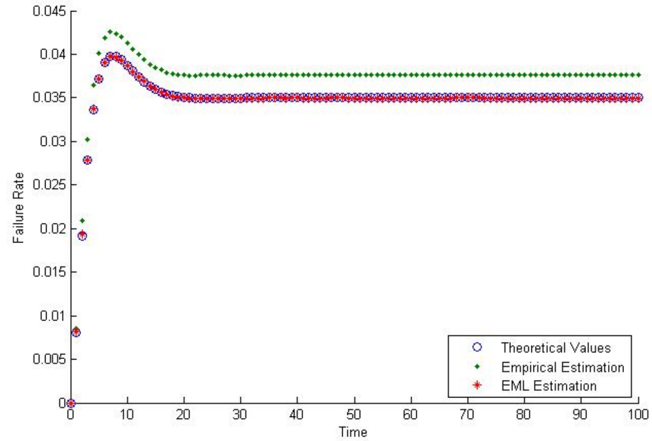

Fig. 4. RG-failure rate plot.

	True Value	Empirical Estimation	EML Estimation
A_∞	0.8589	0.8556	0.8566
$MTTF$	29.7578	27.8198	29.2270
$MTTR$	5.0000	4.8017	5.0067

TABLE I
ESTIMATION OF STEADY-STATE AVAILABILITY AND MEAN HITTING TIMES.

the steady-state availability and mean hitting times

V. CONCLUSION

In the case of a single observed trajectory, the backward recurrence time at time M, is neglected from empirical estimators. In contrast, in the case of multiple observations, significant difference between the two estimations is observed. Even in the case of a large number of trajectories, when the time M is small, the estimated values of all the reliability measures differ, making the empirical estimation to seem less accurate than the EML estimation, which appear to be almost identical to the theoretical values.

The time interval until the system reaches the steady state is of great significance for real data applications and provide important information for the evolution of a system. For this interval, the differences of the two estimators in Figures 1 and 2 seem even wider.

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