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Delta, gamma and bucket hedging of interest rate derivatives

ROBERT A. JARROW

Johnson Graduate School of Management, Cornell University, Ithaca, NY 14853, USA

STUART M. TURNBULL*

School of Business, Queen's University, Kingston, Ontario, Canada K7L 3N6

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The paper describes a framework for delta and gamma hedging an interest rate portfolio using a multifactor form of the Heath *et al.* (1992) model. A formal description of bucket hedging is given along with a discussion of some of the issues surrounding the choice of bucket lengths. Given that a small number of factors can describe the evolution of the term structure, the bucket deltas are defined in terms of these factors. The hedging of corporate bonds is also addressed.

Keywords: delta hedging, gamma hedging, bucket hedging, interest rate derivatives

1. Introduction

The theory of delta and gamma hedging is well developed for equity and foreign currency derivatives. At the applied level, the theory for delta and gamma hedging interest rate instruments is far less developed, for the reason that it is necessary to model in an arbitrage-free way movements in the whole-term structure of interest rates. The analysis increases in complexity when it is assumed that there is more than one factor describing the evolution of the term structure of interest rates, and when there are multiple term structures. The need for a consistent approach is described in Wakeman and Tuffli (1991).

In an attempt to improve interest rate risk management techniques, practitioners have divided the term structure of interest rates into intervals called *buckets.* A delta is calculated for each bucket, so that if that part of the term structure moves by a certain amount the change in the dollar value of the cash flows within the bucket can be calculated. A limitation of this approach is that there is no explicit recognition of the fact that buckets are, in general, correlated. In a typical application there are many buckets, each with an associated delta, and the number of buckets is usually greater than the number of independent hedging instruments, so that it would appear that complete delta hedging becomes impossible. A least squares methodology is then advocated to determine how to hedge. Such a conclusion contradicts the assumption of complete markets in

* Current holder of the Bank of Montreal Chair in Banking and Finance, and Research Associate of the Institute for Policy Analysis, University of Toronto.

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the interest rate model used to determine the bucket deltas. A further complication occurs when there are multiple term structures. A normal swap portfolio contains Treasury instruments, LIBOR instruments and corporate bonds with different credit rates. To hedge such a portfolio in a consistent way requires a model which can price and hedge Treasury and corporate instruments given the initial term structures.

The purpose of this paper is to describe a framework for delta and gamma hedging interest rate portfolios. The basic model for hedging default-free Treasury instruments is presented in Section 2. A special case of the Heath *et al.* (1992) model is used. In Section 3 it is assumed that there is only one factor, and delta and gamma are defined in terms of this one factor. An example is given for hedging an interest cap. In Section 4 it is assumed that there are two factors, the extension to an arbitrary number being immediate. Definitions for the deltas and gammas are given in terms of the factors. A formal description of bucket hedging is given in Section 5, along with a discussion of some of the issues surrounding the choice of bucket lengths. Given that a small number of factors can describe the evolution of the term structure, the bucket deltas are defined in terms of these factors. This avoids the need to introduce *ad hoc* hedging criteria such as least squares. The hedging of corporate bonds is addressed in Section 6, using the Jarrow and Turnbull (1993) model. A summary is given in Section 7.

2. Basic model

Following Heath *et al.* (1992), we model the evolution of the forward rates. Let $B(t, T)$ denote the price at time t of a zero-coupon pure discount bond which pays one dollar for certain at time $T(\geq t)$. The instantaneous forward rate at time t for date $T, f(t, T)$, is defined by

$$f(t, T) = -\frac{\partial}{\partial T}[\ln B(t, T)]$$

which implies

$$B(t, T) = \exp\left[-\int_t^T f(t, v)\mathrm{d}v\right]$$

Let $\{W_1(t), W_2(t)\}$ be two independent Brownian motions initialized at zero. We assume that for fixed but arbitrary $T \in [0, \Gamma], f(t, T)$ is given by

$$\mathrm{d}f(t, T) = \alpha(t, T)\mathrm{d}t + \sigma(t, T)^*\mathrm{d}W(t), \tag{1}$$

where $\mathrm{d}W(t)^* = \{\mathrm{d}W_1(t), \mathrm{d}W_2(t)\}$ and $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$ are locally bounded adapted processes in $\mathbb{R}$ and $\mathbb{R}^2$ respectively. The fixed initial forward rate curve is described by $[f(0, T), T \geq 0]$. Equivalently we can write (1) in the form

$$f(t, T) = f(0, T) + \int_0^t \alpha(u, T)\mathrm{d}u + \int_0^t \sigma(u, T)^*\mathrm{d}W(u)$$

for $T \in [0, \Gamma]$ and for all $0 \leq t \leq T$.

The instantaneous spot interest rate is defined by $r(t) = f(t, t)$, and the spot rate process satisfies

$$r(t) = f(0, t) + \int_0^t \alpha(u, t)\mathrm{d}u + \int_0^t \sigma(u, t)^*\mathrm{d}W(u)$$

Given an initial investment of one dollar, the amount generated by continuously reinvesting in the instantaneous spot rate is defined by

$$A(t) = \exp\left[\int_0^t r(u)\mathrm{d}u\right]$$

$t \geq 0$. $A(t)$ is referred to as the *money market account*. Define the relative bond price by

$$Z(t, T) = B(t, T)/A(t)$$

Under the existing probability P, relative prices are a semi-martingale. From the work of Heath *et al.* (1992) it is shown that there exists an alternative distribution, Q, under which normalized prices follow a martingale. This alternative distribution does not introduce any new or eliminate any existing zero-probability events and is unique. The existence of the distribution implies that there are no arbitrage possibilities and the uniqueness of the distribution implies that markets are complete. Therefore,

$$Z(t, T) = \boldsymbol{E}[Z(T, T)|\mathscr{F}_t]$$

where the expectation is under the measure Q, and $\mathscr{F}_t$ denotes the information set at t. It follows that under the probability measure Q

$$f(t,T) - f(0,T) = \sum_{i=1}^{2} \int_0^t \sigma_i(u,T) \int_u^T \sigma_i(u,v)\mathrm{d}v\mathrm{d}u + \int_0^t \sigma_i(u,T)\mathrm{d}\tilde{W}_i(u) \tag{2}$$

and

$$r(t) - f(0,t) = \sum_{i=1}^{2} \int_0^t \sigma_i(u,t) \int_u^t \sigma_i(u,v)\mathrm{d}v\mathrm{d}u + \int_0^t \sigma_i(u,t)\mathrm{d}\tilde{W}_i(u) \tag{3}$$

where $\tilde{W}_t(t)$ is a Brownian motion under Q. The value at date t of a zero-coupon bond which matures at date T is given by

$$B(t,T) = \frac{B(0,T)}{B(0,t)}\exp\left[-\sum_{i=1}^{2}\left(\int_t^T \mathrm{d}s \int_0^t \sigma_i(u,s) \int_u^s \sigma_i(u,v)\mathrm{d}v\mathrm{d}u + \int_t^T \mathrm{d}s \int_0^t \sigma_i(u,s)\mathrm{d}\tilde{W}_i(u)\right)\right] \tag{4}$$

and

$$\mathrm{d}B(t,T) = B(t,T)\left[r(t)\mathrm{d}t - \sum_{i=1}^{2}\int_t^T \sigma_i(t,u)\mathrm{d}u\mathrm{d}\tilde{W}_i(t)\right] \tag{5}$$

To facilitate the derivation of closed-form solutions, two simplifying assumptions are made. First, it is assumed that in equation (1), $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$ are locally bounded deterministic functions taking values in $\mathbb{R}$. This assumption implies that under the measures P and Q the forward rates and spot rates are normally distributed. This is a strong assumption for it implies that (nominal) interest rates can be negative. Second, it is assumed that

$$\sigma_i(t,T) = \sigma_i \exp\left[-\lambda_i(T-t)\right] \tag{6}$$

where $\sigma_i > 0$, $\lambda_i \geq 0$, $i = 1, 2$. Under the Q measure for $\lambda_i > 0$, $i = 1, 2$,

$$r(t) - f(0,t) = \sum_{i=1}^{2}\left\{\left(\frac{\sigma_i^2}{2\lambda_i^2}\right)[1 - \exp(-\lambda_i t)]^2 + \int_0^t \sigma_i \exp[-\lambda_i(t-u)]\mathrm{d}\tilde{W}_i(u)\right\} \tag{7}$$

Let $F(t;n)$ denote the futures price at time $t(\leq n)$ of a futures contract which matures at time n. The futures contract is written on a Treasury bill which matures at time $T(\geq n)$. At the maturity of the futures contract

$$F(n;n) = B(n,T)$$

At time $t(\leq n)$

$$F(t;n) = E[B(n,T)|\mathscr{F}_t]$$

Using Equation (4) then it follows that

$$F(t;n) = F(0;n)\exp\left[-\sum_{i=1}^{2}\left(\left(\frac{1}{2}\right)\int_0^t \mathrm{d}u\left[\int_n^T \sigma_i(u,s)\mathrm{d}s\right]^2 + \int_0^t \mathrm{d}\tilde{W}_i(u)\int_n^T \sigma_i(u,s)\mathrm{d}s\right)\right] \tag{8}$$

3. Single factor

It is assumed that there is only one Brownian motion. It is shown in Appendix A that, given the assumption of normality and Equation (6), the value of a Treasury bill at time t can be written in the form

$$B(t,T) = \exp[-X(t;t,T)R(t) + a(t,T)], \tag{9}$$

where

$$R(t) \equiv r(t) - f(0,t);\ X(t;t,T) \equiv \{1 - \exp[-\lambda(T-t)]\}/\lambda$$

and

$$a(t,T) \equiv \ln\left[\frac{B(0,T)}{B(0,t)}\right] - \left(\frac{\sigma^2}{4\lambda}\right)X(t;t,T)^2[1 - \exp(-2\lambda t)]$$

Musiela *et al.* (1993) interpret the term $X(t;t,T)$ as a duration measure. While Equation (9) is similar in form to the expression derived by Vasicek (1977), it differs in an important way. Equation (9) is derived taking the initial term structure as exogeneous, while for the Vasicek model the initial term structure is endogeneous. The Vasicek model specifies a process for the instantaneous spot interest rate, which is assumed to be normally distributed and assumes that the market price of risk is constant. Given values for the drift and variance for the spot interest rate, and the market price of risk, the model can be used to generate the term structure of interest rates.

Equation (9) can be viewed as a function of t and $R(t)$. Therefore, if t and R change by the discrete and finite amounts Δt and ΔR respectively, then using Taylor's series the change in the value of the treasury bill is

$$\Delta B(t,T) = \frac{\partial B}{\partial t}\Delta t + \frac{\partial B}{\partial R}\Delta R + \frac{1}{2}\frac{\partial^2 B}{\partial R^2}(\Delta R)^2 + \cdots$$

where

$$\frac{\partial B}{\partial R} = -B(t,T)X(t;t,T) \leq 0 \tag{10}$$

and

$$\frac{\partial^2 B}{\partial R^2} = B(t,T)X(t;t,T)^2 \geq 0 \tag{11}$$

Equation (11) is a measure of the Treasury bill's convexity.

The value of a Treasury bond at time t is given by

$$B_c(t) = \sum_{j=1}^{N} c_j B(t,T_j) \tag{12}$$

where N represents the number of remaining payments which occur at $\{T_1, \ldots, T_N\}$; and c_j denotes the magnitude of the payment which may be coupon and/or principal at time T_j. Equation (12) can be viewed as a function of t and $R(t)$. Therefore, using Taylor's series we can write

$$\Delta B_c(t) = \frac{\partial B_c}{\partial t}\Delta t + \frac{\partial B_c}{\partial R}\Delta R + \frac{1}{2}\frac{\partial^2 B_c}{\partial R^2}(\Delta R)^2 + \cdots$$

where

$$\frac{\partial B_c}{\partial R} = -\sum_{j=1}^{N} c_j B(t,T_j)X(t;t,T_j) \leq 0 \tag{13}$$

and

$$\frac{\partial^2 B_c}{\partial R^2} = \sum_{j=1}^{N} c_j B(t,T_j)X(t;t,T_j)^2 \geq 0. \tag{14}$$

For $\lambda = 0$, implying no variance reduction and $X(t;t,T_j) = T_j - t$, then (13) is proportional to the Redington definition of duration, a result which is to be expected given the work of Ingersoll *et al.* (1978).

The value of a Treasury bill futures contract can be written in the form

$$F(t;n) = F(0;n)\exp[-X(t;n,T)R(t) + a_F(t,T)] \tag{15}$$

where

$$X(t;n,T) \equiv \{\exp[-\lambda(n-t)] - \exp[-\lambda(T-t)]\}/\lambda$$

and

$$a_F(t,T) \equiv \left(\frac{\sigma^2}{2\lambda^2}\right)X(t;n,T)[1-\exp(-\lambda t)]^2 - \left(\frac{\sigma^2}{4\lambda}\right)X(t;n,T)^2[1-\exp(-2\lambda t)]$$

Musiela *et al.* (1993) interpret $X(t; n, T)$ as a duration measure for the future contract. Given (15), then using Taylor's series

$$\Delta F(t,n) = \frac{\partial F}{\partial t}\Delta t + \frac{\partial F}{\partial R}\Delta R + \frac{1}{2}\frac{\partial^2 F}{\partial R^2}(\Delta R)^2 + \cdots$$

where

$$\frac{\partial F}{\partial R} = -F(t,T)X(t;n,T) \leq 0 \tag{16}$$

and

$$\frac{\partial^2 F}{\partial R^2} = F(t,n)X(t;n,T)^2 \geq 0 \tag{17}$$

Consider a European Treasury bill call option which matures at date m. The option is written on a Treasury bill which matures at date $T_m(\geq m)$. The strike price of the option is K. It is shown in Heath *et al.* (1992) that the value of the call option is given by

$$c(t;m) = B(t,T_m)N(d_1) - KB(t,m)N(d_2) \tag{18}$$

where $d_1 \equiv \{\ln[B(t,T_m)/KB(t,m)] + x_c^2\sigma_c^2/2\}/x_c\sigma_c$; $d_2 = d - x_c\sigma_c$, $x_c \equiv [1 - \exp -\lambda(T_m - m)]/\lambda$, $\sigma_c^2 \equiv (\sigma^2/2\lambda)\{1 - \exp[-2\lambda(m-t)]\}$ and $N(\cdot)$ is the cumulative normal distribution function. Because $B(t,T_m)$ and $B(t,m)$ are functions of $R(t)$ and t, we can trust $c(t;m)$ as being a function of t and $R(t)$, so that using Taylor's series gives

$$\Delta c = \frac{\partial c}{\partial t}\Delta t + \frac{\partial c}{\partial R}\Delta R + \frac{1}{2}\frac{\partial^2 c}{\partial R^2}(\Delta R)^2 + \cdots$$

where

$$\frac{\partial c}{\partial R} = -X(t;t,T_m)B(t,T_m)N(d_1) + X(t;t,m)KB(t,m)N(d_2) \leq 0 \tag{19}$$

and

$$\begin{aligned}\frac{\partial^2 c}{\partial R^2} &= X(t;t,T_m)^2B(t,T_m)N(d_1) - KX(t;t,m)^2B(t,m)N(d_2)\\ &\quad + [-X(t;t,T_m) + X(t;t,m)]^2B(t,T_m)g(d_1)/(x_c\sigma_c)\\ &\geq 0\end{aligned} \tag{20}$$

where $g(\cdot)$ is the normal density function.

For a put option,

$$p(t;m) = KB(t,m)N(-d_2) - B(t,T_m)N(-d_1) \tag{21}$$

$$\frac{\partial p}{\partial R} = -X(t;t,m)KB(t,m)N(-d_2) + \dot{X}(t;t,T_m)B(t,T_m)N(-d_1) \tag{22}$$

and

$$\frac{\partial^2 p}{\partial R^2} = X(t;t,m)^2 KB(t,m)N(-d_2) - X(t;t,T_m)^2 B(t,T_m)N(-d_1) \\ + [-X(t;t,T_m) + X(t;t,m)]^2 B(t,T_m)g(-d_1)/x_c\sigma_c \quad (23)$$

Delta hedging

Consider a financial institution which has written an interest rate cap starting at date m, and ending at date T_m. The payoff to the cap is defined by

$$\text{cap}(m) \equiv \max\left\{\frac{R_c - k}{1 + R_c(\Gamma/360)}, 0\right\}(\Gamma/360) \text{ principal}$$

where $\Gamma \equiv T_m - m$ defines the length of the cap, k is the cap rate, and R_c is the Γ period interest rate at date m. The payoff to the cap can be viewed as a payoff to a call option written on the Γ period interest rate. By definition of R_c,

$$B(m, T_m) \equiv 1/[1 + R_c(\Gamma/360)]$$

Therefore we can write the payoff in terms of Treasury bill prices:

$$\text{cap}(m) = \Theta \max\{K - B(m, T_m), 0\}$$

where $K \equiv 1/[1 + k(\Gamma/360)]$; and $\Theta = \text{principal } [1 + k(\Gamma/360)]$. The payoff to the cap can be viewed as a payoff to a put option written on the Γ period Treasury bill. Thus, using (21)

$$\text{cap}(t) = \Theta[KB(t,m)N(-d_2) - B(t,T_m)N(-d_1)] \quad (24)$$

The assumption of market completeness implies that with continuous delta hedging it is possible to hedge this cap perfectly. If hedging is done on a discrete-time basis, this is no longer the case. There are many ways to hedge this cap. We want to consider two possible strategies.

Hedging strategy 1

The first strategy consists of hedging the liability with a Treasury bill which matures at time T_m and the money market account. The value of the hedged portfolio is:

$$V(t) = b(t) + hB(t, T_m) - \text{cap}(t) = 0, \qquad t \leq m$$

where $b(t)$ is the dollar amount invested in the money market account; and h is the number of Treasury bills. The change in the value of the portfolio can be written

$$\Delta V(t) = \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial R}\Delta R + \frac{1}{2}\frac{\partial^2 V}{\partial R^2}(\Delta R)^2 + \cdots \quad (25)$$

To be delta neutral we require that the portfolio be insensitive to changes in the factor R; that is, we require the delta of the portfolio to be zero, which implies

$$0 = h\ \text{delta}_B(t, T_m) - \text{delta}_c$$

where delta_c is the delta of the cap and $\text{delta}_B(t, T_m)$ is the delta of the Treasury bill which matures at date T_m. From (24) and (22) the delta of the cap is defined by

$$\text{delta}_c = \Theta \frac{\partial p}{\partial R}(t, m) \tag{27}$$

and from (10) the delta of the Treasury bill is defined by

$$\text{delta}_B(t, T_m) \equiv \frac{\partial B}{\partial R}(t, T_m) \tag{28}$$

The hedge ratio is given by

$$h(t) = \Theta \left[\frac{X(t; t, m)B(t, m)}{X(t; t, T_m)B(t, T_m)} KN(-d_2) - N(-d_1) \right]$$

and

$$b(t) = \Theta B(t, m)KN(-d_2)[1 - X(t; t, m)/X(t; t, T_m)]$$

Hedging strategy 2

In the second strategy the cap is hedged with Treasury bills which mature at dates m and T_m, and the value of the hedged portfolio is

$$V(t) \equiv h_1 B(t, m) + h_2 B(t, T_m) - \text{cap}(t) = 0$$

where h_1 is the number of Treasury bills which mature at date m; and h_2 is the number of Treasury bills which mature at date T_m. To be delta neutral implies

$$0 = h_1.\text{delta}_B(t, m) + h_2.\text{delta}_B(t, T_m) - \text{delta}_c(t)$$

Solving for h_1 and h_2 gives

$$h_1 = \Theta KN(-d_2) \qquad h_2 = -\Theta N(-d_1) \tag{29}$$

In a continuous-time model either hedging strategy will perfectly hedge the option. In a discrete-time context, most professional traders would not be indifferent to the two strategies, given that they have different gammas.

Gamma hedging

By construction, the delta of the hedged portfolio is zero. Changes in the spot interest rate will affect the portfolio's delta. From (25) the change in the value of the delta-neutral portfolio is

$$\Delta V = \frac{\partial V}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 V}{\partial R^2}(\Delta R)^2 + \cdots$$

Table 1. Delta and gamma for hedge portfolios.

Strategy 1

ΔR	Yield on Treasury bill[a]	Value of portfolio	Delta of portfolio	Gamma of portfolio
−0.0200	6.21	−5.09	502.63	−23132.56
−0.0175	6.39	−3.90	443.66	−24017.94
−0.0150	6.57	−2.87	382.67	−24751.12
−0.0125	6.75	−1.99	320.05	−25310.09
−0.0100	6.93	−1.27	256.28	−25676.18
−0.0075	7.12	−0.71	191.85	−25834.79
−0.0050	7.30	−0.31	127.29	−25776.01
−0.0025	7.48	−0.07	63.15	−25495.08
−0.0000	7.66	0.00	0.00	−24992.62
0.0025	7.84	−0.07	−61.62	−24274.68
0.0050	8.02	−0.30	−121.20	−23352.53
0.0075	8.21	−0.68	−178.23	−22242.27
0.0100	8.39	−1.19	−232.27	−20964.25
0.0125	8.57	−1.83	−282.93	−19542.33
0.0150	8.75	−2.60	−329.88	−18003.04
0.0175	8.92	−3.48	−372.87	−16374.60
0.0200	9.12	−4.46	−411.71	−14685.98

Strategy 2

ΔR	Yield on Treasury bill[a]	Value of portfolio	Delta of portfolio	Gamma of portfolio
−0.0200	6.21	−5.70	565.17	−26455.97
−0.0175	6.39	−4.37	497.96	−27291.01
−0.0150	6.57	−3.21	428.84	−27974.42
−0.0125	6.75	−2.23	358.23	−28484.21
−0.0100	6.93	−1.42	286.58	−28801.69
−0.0075	7.12	−0.79	214.40	−28912.25
−0.0050	7.30	−0.35	142.20	−28805.99
−0.0025	7.48	−0.08	70.55	−28478.13
−0.0000	7.66	0.00	0.00	−27929.30
0.0025	7.84	−0.08	−68.91	−27165.52
0.0050	8.02	−0.34	−135.65	−26198.07
0.0075	8.21	−0.76	−199.74	−25043.04
0.0100	8.39	−1.33	−260.73	−23720.78
0.0125	8.57	−2.06	−318.23	−22255.15
0.0150	8.75	−2.92	−371.91	−20672.65
0.0175	8.92	−3.91	−421.52	−19001.51
0.0200	9.12	−5.03	−466.86	−17270.70

[a] This is the continuously compounded yield on a $4\frac{1}{2}$ year Treasury bill.

The gamma of the portfolio is defined by

$$\text{gamma} \equiv \frac{\partial^2 V}{\partial R^2}$$

The idea behind the definition is to capture 'large' changes in R over a small interval. Ideally, one would normally want a positive gamma, so that ΔV is positive.

For the first strategy, the gamma of the portfolio using (11) and (23) is

$$\text{gamma}_{\text{V}} = h\frac{\partial^2 B}{\partial R^2}(t, T_m) - \Theta\frac{\partial^2 p}{\partial R^2}$$

the second term being the gamma of the put option. Substituting for h gives

$$\text{gamma}_{\text{V}} = \Theta\left\{KB(t,m)N(-d_2)X(t;t,m)[X(t;t,T_m) - X(t;t,m)] - \frac{[X(t;t,T_m) - X(t;t,m)]^2 B(t,T_m)g(-d_1)}{x_c\sigma_c}\right\}$$

For the second strategy, the gamma of the portfolio is

$$\begin{aligned}\text{gamma}_{\text{V}} &= h_1\frac{\partial^2 B}{\partial R^2}(t,m) + h_2\frac{\partial^2 B}{\partial R^2}(t, T_m) - \Theta\frac{\partial^2 p}{\partial R^2} \\ &= -\Theta\frac{[X(t;t,T_m) - X(t;t,m)]^2 B(t,T_m)g(-d_1)}{x_c\sigma_c} < 0\end{aligned}$$

Given that Θ is positive and $X(t;t,T_m) - X(t;t,m) > 0$, then the gamma for the first strategy is greater than the gamma for the second strategy, which is negative. In Table 1 the value of the hedged portfolio, its delta and gamma are shown for perturbations in the initial term structure. For $\Delta R = 0$ there is no perturbation, and the value of the hedged portfolio and its delta are zero. For ΔR not equal to zero, the value of the hedged portfolio and the value of its delta are different from zero.

By introducing an additional asset in the hedge portfolio, the portfolio can be designed to be delta and gamma neutral. Suppose we hedge the cap using Treasury bills which mature at dates m and T_m and the money market account. To be delta-neutral requires

$$0 = h_1 \text{ delta}_{\text{B}}(t,m) + h_2 \text{ delta}_{\text{B}}(t, T_m) - \text{delta}_{\text{c}}(t)$$

and to be gamma neutral

$$0 = h_1 \text{ gamma}_{\text{B}}(t,m) + h_2 \text{ gamma}_{\text{B}}(t, T_m) - \text{gamma}_{\text{c}}(t)$$

The investment in the money market account is given by

$$0 = b(t) + h_1 B(t,m) + h_2 B(t, T_m) - \text{cap}(t)$$

Solving for $b(t)$, h_1 and h_2 gives the delta/gamma-neutral portfolio.

4. Multiple factors

With one factor the form of perturbations to the term structure of interest rates is quite limited. We extend the analysis to two factors: the extension to an arbitrary number is straightforward.

From (2)

$$df(t,T) = \sum_{i=1}^{2}(\sigma_i^2/\lambda_i)\exp[-\lambda_i(T-t)]\{1-\exp[-\lambda_i(T-t)]\}dt + \sum_{i=1}^{2}\sigma_i \exp[-\lambda_i(T-t)]d\tilde{W}_i(t) \tag{30}$$

Without loss of generality, it is assumed that $0 \leq \lambda_1 < \lambda_2$. It is shown in Appendix B that the value of a Treasury bill at time t can be written in the form

$$B(t,T) = \exp[-X_1(t;t,T)R_1(t) - X_2(t;t,T)R_2(t) + a(t,T)] \tag{31}$$

where

$$X_i(t;t,T) \equiv \frac{\{1-\exp[-\lambda_i(T-t)]\}}{\lambda_i}; \qquad R(t) \equiv r(t) - f(0,t) = R_1(t) + R_2(t)$$

$$R_i(t) \equiv (\sigma_i^2/2\lambda_i^2)[1-\exp(-\lambda_i t)]^2 + \sigma_i\int_0^t \exp[-\lambda_i(t-u)]d\tilde{W}_i(u), \qquad i = 1,2,$$

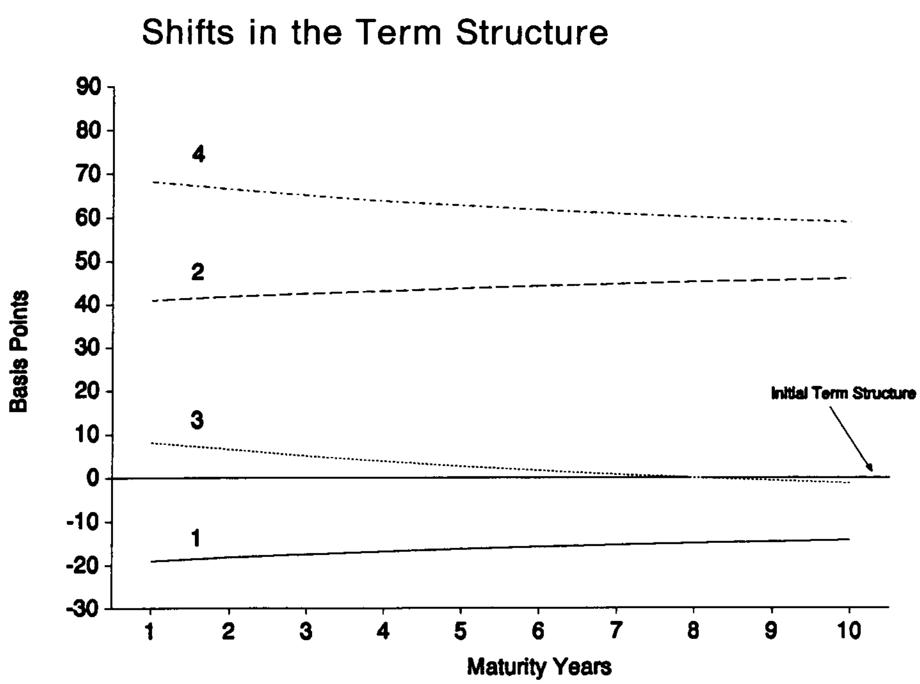

Fig. 1. $\sigma_1 = 1.66\%$, $\sigma_2 = 1.0\%$; $\lambda_1 = 0.2$, $\lambda_2 = 0$; $1 = \Delta R_1 = -0.001$ $\Delta R_2 = -0.001$; $2 = \Delta R_1 = -0.001$ $\Delta R_2 = 0.005$, $3 = \Delta R_1 = 0.002$ $\Delta R_2 = -0.001$, $4 = \Delta R_1 = 0.002$ $\Delta R_2 = 0.005$.

and

$$a(t,T) \equiv -\int_t^T f(0,u)\mathrm{d}u - \sum_{i=1}^{2}\left(\frac{\sigma_i^2}{4\lambda_i}\right)X_i(t;t,T)^2[1-\exp(-2\lambda_i t)]$$

A two-factor model implies that it is possible to generate a wider range of perturbations in the term structure than those allowed by a one-factor model. Given an initial upward sloping term structure, perturbations in the two factors can have quite different impacts upon the shape of the term structure, as shown in Figure 1 and Table 2.

The value of a Treasury bill futures contract can be written in the form

$$F(t;n) = F(0,n)\exp[-X_1(t;n,T)R_1(t) - X_2(t;n,T)R_2(t) + a_F(t,T)] \tag{32}$$

where

$$X_i(t;n,T) \equiv \{\exp[-\lambda_i(n-t)] - \exp[-\lambda_i(T-t)]\}/\lambda_i, \qquad i=1,2,$$

and

$$a_F(t,T) \equiv \sum_{i=1}^{2}\left(\frac{\sigma_i^2}{2\lambda_i^2}\right)X_i(t;n,T)[1-\exp(-\lambda_i t)]^2 - \left(\frac{\sigma_i^2}{4\lambda_i}\right)X_i(t;n,T)^2[1-\exp(-2\lambda_i t)]$$

The generalization of the Expression (18) for an European call option becomes

$$c(t) = B(t,T_m)N(d_1) - KB(t,m)N(d_2)$$

where

$$\sigma_c d_1 \equiv \left\{\ln\left[\frac{B(t,T_m)}{KN(t,m)}\right] + \frac{\sigma_c^2}{2}\right\};\ d_2 \equiv d_1 - \sigma_c$$

$$\sigma_c^2 \equiv \sum_{i=1}^{2} X_{ci}(m,T_m)^2(\sigma_i^2/2\lambda_i)[1-\exp -2\lambda_i(m-t)]$$

Table 2. Shifts in the term structure.

Maturity (years)	*Initial yield* (%)	$\Delta R_1 = -0.001$ $\Delta R_2 = -0.001$	$\Delta R_1 = -0.001$ $\Delta R_2 = 0.005$	$\Delta R_1 = 0.002$ $\Delta R_2 = -0.001$	$\Delta R_1 = 0.002$ $\Delta R_2 = 0.005$
		Basis points			
1	7.55	−19.06	40.93	8.12	68.12
2	7.59	−18.24	41.75	6.48	66.48
3	7.62	−17.52	42.48	5.03	65.03
4	7.65	−16.88	43.11	3.76	63.76
5	7.68	−16.32	43.67	2.64	62.64
6	7.70	−15.82	44.17	1.64	61.64
7	7.72	−15.38	44.61	0.76	60.76
8	7.74	−14.99	45.01	−0.02	59.97
9	7.76	−14.64	45.36	−0.72	59.27
10	7.77	−14.32	45.67	−1.35	58.64

$\sigma_1 = 1.66\%$, $\lambda_1 = 0.2$; $\sigma_2 = 1.0\%$, $\lambda_2 = 0$.

and

$$X_{ci}(m, T_m) \equiv \frac{[1 - \exp -\lambda_i(T_m - m)]}{\lambda_i}, \qquad i = 1,2$$

The generalization of the expression for a put option is immediate, so details are omitted.

Equation (31) can be viewed as a function of t, $R_1(t)$ and $R_2(t)$, so that

$$\Delta B(t,T) = \frac{\partial B}{\partial t}\Delta t + \frac{\partial B}{\partial R_1}\Delta R_1 + \frac{\partial B}{\partial R_2}\Delta R_2 + \frac{1}{2}\left[\frac{\partial^2 B}{\partial R_1^2}(\Delta R_1)^2 + 2\frac{\partial^2 B}{\partial R_1 \partial R_2}(\Delta R_1)(\Delta R_2) + \frac{\partial^2 B}{\partial R_2^2}(\Delta R_2)^2\right] + \cdots$$

where

$$\frac{\partial B}{\partial R_i} = -X_i(t; t, T)B(t,T) \leq 0$$

and

$$\frac{\partial^2 B}{\partial R_i \partial R_j} = X_i(t; t, T)X_j(t; t, T)B(t,T) \geq 0, \qquad i, j = 1,2.$$

Expressions for deltas and gammas for the other instruments are given in Table 3.

Given that there are two factors, to be delta-neutral implies that for the hedged portfolio, the delta with respect to each factor is zero. For gamma there are now three terms, reflecting changes in $(\Delta R_1)^2$, $(\Delta R_2)^2$ *and* $(\Delta R_1)(\Delta R_2)$.

Now consider again the problem of hedging the cap, introduced in Section 3. Suppose we hedge with Treasury bills which mature at dates m and T_m and the money market account:

$$V(t) = b(t) + h_1 B(t,m) + h_2 B(t,T_m) - \text{cap}(t) = 0$$

Table 3. Delta and gamma specifications

Treasury bill	
delta	$-X_i(t; t, T)B(t,T)$
gamma	$X_i(t; t, T)X_j(t; t, T)B(t,T)$
where	$X_i(t; t, T) \equiv \{1 - \exp[-\lambda_i(T - t)]\}/\lambda_i$
Treasury bill futures	
delta	$-X_i(t; n, T)F(t; n)$
gamma	$X_i(t; n, T)X_j(t; n, T)F(t; n)$
where	$X_i(t; n, T) \equiv \{\exp[-\lambda_i(n - t)] - \exp[-\lambda_i(T - t)]\}/\lambda_i$
European Treasury bill options	
Call option	
delta	$-X_i(t; t, T_m)B(t, T_m)N(d_1) + X_i(t; t, m)KB(t, m)N(d_2)$
gamma	$X_i(t; t, T_m)X_j(t; t, T_m)B(t, T_m)N(d_1) - X_i(t; t, m)X_j(t; t, m)KB(t, m)N(d_2)$ $+[X_i(t; t, m) - X_i(t; t, T_m)][X_j(t; t, m) - X_j(t; t, T_m)]B(t, T_m)g(d_1)/\sigma_c$
Put option	
delta	$-X_i(t; t, m)KB(t, m)N(-d_2) + X_i(t; t, T_m)B(t, T_m)N(-d_1)$
gamma	$X_i(t; t, m)X_j(t; t, m)KB(t, m)N(-d_2) - X_i(t; t, T_m)X_j(t; t, T_m)B(t, T_m)N(-d_1)$ $+[X_i(t; t, m) - X_i(t; t, T_m)][X_j(t; t, m) - X_j(t; t, T_m)]B(t, T_m)g(-d_1)/\sigma_c$

$i, j = 1,2.$

To be delta neutral with respect to the ith ($i = 1, 2$) factor we require

$$0 = -h_1 X_i(t; t, m) B(t, m) - h_2 X_i(t; t, T_m) B(t, T_m) - \Theta[-X_i(t; t, m) K B(t, m) N(-d_2) + X_i(t; t, T_m) B(t, T_m) N(-d_1)]$$

where

$$b(t) = 0, \qquad h_1 = \Theta K N(-d_2), \qquad h_2 = -\Theta N(-d_1) \tag{33}$$

To implement the theory of delta and gamma hedging as described in this paper, it is necessary to be able to measure the terms $\{\sigma_i, \lambda_i\}$, $i = 1, 2$. There are two approaches. The first uses historical data. Musiela *et al.* (1992) describes an algorithm for the computation of these terms. The second approach uses market prices and the theoretical pricing model to infer the parameters. Practitioners typically fit the parameters to the prices of different instruments, such as caps and swaptions.

5. Bucket hedging

Suppose we divide the term structure into n buckets $[t, t_1]$, $[t_1, t_2]$, $\ldots [t_{n-1}, t_n]$. Over the jth bucket the average forward rate is defined by

$$Y_j(t) \equiv \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} f(t, u) \mathrm{d}u, \qquad j = 1, \ldots, n, \qquad t_0 \equiv t \tag{34}$$

It will also be useful to define

$$Z_j(t) \equiv (t_j - t_{j-1}) Y_j(t), \qquad j = 1, \ldots n \tag{35}$$

From (2),

$$Z_j(t) = \int_{t_{j-1}}^{t_j} f(0, s) \mathrm{d}s + \sum_{i=1}^{2} \int_{t_{j-1}}^{t_j} \mathrm{d}s \int_0^t \sigma_i(u, s) \int_u^s \sigma_i(u, v) \mathrm{d}v \mathrm{d}u + \int_{t_{j-1}}^{t_j} \mathrm{d}s \int_0^t \sigma_i(u, s) \mathrm{d}\tilde{W}_i(u) \tag{36}$$

implying

$$\mathrm{d}Z_j(t) = \sum_{i=1}^{2} \mathrm{d}t \int_{t_{j-1}}^{t_j} \sigma_i(t, s) \mathrm{d}s \int_t^s \sigma_i(t, v) \mathrm{d}v + \mathrm{d}\tilde{W}_i(t) \int_{t_{j-1}}^{t_j} \sigma_i(t, s) \mathrm{d}s \tag{37}$$

By definition, the value of a zero-coupon bond which matures at date t_g, $g = 1, \ldots, n$,

$$B(t, t_g) = \exp\left[-\int_t^{t_g} f(t, u) \mathrm{d}u\right] = \exp\left[-\sum_{j=1}^{g} Z_j(t)\right]$$

so that

$$\Delta B(t, t_g) = \frac{\partial B}{\partial t}\Delta t - B(t, t_g)\sum_{j=1}^{g}\Delta Z_j(t) + \frac{1}{2}B(t, t_g)\sum_{j=1}^{g}\sum_{l=1}^{g}\Delta Z_j(t)\Delta Z_l(t) + \cdots \tag{38}$$

Defining bucket length

In using the bucket approach to hedging, the cash flows in a portfolio must be assigned to a particular bucket. Consider a portfolio with n_p cash flows. Cash flow c_j occurs at date T_j, $j = 1, \ldots, n_p$. The value of this portfolio is

$$V_p(t) \equiv \sum_{j=1}^{n_p} c_j B(t, T_j) \tag{39}$$

Without loss of generality, it is assumed that $T_{n_p} \leq t_n$. It is assumed that cash flows $1, \ldots, n_1$ occur in the interval $(0, t_1]$, cash flows $n_1 + 1, \ldots, n_2$ occur in $(t_1, t_2]$, etc.

The value of this portfolio in terms of the buckets is

$$\bar{V}_p(t) \equiv \sum_{j=1}^{n} \bar{c}_j B(t, t_j) \tag{40}$$

where $\bar{c}_j$ is defined such that

$$\bar{c}_j B(t, t_j) \equiv \sum_{l=n_{j-1}+1}^{n_j} c_l B(t, T_l), \qquad j = 1, \ldots, n, \quad n_0 \equiv 0 \tag{41}$$

which implies

$$\bar{V}_p(t) = V_p(t) \tag{42}$$

While Equation (41) ensures that the value of the portfolio using buckets equals the true value of the portfolio as given by Equation (39), it introduces errors in calculating the portfolio's delta and gamma over each bucket.

To illustrate the problem, assume that a single factor describes the evolution of the term structure, then using (10) and (11) the delta of the portfolio over the jth bracket is

$$\text{delta}_{pj} \equiv -\sum_{l=n_{j-1}+1}^{n_j} c_l X(t; t, T_l) B(t, T_l)$$

and gamma

$$\text{gamma}_{pj} \equiv \sum_{l=n_{j-1}+1}^{n_j} c_j X(t; t, T_l)^2 B(t, T_l)$$

Table 4.

Maturity (years)	Term structure A: T-bill price	Yield[a] (%)	Forward rate[b] (%)	Term structure B: T-bill price	Yield[a] (%)	Forward rate[b] (%)	Term structure C: T-bill price	Yield[a] (%)	Forward rate[b] (%)
1	95.3256	4.79	4.79	92.7359	7.54	7.54	90.9293	9.51	9.51
2	89.8386	5.36	5.93	85.9500	7.57	7.60	83.2760	9.15	8.79
3	84.0849	5.78	6.62	79.6344	7.59	7.63	76.6004	8.89	8.36
4	78.3715	6.09	7.04	73.7686	7.61	7.65	70.6473	8.69	8.09
5	72.8615	6.33	7.29	68.3271	7.62	7.66	65.2620	8.54	7.93
6	67.6350	6.52	7.44	63.2826	7.63	7.67	60.3463	8.42	7.83
7	62.7250	6.66	7.54	58.6081	7.63	7.67	55.8340	8.33	7.77
8	58.1386	6.78	7.59	54.2776	7.64	7.68	51.6777	8.25	7.74
9	53.8691	6.87	7.63	50.2663	7.64	7.68	47.8413	8.19	7.71
10	49.9028	6.95	7.65	46.5510	7.65	7.68	44.2956	8.14	7.70

Spot rate: A, 4%; B, 7.5%; C, 10%.
[a] Defined as a continuously compounded rate.
[b] One year forward rate, expressed as a continuously compounded yield.

From (40) the delta of the jth bucket is

$$\text{delta}_{B_j} \equiv -\bar{c}_j X(t; t, t_j) B(t, t_j)$$

and gamma

$$\text{gamma}_{B_j} \equiv \bar{c}_j X(t; t, t_j)^2 B(t, t_j)$$

The relative error for delta is defined by

$$\text{delta error}_j \equiv (\text{delta}_{B_j} - \text{delta}_{P_j})/\text{delta}_{P_j}$$

and for gamma

$$\text{gamma error}_j \equiv (\text{gamma}_{B_j} - \text{gamma}_{P_j})/\text{gamma}_{P_j}$$

In Table 4 three term structures are described. Term structure A is upward sloping, term structure B is approximately flat, and Term structure C is inverted. For simplicity, it is assumed that the cash flows in the portfolio are all equal and occur every month. Table 5 shows the types of errors which occur for buckets of different lengths. In part A the interval length is 3 months and in part B 6 months. Three observations can be made. First, the shorter the bucket interval, the lower are the delta and gamma errors. Second, the magnitude of the errors is minimal at the long end of the term structure. Both of these observations are to be expected given the functional dependence of delta and gamma upon maturity. Third, the results are relatively insensitive to the shape of the term structure. These results imply that at the short end of the term structure short bucket intervals should be used, while relatively long intervals can be used at the long end.

Application

The change in the value of the portfolio, given an appropriate choice of bucket lengths, is given by

Table 5. Delta and gamma errors using buckets

Part A: Every 3 months

	Term structure A		Term structure B		Term structure C	
Interval (months)	Delta error (%)	Gamma error (%)	Delta error (%)	Gamma error (%)	Delta error (%)	Gamma error (%)
0–3	48.12	89.81	48.24	90.10	48.34	90.30
3–6	17.97	36.21	18.00	36.29	18.03	36.34
6–9	10.51	21.22	10.52	21.26	10.54	21.29
9–12	7.15	14.42	7.16	14.44	7.16	14.45
12–15	5.25	10.57	5.25	10.58	5.26	10.59
15–18	4.03	8.11	4.04	8.12	4.04	8.12
18–21	3.19	6.42	3.20	6.43	3.20	6.43
21–24	2.59	5.19	2.59	5.19	2.59	5.20
24–27	2.13	4.27	2.13	4.27	2.13	4.27
27–30	1.77	3.55	1.77	3.55	1.77	3.55
57–60	0.39	0.78	0.39	0.78	0.39	0.78
117–120	0.03	0.06	0.03	0.06	0.03	0.06

Part B: Every 6 months

	Term structure A		Term structure B		Term structure C	
Interval (months)	Delta error (%)	Gamma error (%)	Delta error (%)	Gamma error (%)	Delta error (%)	Gamma error (%)
0–6	66.17	127.39	66.50	128.13	66.74	128.66
6–12	21.13	43.60	21.19	43.74	21.24	43.84
12–18	11.28	23.07	11.31	23.12	11.32	23.16
18–24	7.06	14.34	7.07	14.36	7.08	14.38
24–30	4.77	9.64	4.77	9.65	4.78	9.66
30–36	3.37	6.78	3.37	6.79	3.37	6.80
36–42	2.44	4.92	2.44	4.92	2.45	4.92
42–48	1.81	3.63	1.81	3.63	1.81	3.63
48–54	1.35	2.72	1.35	2.72	1.36	2.72
54–60	1.02	2.06	1.02	2.06	1.03	2.06
114–120	0.08	0.15	0.08	0.15	0.08	0.15

The error is defined to be (bucket value – true value)/true value.

using (38) and (40) and can be written in the form

$$\Delta \bar{V}_p(t) = \sum_{j=1}^{n} \bar{c}_j \frac{\partial B}{\partial t}(t, t_j)\Delta t - \sum_{j=1}^{n} \Delta Z_j \sum_{l=j}^{n} \bar{c}_l B(t, t_l)$$
$$+ \frac{1}{2}\sum_{j=1}^{n} \bar{c}_j B(t, t_j) \sum_{l=1}^{j} \sum_{k=1}^{j} \Delta Z_l \Delta Z_k + \cdots \qquad (43)$$

The delta with respect to the jth bucket is defined as

$$\text{delta}_{Z_j} \equiv -\sum_{l=j}^{n} \bar{c}_l B(t, t_l) \tag{44}$$

For very small changes in the bucket factors and for $\Delta t = 0$, then

$$\Delta \bar{V}_p(t) = \sum_{j=1}^{n} \text{delta}_{Z_j} \Delta Z_j \tag{45}$$

Equation (45) is often used by practitioners to answer the question, 'What is the impact upon the value of my portfolio if the jth bucket factor increases by 100 basis points?'

To illustrate the use of bucket hedging, consider the problem described in the last section of hedging an interest rate cap. For simplicity it is assumed that the cap matures at t_2 and is written on a Treasury bill that matures at t_3. From equation (24) and using (35), the value of the cap can be written in the form

$$\text{cap}(t) = \Theta \exp(-Z_1 - Z_2)[KN(-d_2) - \exp(-Z_3)N(-d_1)]$$

where $d_1 \equiv (-Z_3 - \ln(K) + x_c^2\sigma_c^2/2)/(x_c\sigma_c)$ and $d_2 \equiv d_1 - x_c\sigma_c$. The deltas for the cap with respect to each bucket factor are given by

$$\frac{\partial\, \text{cap}(t)}{\partial Z_i} = -\text{cap}(t), \qquad i = 1, 2$$

and

$$\frac{\partial\, \text{cap}(t)}{\partial Z_3} = \Theta B(t, t_3) N(-d_1)$$

Given that there are three bucket factors, (Z_1, Z_2, Z_3), the hedge portfolio consists of an investment in the money market account and h_i Treasury bills, $B(t, t_i)$ which mature at t_i, $i = 1, 2, 3$:

$$V(t) = b(t) + \sum_{i=1}^{3} h_i B(t, t_i) - \text{cap}(t) = 0$$

To be delta neutral with respect to Z_1 requires

$$0 = -\sum_{i=1}^{3} h_i B(t, t_i) + \text{cap}(t)$$

which implies $b(t) = 0$. To be delta neutral with respect to Z_2 requires

$$0 = -\sum_{i=2}^{3} h_i (B, t_i) + \text{cap}(t)$$

which implies $h_1 = 0$. To be delta neutral with respect to Z_3 requires

$$0 = -h_3 B(t, t_3) - \Theta B(t, t_3) N(-d_1)$$

Hence

$$h_3 = -\Theta N(-d_1), \qquad h_2 = \Theta K N(-d_2) \tag{46}$$

Note that this hedge is identical to that described by (33).

Dimensionality

Typical application of bucket analysis is to examine the change in the value of the portfolio caused by changing each factor by a fixed amount and then aggregating to determine the total change in the value of the portfolio. Such an approach ignores the fact that bucket factors are correlated. Changing each factor by the same amount and then aggregating is equivalent to determining the change in the value of the portfolio from a parallel shift in the term structure. These limitations can be circumvented by recognizing that a small number of factors can be used to describe the evolution of the term structure.

Consider the two factor model described by (30). For a small interval, the change in the jth bucket, using (37)

$$\Delta Z_j(t) \simeq \alpha_j \Delta t + \sum_{i=1}^{2} X_i(t; t_{j-1}, t_j) \Delta \tilde{W}_i(t)$$

where α_j is a drift term. Specifying values for $\Delta \tilde{W}_1(t)$ and $\Delta \tilde{W}_2(t)$ implies a value for $\Delta Z_j(t)$, $j = 1, \ldots, n$, so that correlation between the bucket factors is explicitly recognized. Provided λ_1 or λ_2 is positive, then shifts in the term structure will not be parallel.

Substituting the above equation into (38) gives

$$\begin{aligned}
\Delta B(t, t_g) = {} & \alpha_B \Delta t - B(t, t_g) \sum_{i=1}^{2} \Delta \tilde{W}_i(t) \sum_{j=1}^{g} X_i(t; t_{j-1}, t_j) \\
& + [B(t, t_g)/2] \left\{ \sum_{i=1}^{2} \Delta \tilde{W}_i(t)^2 \sum_{l=1}^{g} \sum_{k=1}^{g} X_i(t; t_{l-1}, t_l) X_i(t, t_{k-1}, t_k) \right. \\
& + \Delta \tilde{W}_1(t) \Delta \tilde{W}_2(t) \sum_{l=1}^{g} \sum_{k=1}^{g} [X_1(t; t_{l-1}, t_l) X_2(t, t_{k-1}, t_k) \\
& \left. + X_1(t; t_{k-1}, t_k) X_2(t; t_{l-1}, t_l)] \right\}
\end{aligned}$$

where α_B is a drift term. From (43) and using (47), the change in the value of the bucket portfolio can, after some manipulation, be written in the form

$$\begin{aligned}
\Delta \bar{V}_p(t) = {} & \alpha_{\bar{V}_p} \Delta t - \sum_{i=1}^{2} \Delta \tilde{W}_i(t) \sum_{j=1}^{n} X_i(t; t_{j-1}, t_j) \sum_{l=j}^{n} \bar{c}_l B(t, t_l) \\
& + \frac{1}{2} \sum_{i=1}^{2} \Delta \tilde{W}_i(t)^2 \sum_{g=1}^{n} \bar{c}_g B(t, t_g) \sum_{l=1}^{g} \sum_{k=1}^{g} X_i(t; t_{l-1}, t_l) X_i(t; t_{k-1}, t_k)
\end{aligned}$$

$$+\frac{1}{2}\Delta\tilde{W}_1(t)\Delta\tilde{W}_2(t)\sum_{g=1}^{n}\bar{c}_g B(t,t_g)\sum_{l=1}^{g}\sum_{k=1}^{g}[X_1(t;t_{l-1},t_l)X_2(t;t_{k-1},t_k)$$

$$+X_1(t;t_{k-1},t_k)X_2(t;t_{l-1},t_l)] \tag{48}$$

where $\alpha_{\bar{V}_p}$ is a drift term. The delta with respect to the ith factor ($i = 1, 2$) for the jth bucket can be defined

$$\text{delta}_{ij} \equiv -X_i(t;t_{j-1},t_j)\sum_{l=j}^{n}\bar{c}_l B(t,t_l), \qquad i=1,2; \quad j=1,\ldots,n \tag{49}$$

For very small changes in the two factors and for $\Delta t = 0$, then

$$\Delta\bar{V}_p(t) \simeq \Delta\tilde{W}_1(t)\sum_{j=1}^{n}\text{delta}_{1j} + \Delta\tilde{W}_2(t)\sum_{j=1}^{n}\text{delta}_{2j} \tag{50}$$

Equation (50) differs from (45) in two important ways. Given a specification of $\Delta\tilde{W}_1(t)$ and $\Delta\tilde{W}_2(t)$, the change in the value of the jth bucket is

$$\Delta\tilde{W}_1(t).\text{delta}_{1j} + \Delta\tilde{W}_2(t).\text{delta}_{2j}, \qquad j=1,\ldots,n$$

In (45) the arbitrary specification of ΔZ_j, $j = 1, \ldots, n$ fails to recognize the correlation between factors. Second, in (50), if λ_i is positive then perturbations in $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$ do not imply parallel shifts in the term structure.

Three gamma terms are defined

$$\text{gamma}_i \equiv \sum_{g=1}^{n}\bar{c}_g B(t,t_g)\sum_{l=1}^{g}\sum_{k=1}^{g}X_i(t;t_{l-1},t_l)X_i(t;t_{k-1},t_k), \qquad i=1,2 \tag{51a}$$

and

$$\text{gamma}_3 \equiv \sum_{g=1}^{n}\bar{c}_g B(t,t_g)\sum_{l=1}^{g}\sum_{k=1}^{g}[X_1(t;t_{l-1},t_l)X_2(t;t_{k-1},t_k)$$

$$+X_1(t;t_{k-1},t_k)X_2(t;t_{l-1},t_l)] \tag{51b}$$

For delta and gamma hedging for the portfolio equations (50) and (51) can be used.

Summary

In general, buckets are correlated. We have shown how to model this correlation in a multi-factor model of the term structure. This provides us with an important risk management tool. We have also discussed some of the issues that influence the choice of bucket length.

6. Multiple term structures

A typical interest rate portfolio contains derivatives, such as swaps, written on LIBOR interest rates. Eurodollar futures contracts are some of the most actively traded contracts in the United States. LIBOR markets are fairly liquid for contracts with maturities up to about 4 or 5 years. After that, it is necessary to use Treasury instruments for hedging. Institutional details about these markets are given in Saunders (1994).

To use Treasury and Eurodollar instruments for hedging, it is necessary to have a model in which both types of instruments, one which is default free and the other subject to default risk, can be priced and hedged in a manner consistent with the absence of arbitrage. Jarrow and Turnbull (1993) describe such a model, taking the initial forward Treasury and LIBOR term structures as given. A brief description of the model will be given and then it will be incorporated into the analysis.

Model

The model views the value of risky debt as depending upon the underlying term structure and the probability of default. If default occurs, the bond holder receives some payment, which is less than the promised amount. For simplicity we assume that there are two risk classes to which firms belong. The first class is default free and for the second class there is a positive probability of default, which is the same for all firms within the class. Consider a univariate point process $\{N(t)\}$ independent of the Brownian motions $\{W_1(t), W_2(t)\}$. The point process $N(t)$ is constructed as follows. Let $\tau^* : \Omega \to \mathbb{R}$ be a stopping time representing the first time of bankruptcy for a credit risky firm. We assume that τ^* is exponentially disturbed over $[0, \infty)$ with parameter λ. Next $N(t)$ is defined by

$$N(t) \equiv 1(t \geq \tau^*) \equiv \begin{cases} 1, & \text{if } t \geq \tau^* \\ 0, & \text{otherwise} \end{cases}$$

Let $v(t, T)$ denote the value of a zero-coupon credit risky bond which matures at date T. If default does not occur, the bond pays one dollar. If default occurs, the bond holder receives a constant $\exp(-\delta)$, $\delta > 0$. Jarrow and Turnbull (1993) show that given default has not occurred and using the independence assumption

$$v(t, T) = B(t, T)(\exp[-\lambda\mu(T - t)] + \exp(-\delta)\{1 - \exp[-\lambda\mu(T - t)]\}) \tag{52}$$

where μ is a constant which arises because equation (52) is derived using the equivalent probability measure. The term in parentheses on the right-hand side is the expected payoff under the equivalent martingale measure. The term $\lambda\mu$ is inferred from the initial term structures given that we know $v(t, T)$ and $B(t, T)$. The value of the recovery term $\exp(-\delta)$ can be estimated using historical data or inferred using other traded instruments belonging to the same credit class. A more detailed description is given in Jarrow and Turnbull (1993).

Equation (52) can be written in the form

$$\begin{aligned} v(t, T) = B(t, T)(\{&[1 - \exp(-\delta)] \exp[-\lambda\mu(T - t)] + \exp(-\delta)\} \\ &- N(t)[1 - \exp(-\delta)] \exp[-\lambda\mu(T - t)]) \end{aligned}$$

where $N(t)$ equals zero unless there is a jump. The above equation can be viewed as a function of t, $R_1(t)$, $R_2(t)$ and $N(t)$. To simplify notation, let

$$A_1(t,T) \equiv [1 - \exp(-\delta)] \exp[-\lambda\mu(T-t)]$$

and

$$A_2 \equiv \exp(-\delta)$$

The change in the value of the credit risky bond when a jump occurs is defined to be

$$\begin{aligned}\frac{\Delta v}{\Delta N} &\equiv v(\text{jump}) - v(\text{no jump}) \\ &= B(t,T)A_2 - B(t,T)(A_1 + A_2) \\ &= -B(t,T)A_1\end{aligned}$$

Ignoring an interaction term, see Appendix C, the change in the credit-risky bond can be written

$$\begin{aligned}\Delta v = {} & \frac{\partial v}{\partial t}\Delta t + \left[\Delta R_1 \frac{\partial}{\partial R_1} + \Delta R_2 \frac{\partial}{\partial R_2}\right] B(t,T)(A_1 + A_2) \\ & + \frac{1}{2}\left[\Delta R_1^2 \frac{\partial^2}{\partial R_1^2} + 2(\Delta R_1)(\Delta R_2)\frac{\partial^2}{\partial R_1 \partial R_2} + (\Delta R_2)^2 \frac{\partial^2}{\partial R_2^2}\right] B(t,T)(A_1 + A_2) \\ & + \frac{\Delta v}{\Delta N} \mathrm{d}N\end{aligned}$$

The credit-risky bond is subject to interest rate risk and credit risk, and these risks are approximately additive. Treasury or credit-risky instruments can be used to delta and gamma hedge the interest rate risk. To hedge the credit risk, a credit-risky instrument must be used.

Application

The cash flows of the portfolio are divided into two groups: those that are default free and those subject to the risk of default, so that (39) becomes

$$V_p(t) \equiv \sum_{j=1}^{n_p} c_j B(t,T_j) + \sum_{j=1}^{n'_p} c_{D_j} v(t,T_j) \tag{54}$$

where c_{D_j} is the promised cash flow at T_j subject to the risk of default; and n'_p is the total number of default risky cash flows. The generalizations of the bucket cash flows (40) and (41) are

$$\bar{V}_p(t) \equiv \sum_{j=1}^{n} \bar{c}_j B(t,t_j) + \sum_{j=1}^{n} \bar{c}_{D_j} v(t,t_j) \tag{55}$$

where $\bar{c}_{N_j}$ is defined by

$$\bar{c}_{D_j} v(t,t_j) \equiv \sum_{t=n'_{j-1}+1}^{n'_j} c_{D_j} v(t,T_l) \tag{56}$$

Given (41) and (56), then

$$\bar{V}_p(t) = V_p(t) \tag{57}$$

The remaining analysis will concentrate only on the cash flows subject to default risk. Define

$$\bar{V}_{DP}(t) \equiv \sum_{j=1}^{n} \bar{c}_{D_j} v(t, t_j)$$

which is the current value of all cash flows subject to default risk. The change in the value of this portfolio can be written using equations (48) and (52)

$$\begin{aligned}
\Delta \bar{V}_{DP}(t) = {} & \alpha_{V_{DP}} \Delta t - \sum_{i=1}^{2} \Delta \tilde{W}_i(t) \sum_{j=1}^{n} X_i(t; t_{j-1}, t_j) \sum_{l=j}^{n} \bar{c}_{Dl} v(t-, t_l) \\
& + \frac{1}{2} \sum_{i=1}^{2} \Delta \tilde{W}_i(t)^2 \sum_{g=1}^{n} \bar{c}_{Dg} v(t-, t_g) \sum_{l=1}^{g} \sum_{k=1}^{g} X_i(t; t_{l-1}, t_l) X_i(t; t_{k-1}, t_k) \\
& + \frac{1}{2} \Delta \tilde{W}_1(t) \Delta \tilde{W}_2(t) \sum_{g=1}^{n} \bar{c}_{Dg} v(t-, t_g) \sum_{l=1}^{g} \sum_{k=1}^{g} [X_1(t; t_{l-1}, t_l) X_2(t; t_{k-1}, t_k) \\
& + X_1(t; t_{k-1}, t_k) X_2(t; t_{l-1}, t_l)] \\
& - \sum_{j=1}^{n} \bar{c}_{Dj} B(t, t_j) A_1(t, t_j) \mathrm{d}N(t)
\end{aligned} \tag{58}$$

where $\alpha_{V_{DP}}$ is a drift term. The delta with respect to the ith interest rate factor ($i = 1, 2$) for the jth bucket can be defined

$$\text{delta}_{ij}(W_i) \equiv -X_i(t; t_{j-1}, t_j) \sum_{l=j}^{n} \bar{c}_{Dl} v(t-, t_l), \qquad j = 1, \ldots, n \tag{59}$$

Three gamma terms are defined

$$\text{gamma}_i \equiv \sum_{g=1}^{n} \bar{c}_{Dg} v(t-, t_g) \sum_{l=1}^{g} \sum_{k=1}^{g} X_i(t; t_{l-1}, t_l) X_i(t; t_{k-1}, t_k), \qquad i = 1, 2 \tag{60a}$$

and

$$\begin{aligned}
\text{gamma}_3 \equiv {} & \sum_{g=1}^{n} \bar{c}_{Dg} v(t-, t_g) \sum_{l=1}^{g} \sum_{k=1}^{g} [X_1(t; t_{l-1}, t_l) X_2(t; t_{k-1}, t_k) \\
& + X_1(t; t_{k-1}, t_k) X_2(t; t_{l-1}, t_l)]
\end{aligned} \tag{60b}$$

It should be observed that for delta hedging the interest rate risk, the delta coefficients are defined using the credit-risky term structure – compare equation (59) to equation (49). A similar comment applies to the gamma coefficients – compare equation (60) to equation (50).

7. Summary

This paper provides a framework for delta and gamma hedging Treasury and corporate instruments. A multifactorial model of the term structure is described, and deltas and gamma are defined in terms of these factors. The analysis removes some of the limitations of classical bucket hedging by incorporating it into a general framework that recognizes that buckets are correlated.

In future work this analysis will be extended to incorporate commodity and foreign currency portfolios, implying that one has multiple term structures of forward rates. For risk management it is necessary to model the factor that buckets are correlated along a given term structure curve and across term structures.

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Appendix A: Single factor

For a single factor the expression for the spot interest rate can be written in the form, using (7)

$$r(t) - f(0,t) = (\sigma^2/2\lambda^2)[1 - \exp(-\lambda t)]^2 + \sigma\int_0^t \exp[-\lambda(t-u)]\mathrm{d}\tilde{W}(u) \tag{A1}$$

Define

$$X(t;a,b) \equiv \int_a^b \exp[-\lambda(u-t)\mathrm{d}u$$

Substituting (A1) into (4) and simplifying gives

$$B(t,T) = \exp[-X(t;t,T)[r(t) - f(0,t)] + a(t,T)] \tag{A2}$$

where

$$X(t;t,T) \equiv [1 - \exp -\lambda(T-t)]/\lambda$$

and

$$a(t,T) \equiv -\int_t^T f(0,u)\mathrm{d}u - \left(\frac{\sigma^2}{4\lambda}\right) X(t;t,T)^2[1 - \exp(-2\lambda t)]$$

The expression for the Treasury bill futures contract can be written in the form

$$F(t;n) = F(0,n)\exp\left\{X(t;n,T)\sigma\int_0^t \exp[-\lambda(t-u)]\mathrm{d}\tilde{W}(u) - \left(\frac{\sigma^2}{4\lambda}\right) X(t;n,T)^2[1-\exp(-2\lambda t)]\right\} \tag{A3}$$

where $X(t;n,T) \equiv \{\exp[-\lambda(n-t)] - \exp[-\lambda(T-t)]\}/\lambda$. Substituting (A1) into (A3) gives

$$F(t;n) = F(0;n)\exp[-X(t;n,T)[r(t) - f(0,t)] + a_F(t,T)] \tag{A4}$$

where

$$a_F(t,T) = (\sigma^2/2\lambda^2)X(t;n,T)[1-\exp(-\lambda t)]^2 - (\sigma^2/4\lambda)X(t;n,T)^2[1-\exp(-2\lambda t)]$$

For a Treasury bill option

$$\frac{\partial c}{\partial R} = -X(t;t,T_m)B(t,T_m)N(d_1) + X(t;t,m)KB(t,m)N(d_2) + B(t,T_m)g(d_1)\frac{\partial d_1}{\partial R} - KB(t,m)g(d_2)\frac{\partial d_2}{\partial R}$$

where $g(\cdot)$ is the normal distribution density function. Recognizing that $\partial d_1/\partial R = \partial d_2/\partial R$ and $B(t,T_m)g(d_1) = KB(t,m)g(d_2)$, then

$$\frac{\partial c}{\partial R} = -X(t;t,T_m)B(t,T_m)N(d_1) + KX(t;t,m)B(t,m)N(d_2)$$

Differentiating again gives

$$\frac{\partial^2 c}{\partial R^2} = X(t;t,T_m)^2B(t,T_m)N(d_1) - KX(t;t,m)^2B(t,m)N(d_2) - X(t;t,T_m)B(t,T_m)g(d_1)\frac{\partial d_1}{\partial R} + KX(t;t,m)B(t,m)g(d_2)\frac{\partial d_2}{\partial R}$$

and

$$\frac{\partial d_1}{\partial R} = \frac{[-X(t;t,T_m) + X(t;t,m)]}{x_c\sigma_c} = \frac{\partial d_2}{\partial R}$$

Therefore

$$\frac{\partial^2 c}{\partial R^2} = X(t;t,T_m)^2 B(t,T_m)N(d_1) - X(t;t,m)^2 KB(t,m)N(d_2)$$
$$+ [-X(t;t,T_m) + X(t;t,m)]^2 B(t,T_m)g(d_1)/(x_c\sigma_c)$$

The analysis for put options is similar, so details are omitted.

Now

$$X(t;t,T_m)^2 \geq X(t;t,m)^2$$

and

$$B(t,T_m)N(d_1) \geq KB(t,m)N(d_2)$$
$$\rightarrow \frac{\partial^2 c}{\partial R} \geq 0$$

The money market account is defined by

$$A(t) = \exp \int_0^t r(u)\mathrm{d}u$$

which implies

$$\mathrm{d}A(t) = A(t)r(t)\mathrm{d}t$$

Appendix B: Multiple factors

We will consider two factors, the extension to an arbitrary number being immediate. We can write (7) in the form

$$R(t) = R_1(t) + R_2(t) \tag{B1}$$

where $R(t) \equiv r(t) - f(0,t)$ and

$$R_i(t) \equiv (\sigma_i^2/2\lambda_i^2)[1 - \exp(-\lambda_i t)]^2 + \sigma_i \int_0^t \exp[-\lambda_i(t-u)]\mathrm{d}\tilde{W}_i(u)$$

Substituting (B1) into (4) and simplifying gives

$$B(t,T) = \exp[-X_1(t;t,T)R_1(t) - X_2(t;t,T)R_2(t) + a(t,T)] \tag{B2}$$

where

$$X_i(t;t,T) \equiv \frac{[1 - \exp -\lambda_i(T-t)]}{\lambda_i}, \qquad i = 1,2,$$

and

$$a(t,T) \equiv -\int_t^T f(0,u)\mathrm{d}u - \sum_{i=1}^{2}(\sigma_i^2/4\lambda_i)X_i(t;t,T)^2[1 - \exp(-2\lambda_i t)]$$

The expression for a Treasury bill futures contract can be written in the form

$$F(t;n) = F(0,n)\exp\left(-\left\{\sum_{i=1}^{2} X_i(t;n,T)\int_0^t \sigma_i \exp[-\lambda_i(t-u)]\mathrm{d}\tilde{W}_i(u) + (\sigma_i^2/4\lambda_i)X_i(t;n,T)^2[1-\exp(-2\lambda_i t)]\right\}\right)$$

where $X_i(t;n,T) \equiv [\exp -\lambda_i(n-t) - \exp -\lambda_i(T-t)]/\lambda_i$. Substituting for $R_i(t)$ and simplifying gives

$$F(t;n) = F(0,n)\exp[-X_1(t;n,T)R_1(t) - X_2(t;n,T)R_2(t) + a_F(t,T)]$$

where

$$a_F(t,T) = \sum_{i=1}^{2}(\sigma_i^2/2\lambda_i^2)X_i(t;n,T)[1-\exp(-\lambda_i t)]^2 - (\sigma_i^2/4\lambda_i)X_i(t;n,T)^2[1-\exp(-2\lambda_i t)]$$

Appendix C

We want to consider a change of the form

$$F(R+\Delta R, N+\Delta N) - F(R,N) = [F(R+\Delta R, N) - F(R,N)] + [F(R+\Delta R, N+\Delta N) - F(R+\Delta R, N)]$$

Consider first the Taylor series expression of

$$F(R+\Delta R, N) - F(R,N) = \frac{\partial F}{\partial R}(R,N)\Delta R + \frac{1}{2}\frac{\partial^2 F}{\partial R^2}(R,N)(\Delta R)^2 + \cdots$$

Next consider

$$F(R+\Delta R, N+\Delta N) = F(R, N+\Delta N) + \frac{\partial F}{\partial R}(R, N+\Delta N) + \frac{1}{2}\frac{\partial^2 F}{\partial R^2}(R, N+\Delta N)(\Delta R)^2 + \cdots$$

and

$$F(R+\Delta R, N) = F(R,N) + \frac{\partial F}{\partial R}(R,N)\Delta R + \frac{1}{2}\frac{\partial^2 F}{\partial R^2}(R,N)(\Delta R)^2 + \cdots$$

so that

$$F(R+\Delta R, N+\Delta N) - F(R+\Delta R, N) = F(R, N+\Delta N) - F(R, N)$$

$$+ \left[\frac{\partial F}{\partial R}(R, N+\Delta N) - \frac{\partial F}{\partial R}(R, N)\right]\Delta R$$

$$+ \frac{1}{2}\left[\frac{\partial^2 F}{\partial R^2}(R, N+\Delta N) - \frac{\partial^2 F}{\partial R^2}(R, N)\right](\Delta R)^2 + \cdots$$

Now we can write

$$\left[\frac{\partial F}{\partial R}(R, N+\Delta N) - \frac{\partial F}{\partial R}(R, N)\right]\Delta R$$

$$= \left[\frac{\partial F}{\partial R}(R, N+\Delta N) - \frac{\partial F}{\partial R}(R, N)\right]\Delta N.\Delta R = 0$$

since by assumption ΔN and ΔR are independent and the $\Pr(\Delta N = 1) \approx 0$, the interaction terms can be ignored. Therefore

$$F(R+\Delta R, N+\Delta N) - F(R, N) = \frac{\partial F}{\partial R}(R, N)\Delta R + \frac{1}{2}\frac{\partial^2 F}{\partial R^2}(R, N)(\Delta R)^2 + \cdots$$

$$+ \frac{\Delta F}{\Delta N}.\mathrm{d}N$$