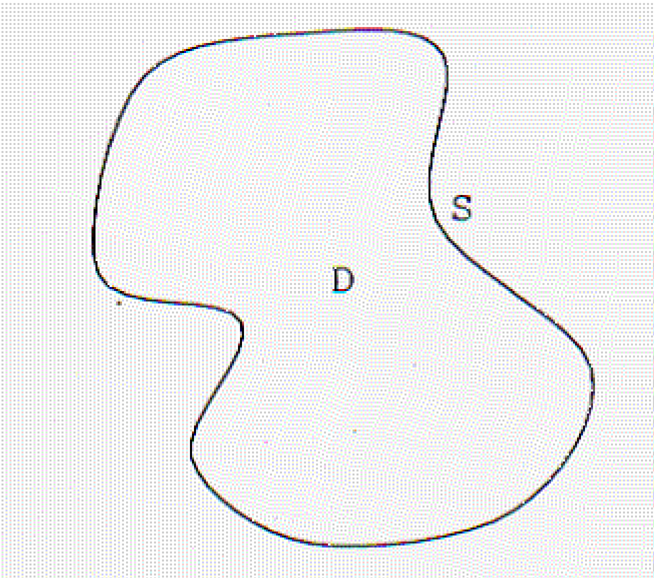
Integral Equation Formulation of the Interior Laplace Problem

In this document we consider a particular boundary value problem[1] that is also the simplest boundary value problem for any serious study of the boundary element method; Laplace's equation within a bounded domain subject to a specified boundary condition. Let the Laplace equation[2] govern an arbitrary closed region D with boundary S, as illustrated in the following figure.



The domain of the interior Laplace problem.

The aim is to solve the Laplace equation

$$\nabla^2 \varphi(\mathbf{p}) = 0 \quad (\mathbf{p} \in D)\,. \tag{1}$$

The boundary condition is assumed to take a general (Robin) form

$$\alpha(\boldsymbol{p})\,\varphi(\boldsymbol{p}) + \beta(\boldsymbol{p})\,v(\boldsymbol{p}) = f(\boldsymbol{p}) \quad (\boldsymbol{p} \in S)\,. \tag{2}$$

where α, β and f are real-valued functions defined on the boundary S. Note that the boundary condition defined by (2) includes the Dirichlet case $(\alpha(\boldsymbol{p}) = 1,\ \beta(\boldsymbol{p}) = 0)$ and the Neumann case $(\alpha(\boldsymbol{p}) = 0,\ \beta(\boldsymbol{p}) = 1)$.

In this section we consider the integral equation fomulations of the interior Laplace equation. An incident field (that is the field that would exist if the boundary S was not present) along with a general boundary condition (2) is included and this leads to more generalised boundary integral equations.

[1] Boundary Value Problema and Boundary Conditions
[2] Laplace Equation

Direct Formulation

The application of Green's second theorem to the Laplace equation gives the following equations

$$\{M\varphi\}_S(\boldsymbol{p}) + \varphi(\boldsymbol{p}) = \{L\,v\}_S(\boldsymbol{p}) \quad (\boldsymbol{p} \in D)\,, \tag{3}$$

$$\{M\varphi\}_S(\boldsymbol{p}) + \frac{1}{2}\varphi(\boldsymbol{p}) = \{L\,v\}_S(\boldsymbol{p}) \quad (\boldsymbol{p} \in S)\,, \tag{4}$$

where $v(q) = \frac{\partial \varphi}{\partial n_q}$ and L and M are two of the Laplace integral operators[3]. Note that the normals to the boundary are taken to be in the outward direction.

The above equations can be utilised to solve the interior Laplace equation in the manner introduced in *Outline of the Boundary Element Method*[4]; the solution of equation (4) gives (approximations to) both φ and v on the boundary S, and - once these boundary functions are known - equation (3) can be utilised to obtain an approximation to $\varphi(\boldsymbol{p})$ for any point $\boldsymbol{p}$ in the domain. In general, first kind equations[5] are found to be difficult or inefficient to solve in that the operator over which the equation is solved and the matrices that arise in their equivalent linear systems are ill-conditioned[6 7]. However, the integral L integral operator has a singular kernel and in practice the problems found with first kind equations do not present themselves in this case.

Further boundary integral equation formulations can be found through differentiating each term of equation (3) with respect to any vector $\boldsymbol{v}_p$ gives

$$\frac{\partial}{\partial v_p}\{M\varphi\}_S(\boldsymbol{p}) + \frac{\partial}{\partial v_p}\varphi(\boldsymbol{p}) = \frac{\partial}{\partial v_p}\{L\,v\}_S(\boldsymbol{p}) \quad (\boldsymbol{p} \in D)\,,$$

or $$\{N\varphi\}_S(\boldsymbol{p};\boldsymbol{v}_p) + \frac{\partial \varphi(\boldsymbol{p})}{\partial v_p} = \{M^t\,v\}_S(\boldsymbol{p};\boldsymbol{v}_p) \quad (\boldsymbol{p} \in D)\,, \tag{5}$$

which for the points on the boundary gives

$$\{N\varphi\}_S(\boldsymbol{p};\boldsymbol{n}_p) = \{M^t\,v\}_S(\boldsymbol{p};\boldsymbol{n}_p) - \frac{1}{2}v(\boldsymbol{p}) \quad (\boldsymbol{p} \in S)\,, \tag{6}$$

using the integral operator notation[3].

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[3] The Laplace Integral Operators
[4] Outline of the Boundary Element Method
[5] Integral Equations
[6] Computational Methods for Integral Equations by L M Delves and J L Mohamed
[7] Condition Number of a Matrix

Th initial stage of the boundary element method can be based on either equation (4) or equation (6), although, for simplicity, equation (4) is more straightforward. It is also possible to use a hybrid of the two equations[8]:

$$\{M + \tfrac{1}{2}I + \mu N\}_S\varphi(\boldsymbol{p}) = \{L + \mu\left(M^t - \tfrac{1}{2}I\right)\}_S v(\boldsymbol{p}) \qquad (\boldsymbol{p} \in S).$$

Indirect Formulation

Indirect integral equation formulations can be obtained by writing φ as a single or double layer potential;

$$\varphi\,(\mathbf{p}) = \{L\,\sigma_0\}_S(\mathbf{p}) \quad \text{or} \quad \varphi\,(\mathbf{p}) = \{M\,\sigma_\infty\}_S(\boldsymbol{p}) \quad (\boldsymbol{p} \in D)\,, \qquad \text{(7a,b)}$$

where the σ_0 and σ_∞ are source density functions defined on S. For points on the boundary the equations become boundary integral equations;

$$\varphi\,(\mathbf{p}) = \{L\,\sigma_0\}_S(\mathbf{p}) \quad \text{or} \quad \varphi\,(\mathbf{p}) = \left\{(M - \frac{1}{2}\,I)\sigma_\infty\right\}_S(\boldsymbol{p}) \quad (\boldsymbol{p} \in S)\,, \qquad \text{(8a,b)}$$

where the jump condition[9] has been taken into account in the second equation.

For Dirichlet problems, φ is known on S and hence either of equations in (7) can be used to find the layer potential σ_0 or σ_∞. The relevant equation in (7) can then be used to find φ in the domain. However, the equations above do not contain $\frac{\partial\varphi}{\partial n}(= v)$and so is insufficient for Neumann (or Robin) problems.

The application of the indirect formulations to the other boundary condidtions can be achieved by differentiating the integral relations (7):

$$\frac{\partial\varphi\,(\mathbf{p})}{\partial v_p} = \frac{\partial}{\partial v_p}\{L\,\sigma_0\}_S(\mathbf{p}) \quad \text{or} \quad \frac{\partial\varphi\,(\mathbf{p})}{\partial v_p} = \frac{\partial}{\partial v_p}\{M\,\sigma_\infty\}_S(\boldsymbol{p}) \quad (\boldsymbol{p} \in D)\,,$$

or, using operator definitions and notation,

$$\frac{\partial\varphi\,(\mathbf{p})}{\partial v_p} = \{M^t\,\sigma_0\}_S(\mathbf{p}; \boldsymbol{v}_p) \quad \text{or} \quad \frac{\partial\varphi\,(\mathbf{p})}{\partial v_p} = \{N\,\sigma_\infty\}_S(\boldsymbol{p}; \boldsymbol{v}_p) \quad (\boldsymbol{p} \in D)\,. \qquad (9)$$

By allowing the point $\mathbf{p}$ to move to the boundar and taking into accoun the jump discontinuitities and $\boldsymbol{v}_p$ become $\boldsymbol{n}_p$, the normal to the boundary at $\mathbf{p}$, the following integral equation formaulations ae obtained:

$$v(\boldsymbol{p}) = \left\{(M^t - \frac{1}{2}\,I)\sigma_0\right\}_S(\mathbf{p}; \boldsymbol{v}_p) \quad \text{or} \quad v(\boldsymbol{p}) = \{N\,\sigma_\infty\}_S(\boldsymbol{p}; \boldsymbol{v}_p) \quad (\boldsymbol{p} \in D)\,. \qquad (10)$$

[8] A Gentle Introduction to the Boundary Element Method in Matlab/Freemat
[9] The Laplace Integral Operators

As for the direct integral equations, a hybrid indirect equation can be formed

$$\varphi(\boldsymbol{p}) = \{(L + \rho M)\sigma_\rho\}_S(\boldsymbol{p}) \quad (\boldsymbol{p} \in D)\,. \tag{11}$$

This gives rise to the following boundary integral equation:

$$\varphi(\boldsymbol{p}) = \{(L + \rho(M - ½I)\sigma_\rho)\}_S(\boldsymbol{p}) \quad (\boldsymbol{p} \in S)\,. \tag{12}$$

Differentiating equation (11) with respect to a vector $\boldsymbol{v}_p$ gives the following equation

$$\frac{\partial\varphi(\boldsymbol{p})}{\partial\boldsymbol{v}_p} = \frac{\partial}{\partial\boldsymbol{v}_p}\{\,(L + \rho M)\sigma_\rho\}_S(\boldsymbol{p}) = \{(M^t + \rho N)\sigma_\rho\}_S(\boldsymbol{p};\boldsymbol{v}_p) \quad (\boldsymbol{p} \in D)\,. \tag{13}$$

Taking the limit as the point $\boldsymbol{p}$ approaches a point on the boundary, and letting $\boldsymbol{v}_p$ be the unit outward normal to the boundary at $\boldsymbol{p}$ (i.e. $\boldsymbol{n}_p$) gives rise to the following boundary integral equation:

$$v(\boldsymbol{p}) = \{(M^t + ½I + \rho N\sigma_\rho)\}_S(\boldsymbol{p};\boldsymbol{n}_p) \quad (\boldsymbol{p} \in S)\,. \tag{14}$$

Field Modification

The potential field need not be a result of the boundary and boundary condition alone; the surface may simply act to modify an existing field. In such cases there is an incident field in the domain, termed $\varphi^i(\boldsymbol{p})$, which is the field that would exist if there were no boundaries. Such problems can also be solved by the boundary element method, it only requires a generalisation of the integral equations and the corresponding alteration of the boundary element methods.

Direct formulation

In the simplest case, the equation (3) may be generalised as follows:

$$\varphi(\boldsymbol{p}) = \varphi^i(\boldsymbol{p}) + \{\,L\,v\,\}_S\,(\boldsymbol{p}) - \{\,M\varphi\}_S\,(\boldsymbol{p}) \quad (\boldsymbol{p} \in D)\,, \tag{15}$$

the solution $\varphi(\boldsymbol{p})$ is equated to the incident field $\varphi^i(\boldsymbol{p})$and modified by the other terms. The boundary integral equation that arises from the formulation (4) is as follows:

$$\{\,M\varphi\}_S\,(\boldsymbol{p}) + \frac{1}{2}\,\varphi(\boldsymbol{p}) = \varphi^i(\boldsymbol{p}) + \{\,L\,v\,\}_S\,(\boldsymbol{p}) \quad (\boldsymbol{p} \in S)\,, \tag{16}$$

Similarly, the incident field can be included to generalise equation (5),

$$\{ N\varphi\}_S\,(\boldsymbol{p};\boldsymbol{v}_p\,) \;+\; \frac{\partial\varphi(\boldsymbol{p})}{\partial v_p} \;=\; \frac{\partial\varphi^i(\boldsymbol{p})}{\partial v_p} + \{\, M^t\, v\,\}_S\,\big(\boldsymbol{p};\boldsymbol{v}_p\big) \quad (\boldsymbol{p}\;\in\; D). \tag{17}$$

Which, taking the point **p** to the surfaceand including the jump discontinuity, returns the following equation

$$\{ N\varphi\}_S\,(\boldsymbol{p};\boldsymbol{n}_p\,) \;=\; \{\, M^t\, v\,\}_S\,\big(\boldsymbol{p};\boldsymbol{n}_p\big) \;+ v^i(\boldsymbol{p}) - \frac{1}{2}\, v(\boldsymbol{p}) \quad (\boldsymbol{p}\;\in\; S)\,. \tag{18}$$

Indirect formulation

Generalising equations (7-8) to include the incident field gives rise to the following integral equations:

$$\varphi\,(\mathbf{p}) \;=\; \varphi^i(\boldsymbol{p}) + \{L\;\sigma_0\,\}_S(\mathbf{p}) \quad \text{or} \quad \varphi\,(\mathbf{p}) \;=\; \varphi^i(\boldsymbol{p}) + \{M\;\sigma_\infty\,\}_S(\boldsymbol{p}) \quad (\boldsymbol{p}\;\in\; D)\,, \tag{19}$$

$$\varphi\,(\mathbf{p}) = \;\varphi^i(\boldsymbol{p}) + \{L\;\sigma_0\,\}_S(\mathbf{p}) \text{ or } \varphi\,(\mathbf{p}) = \varphi^i(\boldsymbol{p}) + \left\{\left(M - \frac{1}{2}\,I\right)\sigma_\infty\right\}_S(\boldsymbol{p})\;(\boldsymbol{p}\;\in\;S). \tag{20}$$

The generalisation of equation (9) to include an incident field gives the following:

$$\frac{\partial\varphi\,(\mathbf{p})}{\partial v_p} = \frac{\partial\varphi^i(\boldsymbol{p})}{\partial v_p} + \;\{M^t\;\sigma_0\,\}_S\big(\mathbf{p};\boldsymbol{v}_p\big) \quad \text{or} \quad \frac{\partial\varphi\,(\mathbf{p})}{\partial v_p} = \frac{\partial\varphi^i(\boldsymbol{p})}{\partial v_p} + \{N\;\sigma_\infty\,\}_S\big(\boldsymbol{p};\boldsymbol{v}_p\big) \quad (\boldsymbol{p}\;\in\; D)\,. \tag{21()}$$

By allowing the point **p** to move to the boundary and taking into accoun the jump discontinuitities and $\boldsymbol{v}_p$ become $\boldsymbol{n}_p$, the normal to the boundary at **p**, the following integral equation formaulations ae obtained:

$$v(\boldsymbol{p}) = \; v^i(\boldsymbol{p}) + \left\{(M^t - \frac{1}{2}\;I)\sigma_0\right\}_S\big(\mathbf{p};\boldsymbol{v}_p\big) \quad \text{or} \quad v(\boldsymbol{p}) \;=\; v^i(\boldsymbol{p}) + \{N\;\sigma_\infty\,\}_S\big(\boldsymbol{p};\boldsymbol{v}_p\big) \quad (\boldsymbol{p}\;\in\; D)\,. \tag{21}$$